# FRACTAL DIMENSIONS FOR CERTAIN CRITICAL RANDOM MATRIX ENSEMBLES

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VI Brunel Workshop on PMT

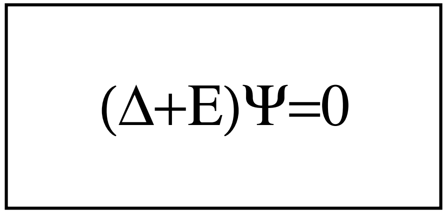
Brunel, 17-18 December 2010

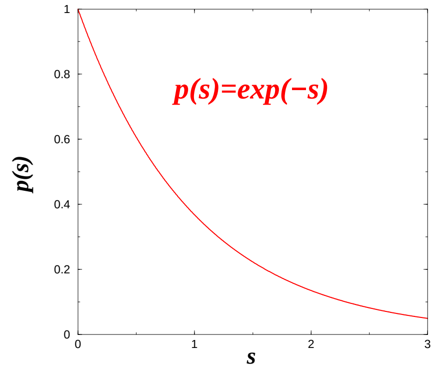
# Outline

- **Critical random matrix ensembles**
- **Perturbation series for fractal dimensions**
  - Strong multifractality
  - Weak multifractality
- **Conjecture:** $\chi = \mathbf{1} - \mathbf{D_1}/\mathbf{d}$
- **Summary**

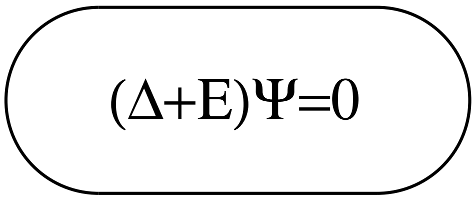
# Well accepted conjectures

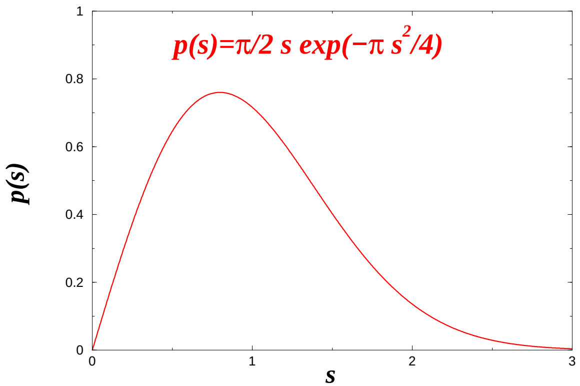
- **Berry, Tabor** (1977):
  **Integrable systems** $\Longrightarrow$ **Poisson statistics**




- **Bohigas, Giannoni, Schmit** (1984):
  **Chaotic systems** $\Longrightarrow$ **Random Matrix Statistics**



# 3-d Anderson model at metal-insulator transition

## 3-d Anderson model

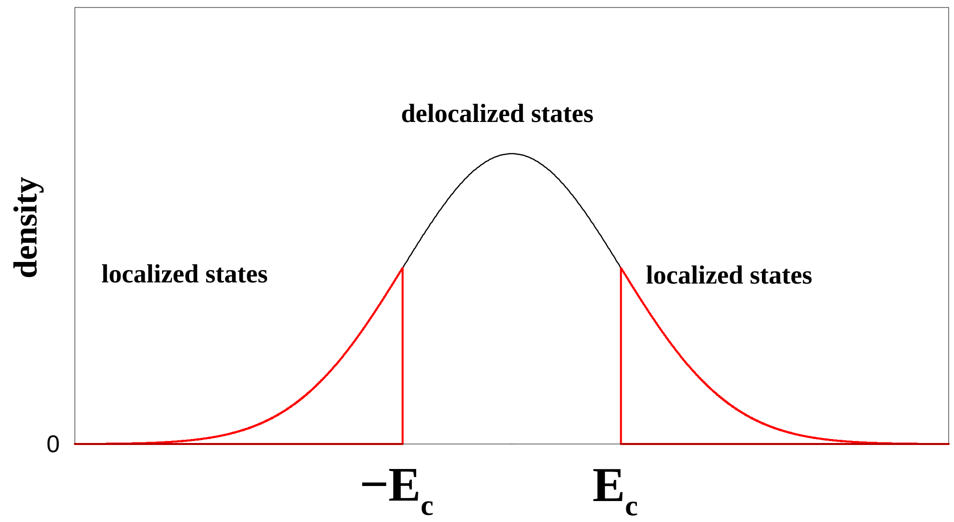
$$H = \sum_i \varepsilon_{\mathbf{i}} a_i^\dagger a_i - \sum_{j=\text{ adjacent to } i} a_j^\dagger a_i$$

$\varepsilon_{\mathbf{i}}$=i.i.d. random variables between $-W/2$ and $W/2$

**Mobility edge**: $\mathbf{E_c(W)}$

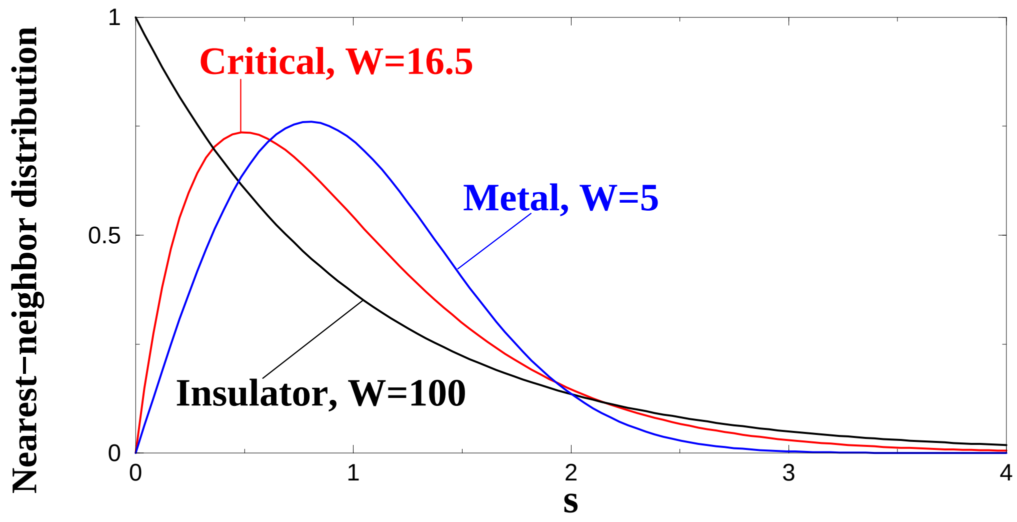
$E_c = 0 \longrightarrow W = W_c = 16.5$

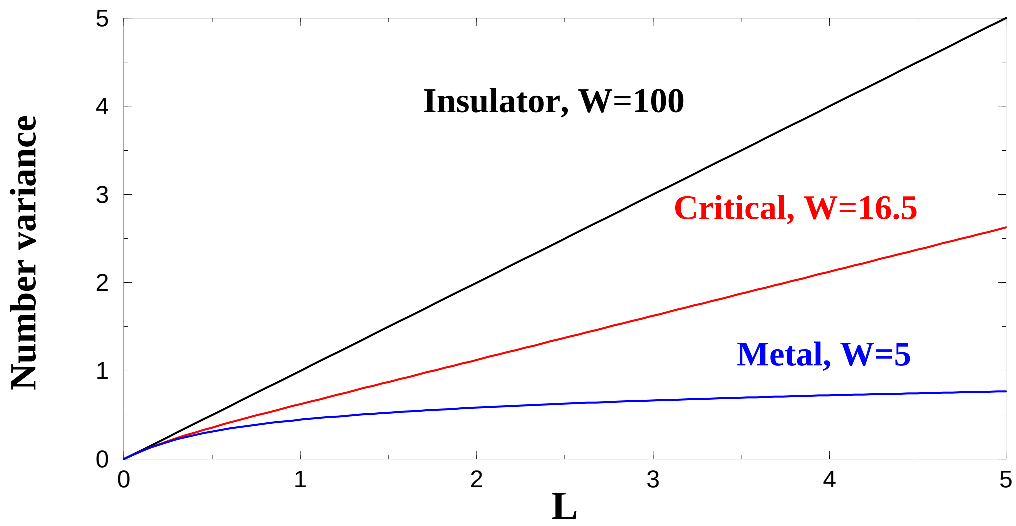
$W > W_c \rightarrow$ all states are localized




- $|E| > E_c$. States are localized. **Poisson statistics** of eigenvalues
- $|E| < E_c$. States are delocalized. **Random matrix statistics**
- $|E| = E_c$. States are neither localized or delocalized. Fractal wave functions. New **intermediate** type of spectral statistics. **Shklovskii et al** (1993)

# Spectral characteristics of 3-d Anderson model at metal-insulator transition



# Characteristic features of intermediate statistics

- **Level repulsion at small distances** as for the **RMT**

$$p(s) \to 0 \ \text{ when } \ s \to 0$$

- **Exponential decrease of $p(s)$ at large distances** as for the **Poisson**

$$p(s) \sim \mathrm{e}^{-as} \ \text{ when } \ s \to \infty$$

- **Linear asymptotics of the number variance**

$$\Sigma^2(L) \equiv \langle (n(L) - \bar{n}(L))^2 \rangle \to \chi L \ \text{ when } \ L \to \infty$$

$\chi =$ **spectral compressibility**
$\chi = 1$ for **Poisson**, $\chi = 0$ for the **RMT**

- **Multi-fractal character of eigenfunctions**

$$\langle |\Psi|^{2q} \rangle \to L^{-(q-1)\mathbf{D_q}} \ \text{ when } \ L \to \infty$$

$D_q = 0$ for the **Poisson**, $D_q = 1$ for the **RMT**

# Random matrix models of intermediate statistics

$$M_{ij} = \varepsilon_{\mathbf{j}} \delta_{ij} + V(i-j)$$

Typically:

$$V(i-j) \sim \frac{g}{|i-j|^{\alpha}}$$

$\varepsilon_{\mathbf{j}}$ = i.i.d. random variables between $-W/2$ and $W/2$

States $i$ and $j$ are in **resonances** provided

$$|\varepsilon_j - \varepsilon_i| \leq |V(i-j)|$$

**Number of resonances** connected with a cite $i$

$$N_{\text{resonances}}(i) \sim \sum_j |V(i-j)|$$

**Levitov** (1990)

- $\alpha > 1 \Longrightarrow$ **localization**
- $\alpha < 1 \Longrightarrow$ **delocalization**
- $\alpha = 1 \Longrightarrow$ **intermediate statistics**

# Critical power law banded random matrices
**(Mirlin et al** (1996))

$N \times N$ Hermitian matrices whose elements, $H_{ij}$, are i.i.d. Gaussian variables (real for $\beta = 1$ and complex for $\beta = 2$) with zero mean $\langle H_{ij} \rangle = 0$ and the **variance** $\langle |H_{ii}|^2 \rangle = 1/\beta$ and

$$\langle |H_{ij}|^2 \rangle = \left( 1 + \frac{(i-j)^2}{b^2} \right)^{-1} \quad \text{for } i \neq j$$

**Perturbation series**: (**Mirlin, Evers** (2000))

- $b \gg 1$: $D_q = 1 - q/(2\pi\beta b)$ , $\chi = 1/(2\pi\beta b)$.

$$\boxed{D_q = 1 - q\chi}$$

- $b \ll 1$: ($c_\beta = 1$ for $\beta = 1$, $c_\beta = \pi/\sqrt{8}$ for $\beta = 2$)

$$D_q = 4bc_\beta \frac{\Gamma(q - 1/2)}{\sqrt{\pi}\Gamma(q)} \ , \ \chi = 1 - 4bc_\beta$$

$$\boxed{D_q = \frac{\Gamma(q-1/2)}{\sqrt{\pi}\Gamma(q)} (1 - \chi)},$$

$$\boxed{D_1 = 1 - \chi}$$

**Absence of universality** for spectral statistics

# Ruijsennars-Schneider ensemble

(**E.B., Schmit, Giraud** (2009))

- Ruijsenaars - Schneider classical integrable model

$$H(p,q)=\sum_{j=1}^{N}\cos(p_j)\prod_{k\neq j}\left(1-\frac{\sin^2\pi a}{\sin^2\frac{\mu}{2}(q_j-q_k)}\right)^{1/2}$$

- Ruijsenaars - Schneider ensemble of random matrices

$N\times N$ unitary matrix related with the **Lax matrix** of this model

$$\mathbf{M_{kp}}=\mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_\mathbf{k}}\frac{\mathbf{1}-\mathrm{e}^{\mathbf{2}\pi\mathrm{i}\mathbf{a}}}{\mathbf{N}[\mathbf{1}-\mathrm{e}^{\mathbf{2}\pi\mathrm{i}(\mathbf{k}-\mathbf{p}+\mathbf{a})/\mathbf{N}}]}$$

$\boldsymbol{\Phi}_\mathbf{m}$ = independent random variables (phases) uniformly distributed in $[0,2\pi]$

$\mathbf{a}$ = parameter

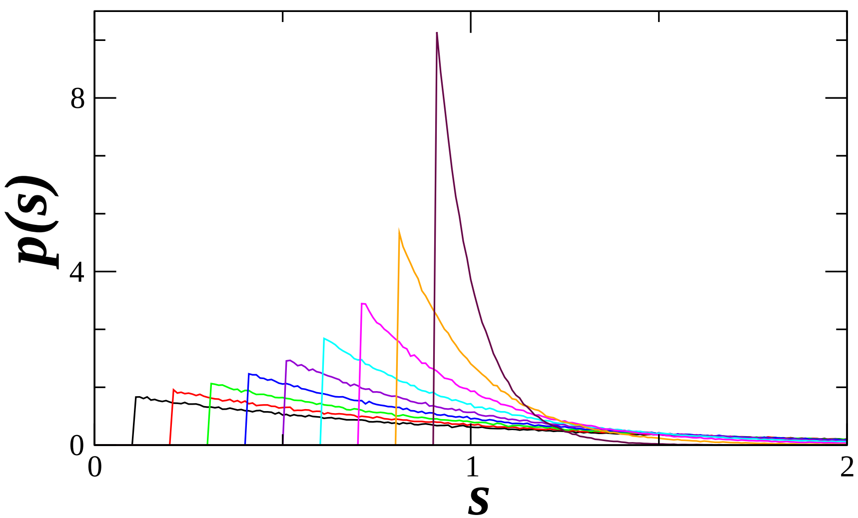
All spectral correlation functions can be calculated analytically using the transfer operator approach

Everything strongly depends on integer part of $\mathbf{a}$

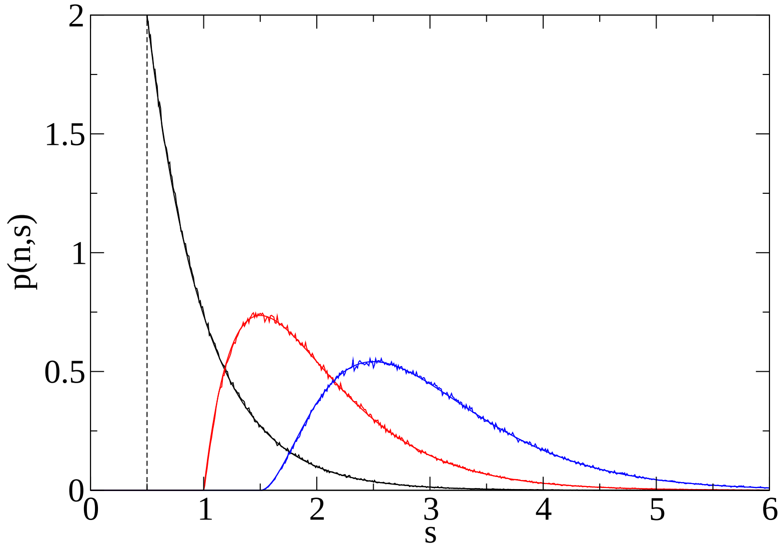
## $0 < a < 1 \longrightarrow$ Poisson distribution shifted by $a$

$$p(s) \equiv p(1,s) = \frac{1}{1-a} e^{-(s-a)/(1-a)}, \; s > a, \quad p(s) = 0, \; s < a$$

$$p(n,s) = \frac{1}{(1-a)^n (n-1)!} e^{-(s-na)/(1-a)}, \; s > na, \quad p(n,s) = 0, \; s < na$$




$a = 0.1, \; 0.2, \; \ldots, \; 0.9$




$a = 0.5$

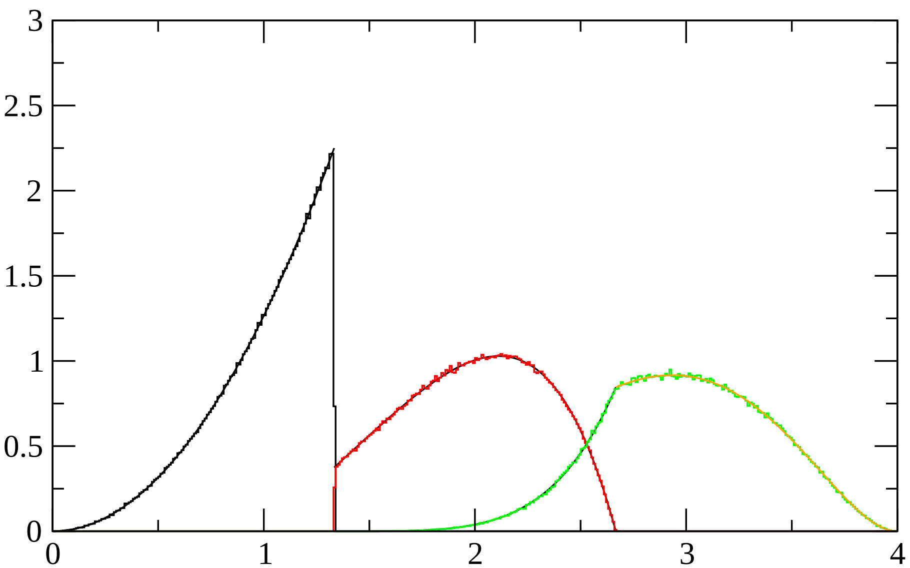
## $\mathbf{1 < a < 2}$

$$p(s) \equiv p(1,s) = 0 \text{ for } s > a$$
$$p(2,s) = 0 \text{ for } s < a \text{ and } s > 2a$$
$$p(3,s) = 0 \text{ for } s < a \text{ and } s > 3a$$

## $\mathbf{a = 4/3}$

$$p(s) = \frac{81}{64}s^2,\ 0 < s < a,\ \ p(2,s) = \left(-\frac{3}{2} + \frac{27}{16}s - \frac{81}{512}s^3\right)\mathrm{e}^{3s/4-1},\ 4/3 < s < 8/3$$

$$p(3,s) = \begin{cases} \left(\frac{3}{4} - \frac{81}{32}s + \frac{81}{512}s^3\right)\mathrm{e}^{3s/4-1} + \frac{81}{64}s^2, & 4/3 < s < 8/3 \\ \left(-\frac{9}{4} + \frac{27}{32}s - \frac{81}{512}s^3\right)\mathrm{e}^{3s/4-1} + 9\mathrm{e}^{3s/2-4}, & 8/3 < s < 4 \end{cases}$$

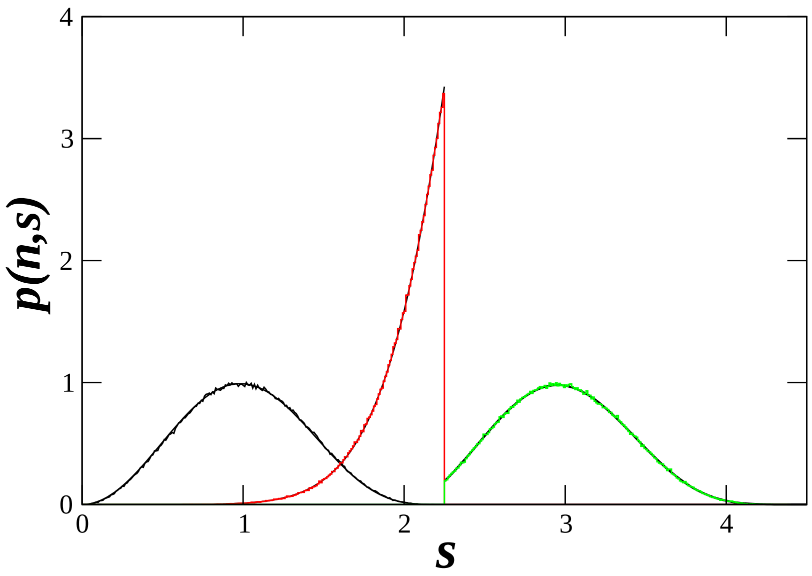


## $2 < a < 3$
## Tedious calculations and complicated expressions

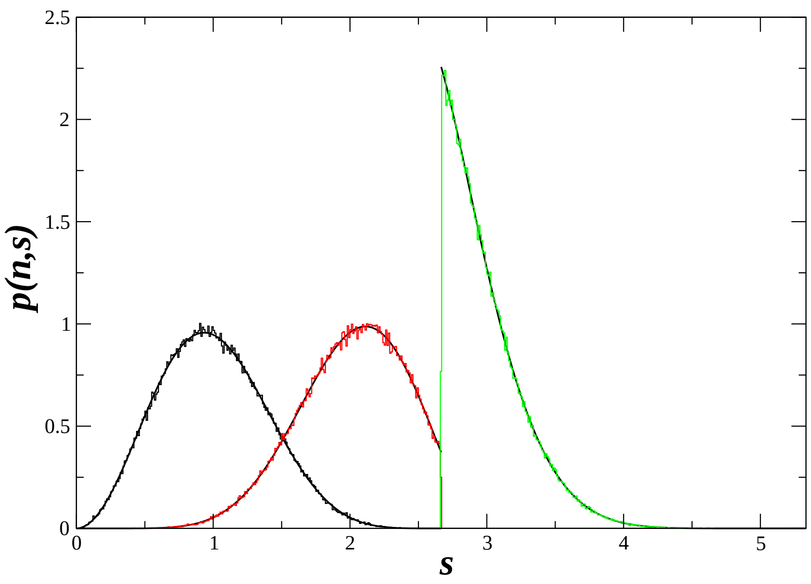
$$p(s) \equiv p(1,s) = 0 \text{ for } s > a$$

$$p(2,s) = 0 \text{ for } s > a$$

$$p(3,s) = 0 \text{ for } s < a \text{ and } s > 2a$$




$a = 9/4$




$a = 8/3$

# Fractal dimensions

Fractal dimensions are **not** yet accessible for analytical calculations

Perturbation series = the **only** analytical way to them

The Ruijsenaars-Schneider ensemble:

$$M_{mn} = \mathrm{e}^{\mathrm{i}\Phi_m} \frac{1 - \mathrm{e}^{2\pi \mathrm{i} \mathbf{a}}}{N\left(1 - \mathrm{e}^{2\pi \mathrm{i}(m-n+\mathbf{a})/N}\right)}$$

Perturbation series are possible around all integer points $\mathbf{a} = k$.

$\mathbf{a} = k + \epsilon$

$$M_{mn} = M_{mn}^{(0)} \left(1 + \frac{\pi \mathrm{i}(N-1)}{N}\epsilon\right) + \epsilon M_{mn}^{(1)} + \mathcal{O}(\epsilon^2)$$

where

$$M_{mn}^{(0)} = \mathrm{e}^{\mathrm{i}\Phi_m}\delta_{n,\,m+k}$$

$$M_{mn}^{(1)} = \mathrm{e}^{\mathrm{i}\Phi_m}(1-\delta_{n,\,m+k})\frac{\pi \mathrm{e}^{-\pi \mathrm{i}(m-n+k)/N}}{N\sin(\pi(m-n+k)/N)}$$

($\delta_{n,\,m+k} = 1$ when $n \equiv m + k \bmod N$ and 0 otherwise)

# Perturbation series for strong multifractality: $|D_q| \ll 1$

$\mathbf{a} \ll \mathbf{1} \longrightarrow M^{(0)}_{mn}$ is diagonal $\longrightarrow$ degenerate perturbation series

- Unperturbed eigenfunctions $\Psi^{(0)}_j(\alpha) = \delta_{j\alpha}$
- Unperturbed eigenvalues $\lambda_\alpha = \mathrm{e}^{\mathrm{i}\Phi_\alpha}$

The first order = the contributions of $\mathbf{2} \times \mathbf{2}$ sub-matrices

$$\begin{pmatrix} M_{mm} & M_{mn} \\ M_{nm} & M_{nn} \end{pmatrix} \equiv \begin{pmatrix} \mathrm{e}^{\mathrm{i}\Phi_m} & \mathrm{e}^{\mathrm{i}\Phi_m} h \\ -\mathrm{e}^{\mathrm{i}\Phi_n} h^* & \mathrm{e}^{\mathrm{i}\Phi_n} \end{pmatrix}, \quad h = \mathbf{a}\frac{\pi \mathrm{e}^{-\pi\mathrm{i}(m-n)/N}}{N\sin(\pi(m-n)/N)}$$

Eigenvalues, $\lambda_{1,2} = \mathrm{e}^{\mathrm{i}\theta_{1,2}}$ and eigenfunctions of this matrix are

$$\theta_{1,2} = \Phi_{m,n} \pm \Delta(\eta), \;\; |u_1| = \frac{|h|}{\sqrt{|h|^2+\Delta^2(\eta)}}\;, \;\; |v_1| = \frac{|\Delta(\eta)|}{\sqrt{|h|^2+\Delta^2(\eta)}}$$

where $\eta = \frac{1}{2}(\Phi_m - \Phi_n), \;\; \Delta(\eta) = \sqrt{\sin^2\eta + |h|^2} - \sin\eta$

# Mean moments of eigenfunctions

$$I_{\mathbf{q}} = \frac{1}{N\rho(E)} \sum_{j,\alpha=1}^{N} \left\langle |\Psi_j(\alpha)|^{2\mathbf{q}} \delta(E - E_\alpha) \right\rangle .$$

$\rho(E)$ = the total mean eigenvalue density. For RSE: $\rho(E) = 1/2\pi$

Fractal dimensions $\boxed{I_{\mathbf{q}} \longrightarrow N^{-(q-1)\mathbf{D_q}}}$

$$I_{\mathbf{q}} = \frac{1}{N\rho(E)} \sum_m \sum_{n \neq m} \int \rho(\Phi_m)\rho(\Phi_n) \mathrm{d}\Phi_m \mathrm{d}\Phi_n \left[ u_1^{2\mathbf{q}} + v_1^{2\mathbf{q}} \right] \delta(E - \theta_1)$$

The dominant contribution is from region $|\eta| \sim |h| \ll 1$

$$I_{\mathbf{q}} = \frac{2\rho(E)}{N} \sum_m \sum_{n \neq m} \int \mathrm{d}\eta \left[ \frac{|h|^{2\mathbf{q}} + |\Delta(\eta)|^{2\mathbf{q}}}{(|h|^2 + \Delta^2(\eta))^{\mathbf{q}}} \right]$$

Substitute $\eta = |h|t$ and extend the integration over $t$ from $-\infty$ to $\infty$

$$I_{\mathbf{q}} = \frac{J(q)}{\pi N} \sum_m \sum_{n \neq m} |h|$$

where

$$J(q) = \int_{-\infty}^{\infty} \left[ \frac{1}{(1+\mathrm{e}^{2t})^q} + \frac{1}{(1+\mathrm{e}^{-2t})^q} \mathbf{-1} \right] \cosh(t)\mathrm{d}t = -\frac{\sqrt{\pi}\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q-1)}$$

One gets

$$\sum_{j=1}^{N-1} \frac{\pi}{N\sin(\pi j/N)} = 2\ln N + 2(\gamma + \ln 2 - \ln\pi)$$

Finally when $N \to \infty$

$$I_{\mathbf{q}} \longrightarrow -2\mathbf{a}\frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q-1)}\ln\mathbf{N}$$

By definition $I_{\mathbf{q}} \longrightarrow N^{-(q-1)\mathbf{D_q}}$, therefore in the first order in $\mathbf{a}$

$$\mathbf{D_q} = 2\mathbf{a}\frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q)}$$

## Perturbation series for week multifractality: $|1 - D_q| \ll 1$

When $\mathbf{a} = \mathbf{k} + \epsilon$ and $\mathbf{k} \neq 0$ the unperturbed matrix

$$M^{(0)}_{mn} = \mathrm{e}^{\mathrm{i}\Phi_m} \delta_{n,\, m+\mathbf{k}}$$

is the shift matrix and its eigenfunctions are extended

The case $\mathbf{k} = \mathbf{1}$

Eigenvalues $\lambda(\alpha)$ and eigenfunctions $\Psi^{(0)}_n(\alpha)$ of $M^{(0)}_{mn}$ are

$$\lambda(\alpha) = \mathrm{e}^{\mathrm{i}\bar{\Phi} + 2\pi \mathrm{i}\alpha/N}, \qquad \Psi^{(0)}_n(\alpha) = \frac{1}{\sqrt{N}} \mathrm{e}^{\mathrm{i}S_n(\alpha)},$$

$$S_n(\alpha) = \frac{2\pi}{N}\alpha(n-1) - \sum_{j=1}^{n-1}(\Phi_j - \bar{\Phi}), \quad \bar{\Phi} = \sum_{j=1}^{N} \Phi_j / N$$

The expansion of the exact eigenfunctions into a series of unperturbed eigenfunctions

$$\Psi_n(\alpha) = \Psi^{(0)}_n(\alpha) + \sum_{\beta} C_{\alpha\beta} \Psi^{(0)}_n(\beta).$$

## The first order in $\epsilon = a - 1$

$$C_{\alpha\beta} = \epsilon \frac{\sum_{mn} \Psi_m^{(0)*}(\beta) M_{mn}^{(1)} \Psi_n^{(0)}(\alpha)}{\lambda(\alpha) - \lambda(\beta)}$$

At the leading order in $\epsilon$

$$\left\langle \sum_{n=1}^{N} |\Psi_n(\alpha)|^{2q} \right\rangle = N^{1-q}\left[1 + \frac{q(q-1)}{2} W(\alpha)\right],$$

$$W(\alpha) = \frac{1}{N} \sum_{n=1}^{N} \left\langle \left[ \sum_{\beta} \mathrm{e}^{\mathrm{i}S_n(\beta) - \mathrm{i}S_n(\alpha)} C_{\alpha\beta} + \text{c.c.} \right]^2 \right\rangle.$$

Direct (but tedious) calculations show strong cancellations and

$$W(\alpha) = \epsilon^2 \frac{\pi^2}{N^3} \sum_{\beta=1}^{N-1} \sum_{n=1}^{N-1} \frac{\sin^2(\pi\beta n/N)}{\sin^2(\pi n/N) \sin^2(\pi\beta/N)}.$$

When $N \to \infty$ the diverging terms correspond to two regions:
**(i)** $\beta \ll N$ with $n/N = \mathcal{O}(1)$, **(ii)** $n \ll N$ with $\beta/N = \mathcal{O}(1)$

$$W(\alpha) \longrightarrow \frac{2\epsilon^2}{N} \sum_{n=1}^{N-1} \frac{g(n/N)}{\sin^2(\pi n/N)},\quad g(y) = \sum_{\beta=1}^{\infty} \frac{\sin^2(\pi\beta y)}{\beta^2} = \frac{\pi^2}{2} y(1-y)$$

# Fractal dimensions for RSE

The remaining sum over $n$ can be transformed into an integral over $y$ and when $N \to \infty$

$$W(\alpha) \longrightarrow 2\epsilon^2 \ln N + \mathcal{O}(1).$$

Combining all terms together one finds

$$D_q = 1 - q(\mathbf{1} - \mathbf{a})^2.$$

For $k \geq 2$ calculations are more tedious but one can prove that

$$\mathbf{D_q} = 1 - q\frac{(\mathbf{a} - \mathbf{k})^2}{\mathbf{k}^2} \text{ when } |\mathbf{a} - \mathbf{k}| \ll 1$$

For comparison when $|\mathbf{a}| \ll 1$

$$\mathbf{D_q} = 2\mathbf{a}\frac{\Gamma\left(q - \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(q)}$$

# Spectral compressibility for RSE

- **0 < a < 1**

$$\chi = (1-a)^2.$$

$$\chi \longrightarrow 1 - \mathbf{2a}, \quad |a| \ll 1$$

- **1 < a < 2**

$$\chi = \left( \frac{a^2}{4} - \frac{4a(1-a)z^2 + a^2 \sinh^2 z}{(2z - \sinh 2z)^2} \sinh^2 z \right) \frac{\sinh^2 z}{z^2}$$

where $z$ is the solution of

$$a = \frac{2z^2 - z \sinh 2z}{z^2 + \sinh^2 z - z \sinh 2z}$$

$z$ is real when $1 < a < 4/3$

$z$ is imaginary when $4/3 < a < 1$

For $a = 4/3$, $\chi = 1/9$

- $\mathbf{2 < a < 3}$

$$\chi = \frac{1}{a(\sin^2 z + z^2 - z\sin 2z)^2}\Big[(a-3)^2(a-2)z^2 - 6(a-2)z^2\sin^2 z$$
$$-(a-3)(a-1)(2a-5)z^3\sin 2z + 2(a-2)(\cos 2z + 2)(a-1)(a-2)z^2\sin^2 z$$
$$-2a(a-2)(2a-3)z\cos z\sin^3 z + a(a-1)^2\sin^4 z\Big]$$

where

$$x = \frac{a\sin^2 z + (a-2)z^2 + (1-a)z\sin 2z}{(a-1)\sin^2 z + (a-3)z^2 + (2-a)z\sin 2z}$$

and

$$\frac{e^x}{x} = \frac{\sin z}{z} e^{z/\tan z}$$

From exact expressions it follows

$$\chi \longrightarrow \begin{cases} 1 - 2\mathbf{a} & |\mathbf{a}| \ll 1 \\ \dfrac{(\mathbf{a}-k)^2}{k^2} & |\mathbf{a}-k| \ll 1 \text{ and } |k| \geq 1 \end{cases}$$

# Fractal dimensions for CrBRME and RSE

| CrBRME | RSE |
|---|---|
| Weak multifractality | |
| $1/b \ll 1$ | $\lvert a-k\rvert \ll 1$ |
| $D_q = 1 - q\,\frac{\mathbf{1}}{\mathbf{2}\pi\,\beta\,\mathbf{b}}$ | $D_q = 1 - q\,\frac{\mathbf{(a-k)^2}}{\mathbf{k^2}}$ |
| $\chi = \frac{\mathbf{1}}{\mathbf{2}\pi\,\beta\,\mathbf{b}}$ | $\chi = \frac{\mathbf{(a-k)^2}}{\mathbf{k^2}}$ |
| Strong multifractality | |
| $b \ll 1$ | $\lvert a\rvert \ll 1$ |
| $D_q = \mathbf{4bc}_\beta \frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q)}$ | $D_q = \mathbf{2a}\frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q)}$ |
| $\chi = 1 - \mathbf{4bc}_\beta$ | $\chi = \mathbf{1 - 2a}$ |

$(c_\beta = 1$ for $\beta = 1$, $c_\beta = \pi/\sqrt{8}$ for $\beta = 2)$

# Universal formulas

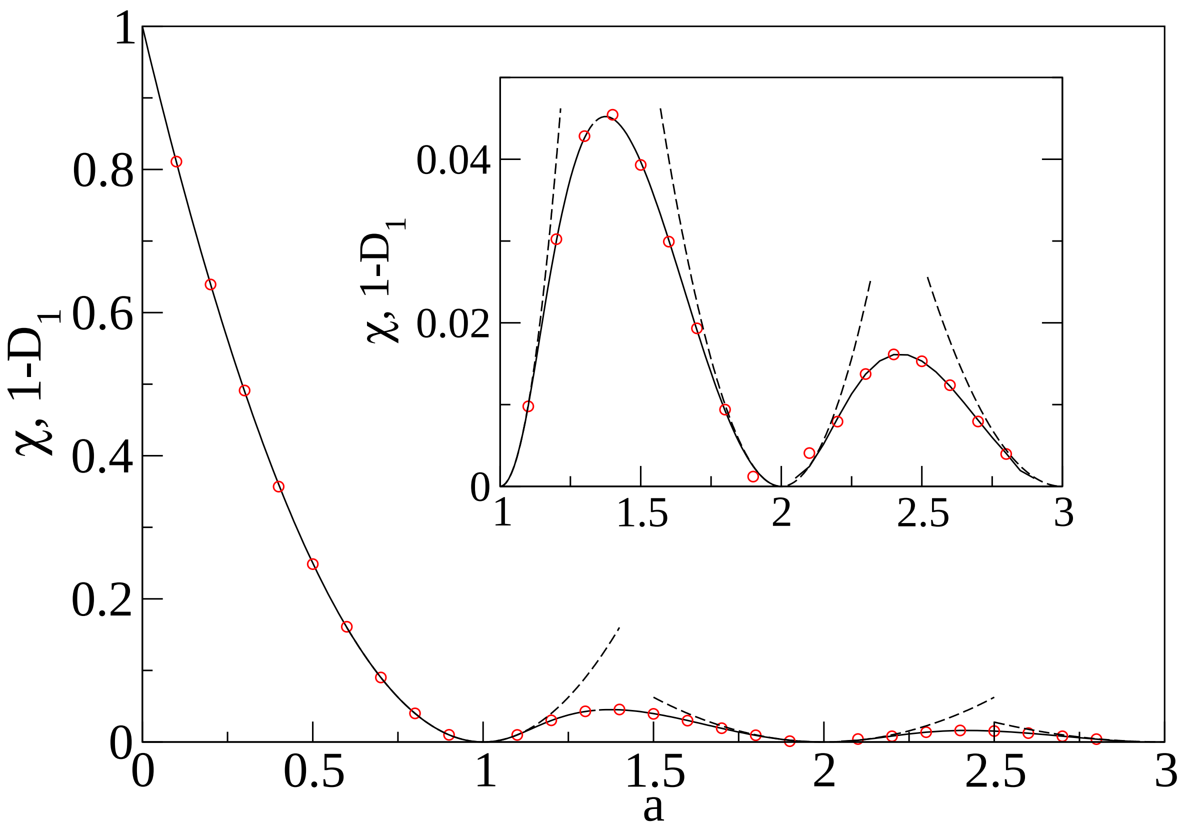
$$D_q = \frac{\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q)}(\mathbf{1}-\chi), \qquad D_q = 1 - q\chi, \qquad \boxed{D_1 = 1-\chi}$$

**Conjecture:** $\boxed{\chi = 1 - D_1/d}$, (**E.B. and Giraud** (2010))

Wave function entropy (information dimension):

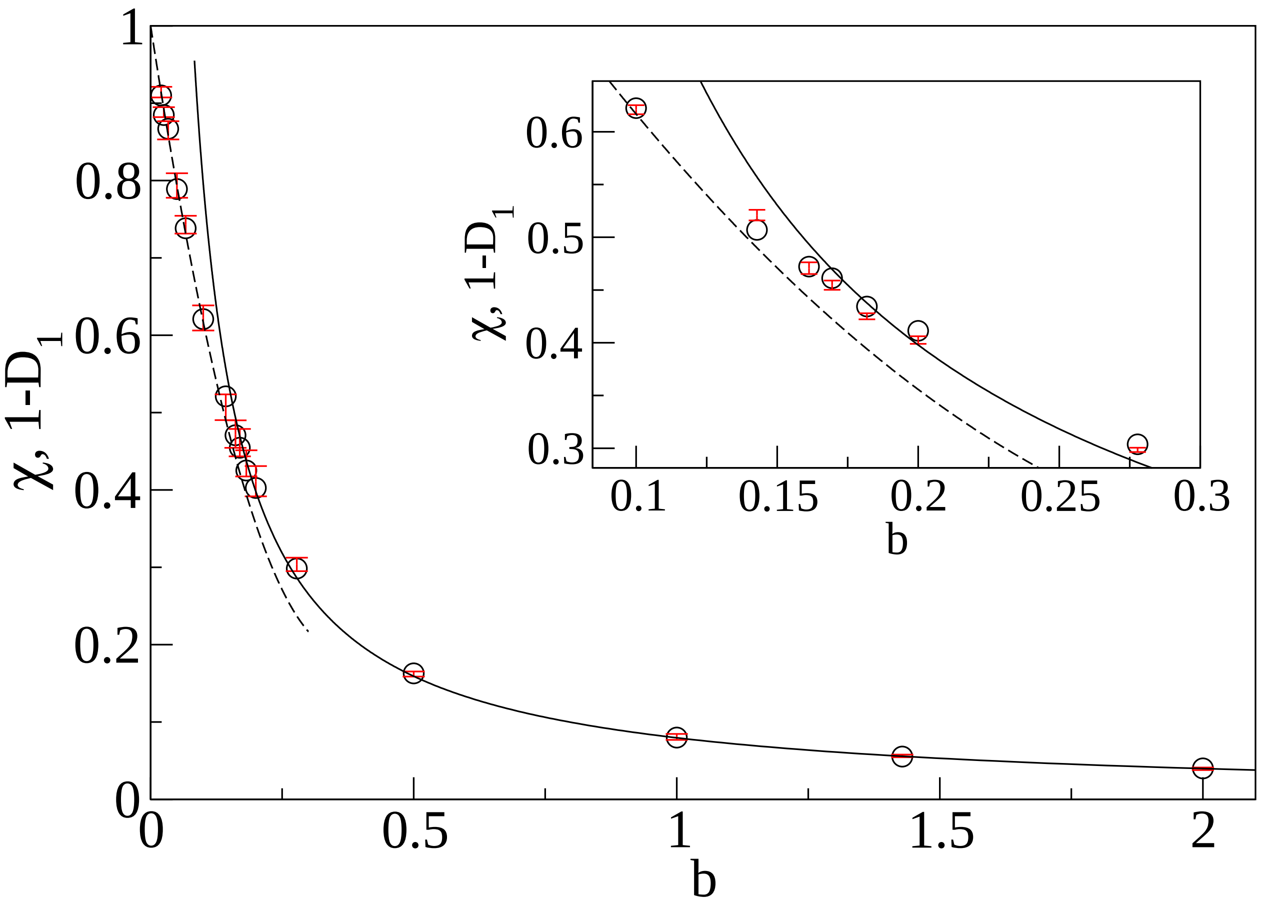
$$-\left\langle \sum_n |\Psi_n(\alpha)|^2 \ln |\Psi_n(\alpha)|^2 \right\rangle \longrightarrow \mathbf{D_1} \ln N$$

**Chalker, Kravtsov, Lerner** (1996): $\boxed{\chi = 1/2 - D_2/2d}$




$\chi$ (solid line) and $1 - D_1$ (red circles) for RSE

# Critical banded random matrices

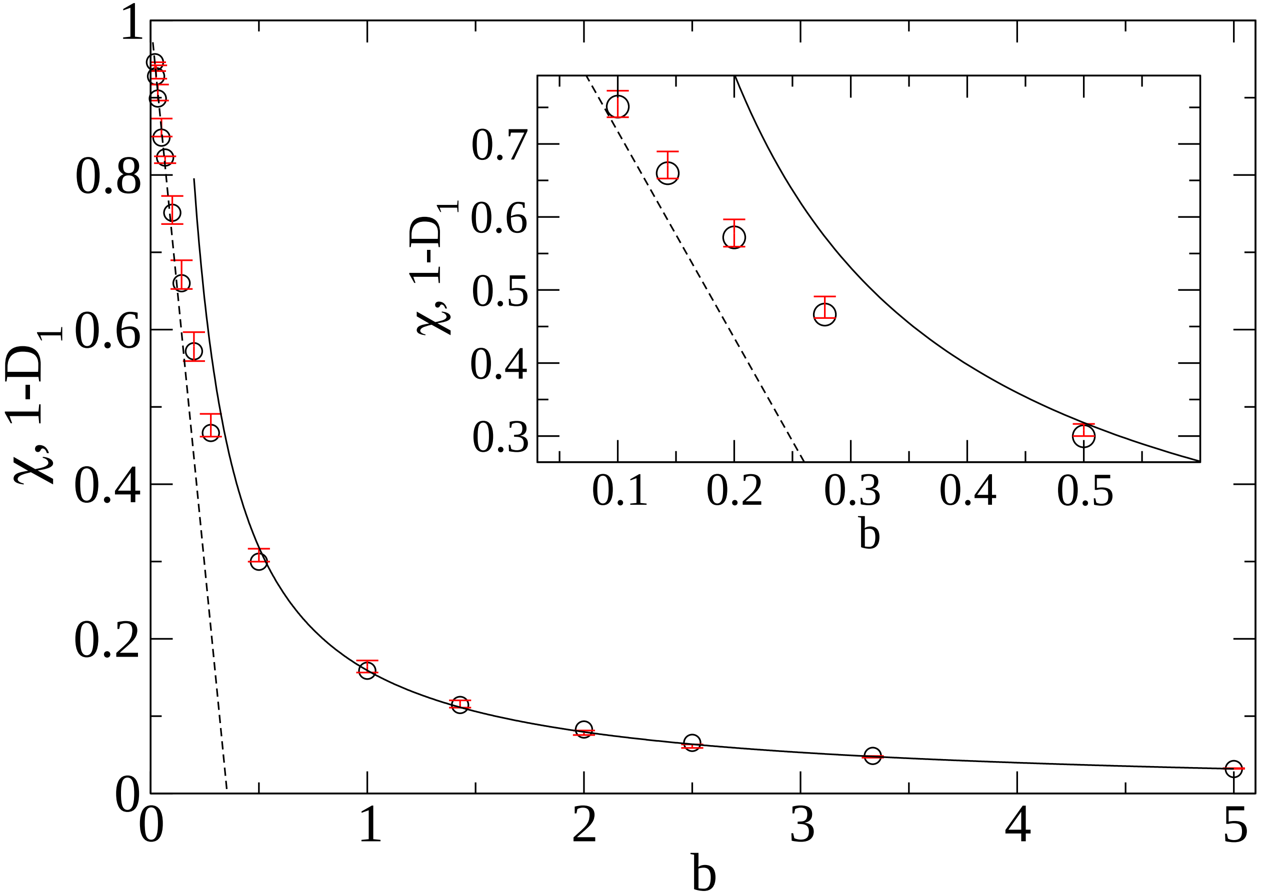



$\chi$ (black circles) and $1 - D_1$ (red error bars) for PLBRM with $\beta = \mathbf{2}$

Perturbation series: $\chi = 1/(4\pi b)$ for $b \gg 1$,

$\chi = 1 - \pi\sqrt{2}b - 4(2/\sqrt{3} - 1)\pi^2 b^2$ for $b \ll 1$

# Critical banded random matrices



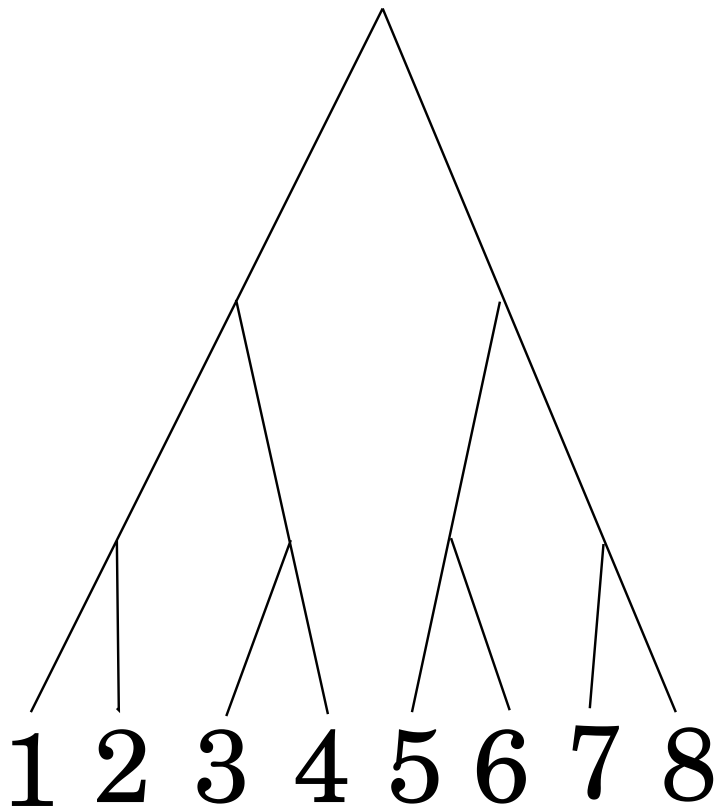

$\chi$ (black circles) and $1 - D_1$ (red error bars) for PLBRM with $\beta = \mathbf{1}$

Perturbation series: $\chi = 1/(2\pi b)$ for $b \gg 1$, $\chi = 1 - 2\sqrt{2}b$ for $b \ll 1$

# Critical ultrametric matrices

(**Fyodorov, Ossipov, Rogriguez** (2009))

$2^K \times 2^K$ Hermitian matrices with independent Gaussian variables with zero mean and $\langle |H_{nn}|^2 \rangle = W^2$. $\langle |H_{mn}|^2 \rangle = 2^{2-d_{mn}} J^2$, $d_{mn} =$ the **ultrametric distance** between $m$ and $n$ along the binary tree

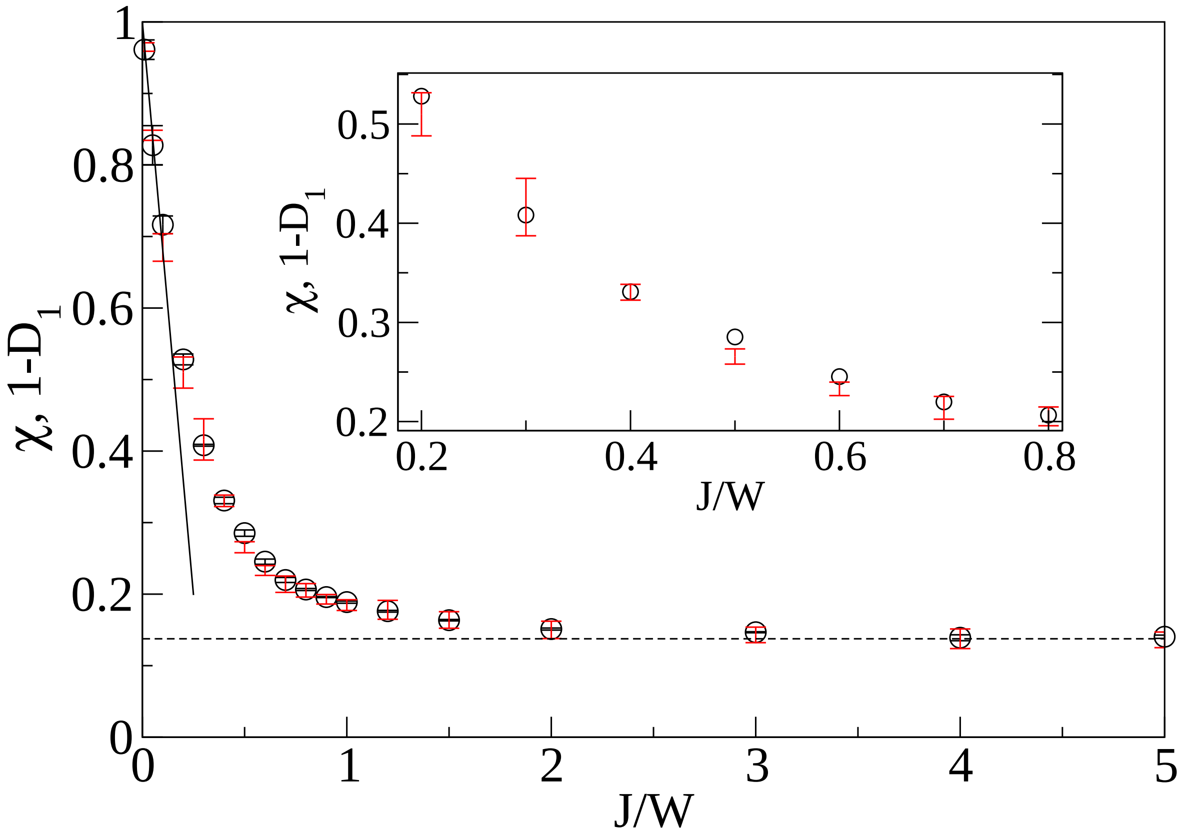



| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 1/2 | 1/2 | 1/4 | 1/4 | 1/4 | 1/4 |
| 1 | | 1/2 | 1/2 | 1/4 | 1/4 | 1/4 | 1/4 |
| 1/2 | 1/2 | | 1 | 1/4 | 1/4 | 1/4 | 1/4 |
| 1/2 | 1/2 | 1 | | 1/4 | 1/4 | 1/4 | 1/4 |
| 1/4 | 1/4 | 1/4 | 1/4 | | 1 | 1/2 | 1/2 |
| 1/4 | 1/4 | 1/4 | 1/4 | 1 | | 1/2 | 1/2 |
| 1/4 | 1/4 | 1/4 | 1/4 | 1/2 | 1/2 | | 1 |
| 1/4 | 1/4 | 1/4 | 1/4 | 1/2 | 1/2 | 1 | |

## Perturbation series for ultrametric ensemble

$$D_q = \frac{\pi \mathbf{J}}{\sqrt{\mathbf{2}}\ln \mathbf{2}\,\mathbf{W}} \frac{\Gamma(q-1/2)}{\sqrt{\pi}\,\Gamma(q)}, \qquad \chi = 1 - \frac{\pi \mathbf{J}}{\sqrt{\mathbf{2}}\ln \mathbf{2}\,\mathbf{W}},$$

$$D_q = \frac{\Gamma(q-1/2)}{\sqrt{\pi}\,\Gamma(q)}(\mathbf{1}-\chi), \qquad \boxed{\chi = 1 - D_1}$$




$\chi$ and $1 - D_1$ for ultrametric matrices

## **Higher dimensional conjecture**: $\boxed{\chi = 1 - D_1/d}$

Standard two-dimensional critical model:

MIT in the quantum Hall effect

via the Chalker-Coddington network model

| | | |
|---|---|---|
| (**Evers et al.** (2008)) | $\longrightarrow$ | $D_1 = 1.7405 \pm 0.0004$ |
| **Conjecture**: $\chi_c = 1 - D_1/2$ | $\longrightarrow$ | $\chi_c = \mathbf{0.1298 \pm 0.0002}$ |
| (**Klesse, Metzler** (1997)) | $\longrightarrow$ | $\chi = \mathbf{0.124 \pm 0.006}$ |

Standard three dimensional critical model:

MIT in 3-d Anderson model

| | | |
|---|---|---|
| (**Rodriguez et al.** (2010)) | $\longrightarrow$ | $D_1 = 1.93 \pm 0.01$ |
| (**Rodriguez et al.** (2009)) | $\longrightarrow$ | $D_0' = 4.027 \pm 0.016$ |
| Symmetry: $D_1 = 2d - D_0'$ | $\longrightarrow$ | $D_1 = 1.973 \pm 0.016$ |
| **Conjecture**: $\chi_c = 1 - D_1/3$ | $\longrightarrow$ | $\chi_c \approx \mathbf{0.34 \ldots 0.36}$ |
| (**Ndawana et al.** (2002)) | $\longrightarrow$ | $\chi = 0.28 \pm 0.06$ |
| (**Ndawana et al.** (2002)) | $\longrightarrow$ | $\chi = \mathbf{0.32 \pm 0.03}$ |

# Summary

- Large varieties of intermediate statistics
- **Lax matrices** of integrable classical systems give new soluble ensembles of random matrices with intermediate statistics
- **Absence of universality for spectral statistics**
- Perturbation series for fractal dimensions
- **Conjecture**: $\chi = 1 - D_1/d$
- New perspectives for intermediate statistics

# Compressibility for ultrametric ensemble

By definition: $\boxed{\chi = 1 + \lim_{L\to\infty} \lim_{N\to\infty} F_{L,N}}$,

$$F_{L,N} = \frac{1}{\bar{\rho}} \int_{-L/\bar{\rho}}^{L/\bar{\rho}} [R_2(E + s/2, E - s/2) - \bar{\rho}^2] \mathrm{d}s.$$

Here $R_2(E_1, E_2)$ is the two-point correlation function

$$R_2(E_1, E_2) = \left\langle \sum_{m\neq n}^{N} \delta(E_1 - \lambda_m)\delta(E_2 - \lambda_n) \right\rangle,$$

$\bar{\rho} = N\rho(E)$ is the mean density.

In the first order of perturbation series it is sufficient to consider **2** $\times$ **2** sub-matrix

$$\begin{pmatrix} H_{mm} & H_{mn} \\ H_{nm} & H_{nn} \end{pmatrix}$$

and then to perform the summation of all pairs $m \neq n$.

Eigenvalues: $\lambda_{1,2} = \xi \pm \sqrt{\eta^2 + h^2}$,

$h = |H_{mn}|$, $\xi = (H_{mm} + H_{nn})/2$, $\eta = (H_{mm} - H_{nn})/2$.

$$R_2(E+s/2,E-s/2) =$$

$$= \sum\nolimits_{n\neq m} \delta(E+s/2-\xi-\sqrt{\eta^2+h^2})\delta(E-s/2+\xi-\sqrt{\eta^2+h^2})\rho(\xi+\eta)\rho(\xi-\eta)2\mathrm{d}\xi\mathrm{d}\eta$$

$$\approx 2\rho^2(E)\sum_{n\neq m}\delta(s-2\sqrt{\eta^2+h^2})\mathrm{d}\eta = \rho^2(E)\sum_{n\neq m}\frac{|s|}{\sqrt{s^2-4h^2}}\theta(s-2h)$$

Integrating over $s$ gives (assuming $\eta \sim |H_{mn}| \ll 1$)

$$F_{L,N} = \frac{2\rho}{N}\left\langle \sum_{n\neq m}\sqrt{\left(\frac{L}{\rho N}\right)^2 - 4|H_{mn}|^2}\ \right\rangle - 2L.$$

The first term exists only when $|H_{mn}| \leq L/(2\rho N)$.

$$H_{mn} = z\, J\, 2^{1-d_{mn}/2}$$

$z$ is the random complex variable $\langle \mathrm{Re}\, z)\rangle = \langle \mathrm{Im}\, z\rangle = 0$ and $\langle (\mathrm{Re}\, z)^2\rangle = \langle (\mathrm{Im}\, z)^2\rangle = 1/\sqrt{2}$. Ultrametric distance $d_{mn} = 2k$ and each value of $k$ is degenerate $2^{k-1}$ times (criticallity)

$$F_{L,N} \approx 2\rho \left\langle \sum_{i=i_0}^{K-1} 2^i \sqrt{\left(\frac{L}{\rho N}\right)^2 - 4\left(\frac{J|z|}{2^i}\right)^2} \right\rangle - 2L$$

with $i_0$ such that $L/(2\rho N) \approx J|z|/2^{i_0}$.

$$\sum_{i=i_0}^{K-1} 2^i = 2^K - 2^{i_0} = N - \frac{2J|z|N}{L\rho}$$

Therefore

$$F_{L,N} \approx 2\rho \left\langle \sum_{i=i_0}^{K-1} 2^i \left[ \sqrt{\left(\frac{L}{\rho N}\right)^2 - 4\left(\frac{J|z|}{2^i}\right)^2} - \frac{L}{\rho N} \right] - 2J|z| \right\rangle$$

Change $i$ to $2J|z|/2^i = L/(x\rho N)$. Then

$$F(L,N) \approx 4\rho \frac{J}{\ln 2} \left\langle |z| \left[ \int_1^{x_m} (\sqrt{1 - 1/x^2} - 1)\mathrm{d}x - 1 \right] \right\rangle$$

where $x_m = L/(4\rho J|z|)$. In the limit $L \to \infty$, $x_m \to \infty$.
$\int_1^{\infty} (\sqrt{1 - 1/x^2} - 1)\mathrm{d}x = 1 - \pi/2$, $\langle |z| \rangle = \sqrt{\pi}/2$, and $\rho(0) = 1/\sqrt{2\pi}W$:

$$\chi = 1 - \frac{\pi J}{\sqrt{2} \ln 2\, W}$$