Eigenvalue statistics in non-Hermitean Wishart Random Matrices at $\beta = 2^*$

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Asymptotic analysis of the spectrum

V BRUNEL Workshop on Random Matrix Theory, BRUNEL UNIVERSITY, WEST LONDON.

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*** Supported by the ISF grant no: 414/08**

OUTLINE

1. Introduction to non-Hermitean Wishart Random Matrices and their applications.
2. Non-Hermitean Wishart with complex entries ($\beta = 2$).
3. Calculation of the joint probability density function for all eigenvalues and the p-point correlation function.
4. Asymptotic analysis of the density of states; various scaling regimes.
    - Interdisciplinary applications (Econophysics, bio-medical etc).
    - Physical applications (QCD).
5. Conclusions and open problems.

Classical Wishart matrices:

1. Introduced by J. Wishart in 1928 in the context of multivariate statistics.
2. Definition: $W = XX^\dagger$, here X is a rectangular $(n \times p)$ matrix with no specific symmetries.
3. Constructed from n sets of uncorrelated, discretized in time, Gaussian random processes $x_\alpha(t_j)$ as

$$X_{\alpha,j} = x_\alpha(t_j), \quad W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j}\bar{X}_{\beta,j}, \qquad \alpha,\ \beta = 1, ...n$$

W is the most random correlation matrix.

Applications

1. **Wishart matrix** *(or most random correlation matrix) acts like a template or as a null hypothesis in testing correlations within a given complex system.*
2. **The empirical correlation matrix** constructed from the experiment (data) contains the information about correlations between time-series. Consider an example from Econophysics. Let $x_\alpha(t)$ represents the time-dependent price changes of the α-th asset (say gold items belonging to a person). The empirical correlation matrix between α-th and β-th assets is defined as:

$$C_{\alpha,\beta} = \sum_{j=1}^{p} x_\alpha(t_j) x_\beta(t_j), \quad \alpha, \beta \in (1, .., n)$$

.

3. If $x_\alpha(t_j)$ and $x_\beta(t_j)$ are uncorrelated Gaussian random processes, then $C_{\alpha,\beta} \to W_{\alpha,\beta}$
4. In the case of correlations, the maximal eigenvalue of C is much bigger than that of the Wishart matrix (of same dimensions and variance chosen for the best fit[1]). It is then expected that the eigenvector corresponding to this large eigenvalue of C contains important correlation information.

[1] See for example; L. Laloux et al, PRL, 83, 1467 (1999)

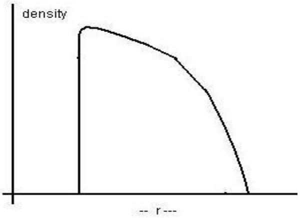
Spectral properties of the Wishart random matrices

1. Wishart matrix $W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j}\bar{X}_{\beta,j}, \quad \alpha,\ \beta = 1,...n \quad (p > n)$.
2. The J. P. D. F of its eigenvalues:
   $P_n^{(\beta)}(\lambda_1,...,\lambda_p) \propto \prod_{j>k=1}^{p} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^{p} \lambda_j{}^{\frac{\beta}{2}(n-p+1)-1} e^{-\lambda_j/2}$

1. The mean density of eigenvalues in the limit $n = N \to \infty$ with $p/N = Q$ kept fixed is $\rho_W(\lambda)$, with $Q = p/N$ is the "aspect ratio", and $\lambda_{max/min} = 1 + 1/Q \pm 2/\sqrt{Q}$.
2. Characteristic eigenvalue gap (no λ for $\lambda < \lambda_{min}$), and upper-edge cut (no λ for $\lambda > \lambda_{max}$) for infinite sized matrices.
3. Hermitean anti-Wishart regime ($p < n, \quad Q \to 1/Q$) does not bring any new qualitative results.

$$\underbrace{\rho_W(\lambda) = \frac{Q}{2\pi}\frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda}}$$

[Marcenko-Pastur law]



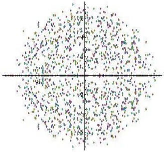
Non-Hermitean Wishart random matrices (New!!)

In problems where time-series generating systems are two different systems and the interest is in studying the correlations between them (e.g. correlations between functioning of the left and right hemispheres of a brain) the correlation matrix takes the form $\tilde{W} = XY^{\dagger}$ [2]. We call this the non-Hermitean Wishart random matrix;

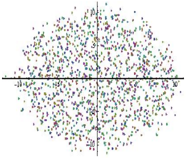
1. $W_{\alpha,\beta} = \sum_{j=1}^{p} X_{\alpha,j}\bar{Y}_{\beta,j}, \quad \alpha,\ \beta = 1, ...n, \quad (p > n)$. The matrix is no longer Hermitean and its spectrum becomes complex valued.
2. What are the differences between well studied Ginibre and non-Hermitean Wishart?
3. In the following we consider non-Hermitean Wishart matrices with complex entries ($\beta = 2$).
4. Strong relation to QCD inspired non-Hermitean Laguerre ensembles. [*J. C. Osborn, PRL,* **93** *222001 (2004); G. Akemann, Nucl.Phys. B* **730** *253(2005); G. Akemann, M.J. Phillips, H.-J. Sommers, arXiv:0911.1276*]

[2] J. Kwapien et al., PRE, 62, 5557 (2000)

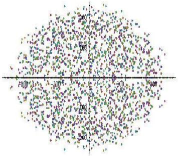
Results from computer experiments



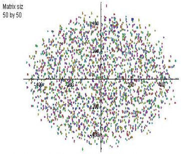
Ginibre, beta = 1, 1500 eigen-values



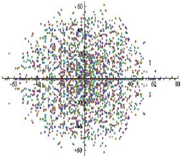
Gìnibre, beta = 2 1500 eigen-values



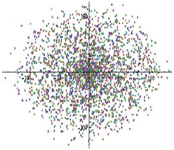
Wishart, beta=1, p - n = nu = 950 2500 eigen-values



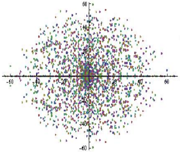
Wishart, beta = 2, p - n = nu = 950, 2500 eigen-values



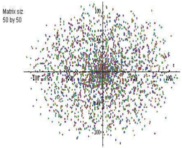
Wishart, beta=1, p - n = nu = 10 2500 eigen-values



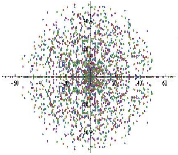
Wishart, beta = 2, p - n = nu = 10 2500 eigen-values



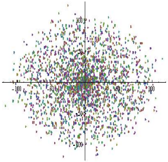
Wishart, beta=1, p - n = nu = 2 2500 eigen-values



Wishart, beta = 2, p - n = nu = 2, 2500 eigen-vaues



Wishart, beta=1, p - n = nu = 0 2500 eigen-values



Wishart, beta = 2, p = n, or, nu = 0, 2500 eigen values

Analytic theory for non-Hermitean Wishart random matrices

Starting point: J.P.D.F. of matrix elements:

$$P_{n,p}(W)DW = \langle \delta^{(2n^2)}(W - XY^\dagger)\rangle_{X,Y} DW = \int \frac{1}{\text{Det}^p(1+Y_nY_n^\dagger)} \text{Exp}[i\text{tr}(W^\dagger Y + WY^\dagger)]DY_n DW.$$

Goal: J. P. D. F. of all eigenvalues and correlation functions.

1. First Schur-decompose W as, $W = UR_nU^\dagger$
2. The Jacobian of transformation is: $DW = |\Delta_n(W)|^2(dU)\prod_{j=1}^n dw_j D\tilde{R}_n$.
3. Using the cyclic property of the trace in the exponent and delta functions from the integrals over R_n, we conclude that X_n is lower triangular matrix, and,
4.

$$R_n = \begin{pmatrix} w_1 & & \tilde{R}_n \\ & \vdots & \\ \phi & & w_n \end{pmatrix}$$

$$P_{n,p}(W)DW = C_U \int |\Delta_N(w)|^2 e^{i(w_j x_{jj}^\dagger + \bar{w}_j x_{jj})} \prod_{j=1}^N dw_j \prod_{j=1}^n d^2x_{jj}$$

$$\times \int \frac{D\tilde{X}_n}{\det^p(\mathbb{I} + X_nX_n^\dagger)}.$$

$$X_n = \begin{pmatrix} x_{11} & & \phi \\ & \vdots & \\ x_{jk} & & x_{nn} \end{pmatrix}$$

We solve by using successive size reduction of the matrix X_n, decompose X_n as:

1. $\mathbb{I} + X_n X_n^\dagger = \mathbb{I} + \begin{pmatrix} X_{n-1} & \phi \\ u^\dagger & x_{nn} \end{pmatrix} \begin{pmatrix} X_{n-1}^\dagger & u \\ \phi & x_{nn}^* \end{pmatrix}$
2. By decomposing $D\tilde{X}_n = (du \;\; du^\dagger) D\tilde{X}_{n-1}$, we derive,

$$P_{n,p}(W)DW = C_U \int |\Delta_n(w)|^2 e^{i(w_j x_{jj}^\dagger + \bar{w}_j x_{jj})} \prod_{j=1}^{n} dw_j \prod_{j=1}^{n} d^2 x_{jj} \int \frac{D\tilde{X}_{n-1}}{\det^p(\mathbb{I} + X_{n-1}X_{n-1}^\dagger)}$$

$$\times \int \frac{du \; du^\dagger}{(\mathbb{I} + u^\dagger u + |x_{nn}|^2 - u^\dagger X_{n-1}^\dagger (\mathbb{I} + X_{n-1}X_{n-1}^\dagger)^{-1} X_{n-1} u)^p}$$

3. With a change of co-ordinates as $v = (1 + |x_{nn}|^2)^{-1/2} (\mathbb{I} - X_{n-1}^\dagger (\mathbb{I} + X_{n-1}X_{n-1}^\dagger)^{-1} X_{n-1})^{1/2} u$, and with some manipulations, $P_{n,p}(W)DW$ reduces to,

$$C_U' \int |\Delta_n(w)|^2 e^{i(w_j x_{jj}^\dagger + \bar{w}_j x_{jj})} \prod_{j=1}^{n} dw_j \frac{\prod_{j=1}^{n} d^2 x_{jj}}{(1 + |x_{nn}|^2)^{p-n+1}} \int \frac{D\tilde{X}_{n-1}}{\det^{p-1}(\mathbb{I} + X_{n-1}X_{n-1}^\dagger)}$$

4. Then p recursions reduce it to a single integral, whose exact solution finally gives us:

The joint probability density function of all eigenvalues and the density of states

1. $P(w_1, ..., w_n) = C \prod_{j=1}^{n} |w_j|^{\nu} K_{\nu}(2|w_j|)|\Delta_n(w)|^2$, and $\nu = p - n$.
2. $C = (\frac{\pi}{2})^{(-n)}(\prod_{j=1}^{n}[\Gamma(j)\Gamma(j+\nu)])^{(-1)}$
3. The Dyson integration theorem brings the density of states after successive integrations as:

$$\rho(|w|) = \frac{2}{\pi}|w|K_{\nu}(2|w|)\underbrace{\sum_{k=0}^{n-1}\frac{|w|^{2k}}{k!(k+\nu)!}}_{F_{p,n}(|w|^2)}$$

4. This is also obtained by, J. C. Osborn, PRL, **93** 222001 (2004), for $\beta = 1$ case see: G. Akemann, M.J. Phillips, H.-J. Sommers. They use a different method called "variational method". But the asymptotic analysis is new here.

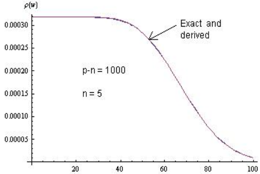
Solution by constructing a differential equation and a scaling ansatz

1. Define; $F_{p,n}(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!(k+\nu)!}$.
2. By differentiating one can construct the differential equation:
   $xF_{p,n}^{''}(x) + (\nu+1)F_{p,n}^{'}(x) - F_{p,n}(x) = -\frac{x^{n-1}}{\Gamma(n)\Gamma(n+\nu)}$
3. Scaling ansatz: $F_{p,n}|_{x=t\nu^c} = g(\nu,n)f_n(t), \quad \nu \to \infty$.

- The solution of the above differential equation gives us: $F_{p,n}(x) = e^{\frac{x}{\nu}} \frac{\Gamma(n, \frac{x}{\nu})}{\nu!\Gamma(n)}$,

  and the mean density takes the compact form,

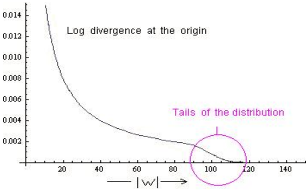
$$\boxed{\rho_{p,n}(|w|) \simeq \frac{1}{\pi\nu} \frac{\Gamma\left(n, \frac{|w|^2}{\nu}\right)}{\Gamma(n)}}$$

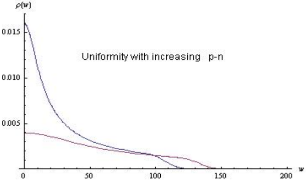


4. For $n = N >> 1$, we have the Girko's law: $\frac{1}{\pi\nu}\Theta(\sqrt{\nu N} - |w|)$.
5. **Alternatively, the same formula can be obtained from the direct analysis of the series $\sum_{k=0}^{n-1} \frac{x^k}{k!(k+\nu)!}$ in the large ν limit.**

Asymptotic analysis:

Case II: Physical problems regime (QCD motivated case, large N but finite ν)





1. For $N \to \infty$ the sum $\sum_{k=0}^{N-1} \frac{x^k}{k!(k+\nu)!} \to |w|^{-\nu} I_\nu(2|w|)$. The mean density in the large N limit is: $\rho_{p,N}(|w|) = \frac{2}{\pi} K_\nu(2|w|) I_\nu(2|w|)$.
2. Behaviour near origin ($|w| \to 0$):
$$\rho_{p,N}(|w|) = \left\{ \begin{array}{ccc} -\frac{2}{\pi}\log(|w|) & \text{for} & \nu = 0 \\ \frac{1}{\pi\nu} = \text{Const.} & \text{for} & \nu \neq 0 \end{array} \right\}$$
3. Behaviour at large w, $1 << w << N$: $\rho_{p,n}(|w|)|_{N\to\infty} = \frac{1}{2\pi|w|}$. This is in contrast with the constant density profile in the Ginibre case: $\rho^{\beta=2}_{Ginibre}(x\sqrt{N}) = \frac{1}{\pi}$ for $x < 1$ and zero otherwise.

Tails of the density (rare fluctuations)

1. Breakdown of $\frac{1}{2\pi|w|}$ – law at $|w| = |w_c|$:
   $2\pi \int_0^{|w|_c} |w|(\frac{1}{2\pi|w|})d|w| = N, \quad \Rightarrow |w|_c = N.$ Thus, something non-trivial should happen at about $|w| \sim N$. To understand it we have to treat the large N limit more carefully.
2. Idea is to utilize the Euler-Maclaurin formula to reduce the sum $\left[F_{p,N} = \sum_{k=0}^{N-1} \frac{x^k}{k!(k+\nu)!}\right]$ to an integral in the large N limit, and solve it using the saddle point approximation.
3. Case I: $|w| < N$

$$\boxed{\rho_{p,N}(|w|) = \frac{1}{4\pi|w|}\left(\text{Erf}(\sqrt{|w|}) - \text{Erf}\left(\frac{N-|w|}{\sqrt{|w|}}\right)\right)}$$

Tails of the density (rare fluctuations)

1. Case II: $|w| > N$

$$\boxed{\rho_{p,N}(|w|) = \frac{1}{4\pi|w|}\left(1 + \mathsf{Erf}\left(\frac{N - |w|}{\sqrt{|w|}}\right)\right)}$$

2. Number of eigenvalues in the tails (i.e., for $|w| > N$)

$$N_{+} \simeq \frac{1}{2}\sqrt{\frac{N}{\pi}} + \frac{1}{8}$$

Conclusions

- An exact treatment of non-Hermitean Wishart random matrices at $\beta = 2$.
- Various large $-N/$ large$-\nu$ scaling regimes analyzed.
- Good agreement with numerical simulations observed.

Open problems

- The non-Hermitean Wishart at $\beta = 1$ symmetry class (work in progress). In this case spectrum is complex valued as in the case $\beta = 2$, but there is a finite probability of real eigenvalues (due to the accumulation of eigenvalues along the real axis).
- In applications where the number of discretizations (p) is less than number of channels (N) (for example, due to unavailability of data points), we are in the non-Hermetian anti-Wishart regime. This is a completely open area.
- **Experimental utilization of the present work remains open, although in such cases people have relied on Ginibre ensembles [see for example, J. Kwapien et al, arXive:physics/0605115]. The present work will improve our understanding of such applications.**