Large Deviations of the Smallest Eigenvalue of Wishart-Laguerre Ensemble

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Outline

- Introduction
  - Random matrices, the Wishart ensemble
  - The smallest eigenvalue
  - Known results: exact approaches for small N, Tracy-Widom for large N
- Coulomb Gas Approach
  - Exact Mapping
  - Continuing theory
  - Saddle-point evaluation (large N)
  - Results
- Future
  - Entropy corrections, etc.

Wishart-Laguerre Ensemble

- Experiment: where N quantities are measure M times

$$X^T = \underbrace{\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1M} \\ \vdots & & & \vdots \\ x_{N1} & x_{N2} & \dots & x_{NM} \end{pmatrix}}_{M \text{ repetitions}} \Bigg\} N \text{ quantities}\,, \quad W = \underbrace{X^T X}_{N\times N \text{ matrix}}$$

- The Wishart-Laguerre ensemble is the random-matrix version, i.e. $x_{ij} \sim N(0,1)$
- Normally $N < M$ (better $N \ll M$)
- When $N = M$, square matrices
- When $N \sim N$, $M - N = \mathcal{O}(1)$, almost square matrices
- Sometimes $N > M$ (anti-Wishart ensemble)

Wishart-Laguerre Ensemble

- We consider Wishart ensemble (Biometrica, 1928)
- Distribution of the $M \times N$ matrix X is Gaussian

$$P(X) \sim \exp\left[-\frac{\beta}{2}\mathrm{Tr}X^{\dagger}X\right]$$

($\beta = 1, 2, 4$ Dyson index)
- Wishart matrix $W = X^T X$
- In recent years, increased interest in so-called generalised β-ensembles (Dumitriu, 2003)

Density of Eigenvalues

- From distribution of Wishart matrices $\Rightarrow$ joint distribution of N eigenvalues

$$\varrho_N(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_0} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

- First interesting object of study is the spectral density

$$\rho_N(\lambda) = \int d\lambda_1 \cdots d\lambda_N \varrho_N(\lambda_1,\ldots,\lambda_N) \left(\frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) \right)$$

The Marčenko-Pastur Law

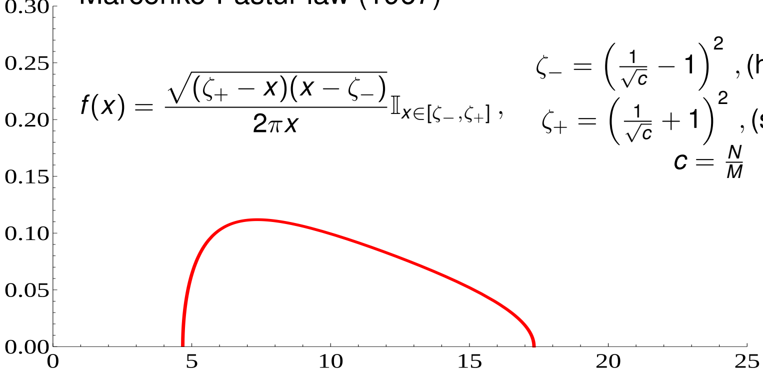
- For large N, $\rho_N(\lambda) = (1/N) f(\lambda/N)$ follows the Marčenko-Pastur law (1967)

$$f(x) = \frac{\sqrt{(\zeta_+ - x)(x - \zeta_-)}}{2\pi x} \mathbb{I}_{x \in [\zeta_-, \zeta_+]}\,,$$

$$\zeta_- = \left(\frac{1}{\sqrt{c}} - 1\right)^2, \text{(hard edge)}$$

$$\zeta_+ = \left(\frac{1}{\sqrt{c}} + 1\right)^2, \text{(soft edge)}$$

$$c = \frac{N}{M}$$



$c = 1/10$, $\zeta_- = 4.678$, $\zeta_+ = 17.324$

The Smallest Eigenvalue

Mathematics

- invertibility of Wishart matrix is controlled by λ_{min}
- Compressive sensing: fluctuations of λ_{min} set bounds on $\#$ of measurements to fully recover a sparse signal

Statistics

- Statistical tests based on W^{-1} (e.g. Hotelling's $T-$square test)

Physics

- Quantum information -measure of entanglement

The Smallest Eigenvalue

Exact expressions for finite N and M using various techniques, e.g.

- Edelman's approach (1991)

$$\rho_{M,N}^{(\min)}(\lambda) = C_{M,N}\lambda^{(M-N-1)/2}e^{-\lambda N/2}g_{M,N}(\lambda)$$

with $g_{M,N}(\lambda)$ polynomials (different expressions for $M-N$ even or odd).
These expressions (and similar ones) difficult to evaluate for large sizes.

For large N, information on the typical fluctuations of the smallest eigenvalue ($c<1$): Tracy-Widom distribution (Feldheim & Sodin, 2010)

$$\lambda_{\min} = \zeta_- - \zeta_-^{2/3}c^{1/6}N^{-2/3}\chi_\beta\,, \quad \chi_\beta \sim TW_\beta$$

Our Goal

Study large fluctuations of the smallest eigenvalue

- simple expressions for rate functions for large deviations.

$$P_N^{(\min)}(t) \sim e^{-\beta N^2 \Phi_+^{(\min)}\left(\frac{t-N\zeta_-}{N}\right)}, \qquad N\zeta_- \leq t < \infty$$

$$P_N^{(\min)}(t) \sim e^{-\beta N \Phi_-^{(\min)}\left(\frac{N\zeta_- - t}{N}\right)}, \qquad 0 \leq t \leq N\zeta_-$$

- valid for for any β and any c

Coulomb Gas approach

- From joint distribution of eigenvalues

$$\varrho_N(\boldsymbol{\lambda}) = \frac{1}{Z_0} e^{-\frac{\beta}{2}\sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

- Coulomb Gas: eigenvalues as a system of charged particles in a 2D world (logarithmic potential), constrained to the real line and external linear-log potential

$$\varrho_N(\boldsymbol{\lambda}) = \frac{e^{-\beta F(\boldsymbol{\lambda})/2}}{Z_0}$$

with

$$F(\boldsymbol{\lambda}) = \sum_{i=1}^N \lambda_i - \left(1 + M - N - \frac{2}{\beta}\right) \sum_{i=1}^N \log \lambda_i - \sum_{i \neq j} \log |\lambda_i - \lambda_j|$$

Coulomb Gas approach

- Quantity to calculate:

$$P_N^{(\min)}(t) \equiv \text{Prob}(\lambda_{\min} \geq t) = \int_t^{\infty} d\lambda \, \rho_N^{(min)}(\lambda) = \frac{Z(t)}{Z_0}$$

  with

$$Z(t) = \int_t^{\infty} \cdots \int_t^{\infty} e^{-\frac{\beta}{2} F(\boldsymbol{\lambda})} d\lambda_1 \cdots d\lambda_N$$

  and $Z_0 = Z(t = 0)$.
- Coulomb gas with hard wall at t.

Continuum approach

- Introduce the density of particles

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i)$$

- Change variables: $(\lambda_1, \ldots \lambda_N) \to \rho(\lambda)$

$$Z(t) = \int_t^\infty d\boldsymbol{\lambda} e^{-\frac{\beta}{2}F(\boldsymbol{\lambda})}$$
$$= \int D\rho\, e^{-\frac{\beta}{2}F[\rho(\lambda)]} \int_t^\infty d\boldsymbol{\lambda}\delta_{(F)}\left[\rho(\lambda) - \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda - \lambda_i)\right]$$

- Rescaling: $\rho(\lambda) = f(\lambda/N)/N$

Analytics - Path integral

To obtain

$$Z(t) = \int Df\, e^{-\frac{\beta}{2}N^2 S[f(x)]}$$

with

$$\begin{aligned} S[f(x)] = &\int_\zeta^\infty dx f(x) x - \left(\alpha + \frac{\beta - 2}{\beta N}\right) \int_\zeta^\infty dx f(x) \log x \\ &- \iint_\zeta^\infty dx dy f(x) f(y) \log|x-y| \\ &+ \frac{2}{\beta N} \int_\zeta^\infty dx\, f(x) \log f(x) + C_1 \left(\int_\zeta^\infty dx f(x) - 1 \right) \end{aligned}$$

with $\alpha = (1-c)/c$.

- No Dyson correction in the entropic term

Analytics- Saddle-point method

Stationary approximation (neglecting $1/N$ terms):

$$0 = \frac{\delta \mathcal{S}[f(x)]}{\delta f(x)}$$

and differentiation with respect x we obtain

$$\frac{1}{2} - \frac{\alpha}{2x} = P \int_{\zeta}^{\infty} dx' \, \frac{f_\star(x')}{x - x'} \, , \quad \zeta \leq x < \infty$$

Tricomi equation

Analytics: Finite Interval Hilbert Transformation

Solution (Mathematical solution + normalisation + positivity):

$$f_\star(x) = \frac{\sqrt{U-x}}{\sqrt{x-\zeta}} \left(\frac{x - \alpha\sqrt{\zeta/U}}{x} \right) \mathbb{I}_{x\in[\zeta,U]}\,, \quad \zeta \geq \zeta_-$$

with $U \equiv U(c,\zeta) = w^2(c,\zeta)$ with

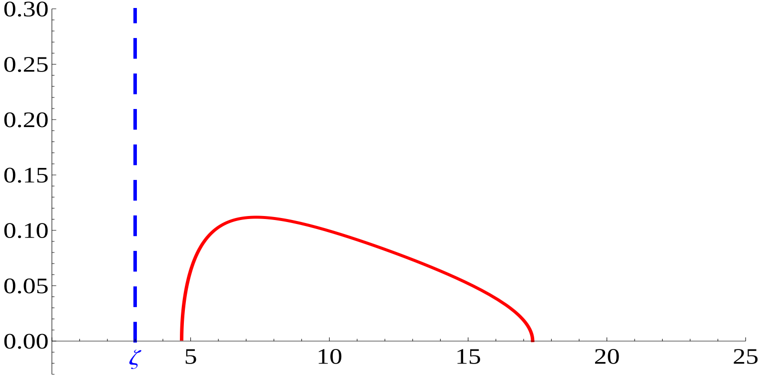
$$w(c,\zeta) = \frac{2p}{3\rho^{1/3}} \cos\left(\frac{\theta+2\pi}{3}\right), \quad p = -[\zeta + 2(2+\alpha)]$$

$$q = 2\alpha\sqrt{\zeta}, \quad \rho = \sqrt{-\frac{p^3}{27}}, \quad \theta = \text{atan}\left(\frac{2\sqrt{B}}{q}\right),$$

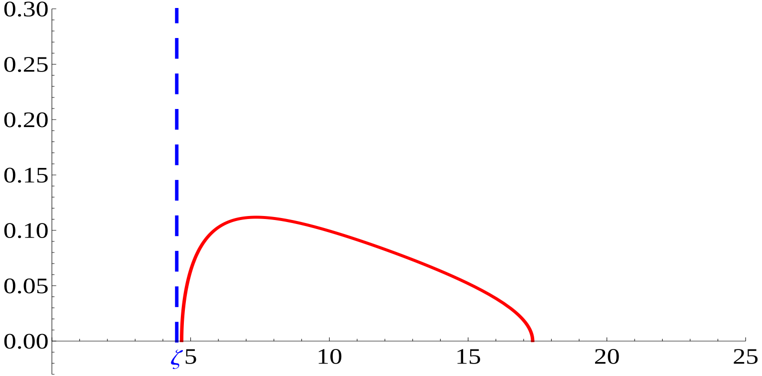
$$B = -\left(\frac{p^3}{27} + \frac{q^2}{4}\right)$$

For $\zeta \leq \zeta_-$ we have $U(c,\zeta_-) = \zeta_+$ $(\sqrt{\zeta_-\zeta_+} = \alpha)$

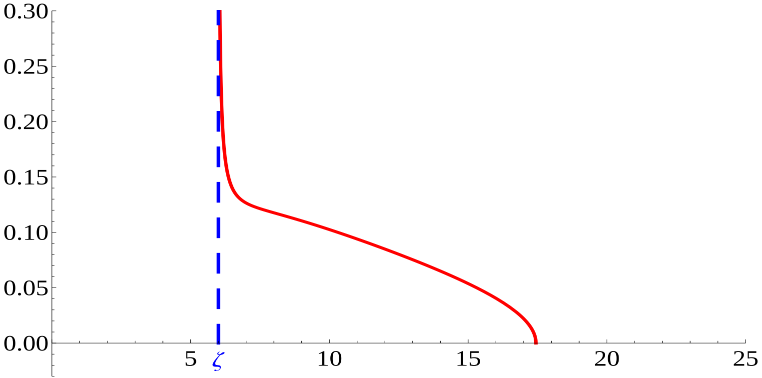
An intuitive representation



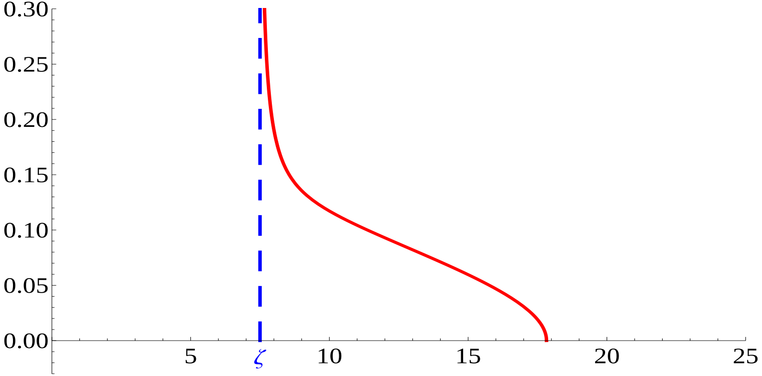
An intuitive representation



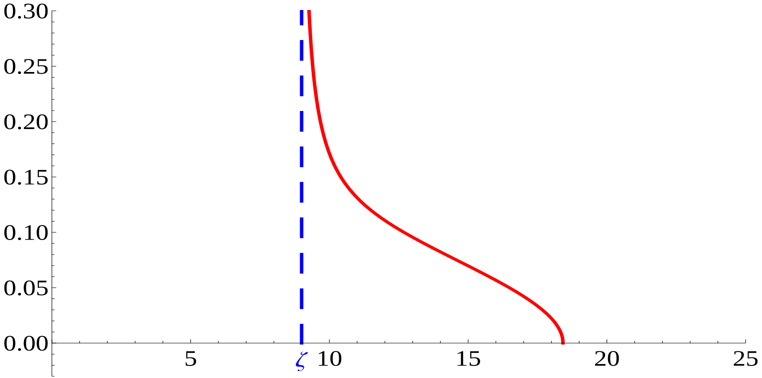
An intuitive representation



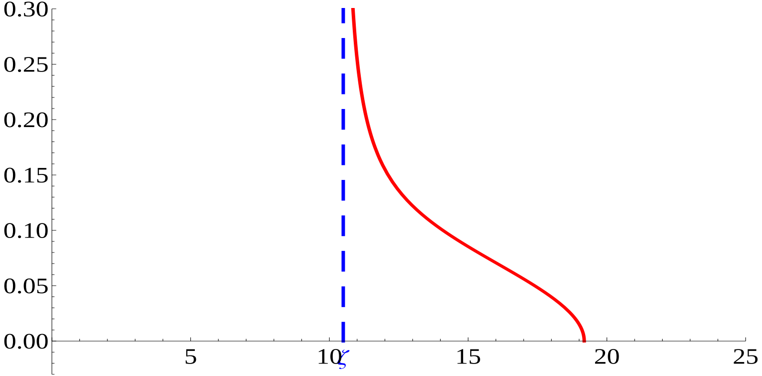
An intuitive representation



An intuitive representation



An intuitive representation



Back to the distribution for $\lambda_{textmin}$

- Recall $Z(t) = \int Df\, e^{-\frac{\beta}{2}N^2 S[f(x)]}$

$$P_N^{(\min)}(t) = \frac{Z(t)}{Z_0} \sim e^{-\beta N^2 \Phi_+^{(\min)}\left(\frac{t-N\zeta_-}{N}\right)}, \qquad N\zeta_- \le t < \infty$$

- Right rate function

$$\Phi_+^{(\min)}(x) = \frac{1}{2}\left[S(x+\zeta_-) - S(\zeta_-)\right], \quad 0 \le x < \infty$$

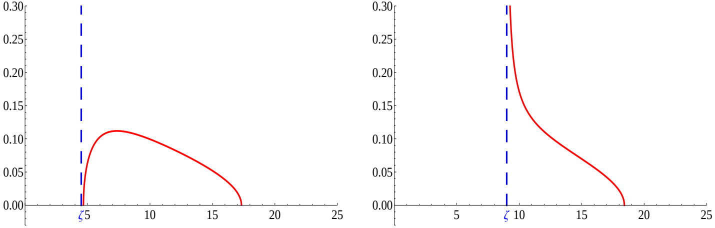
with

$$S(\zeta) = \frac{\zeta + U}{2} - \frac{\Delta^2}{32} - \log\left(\frac{\Delta}{4}\right) + \frac{\alpha}{4}\left(\sqrt{U} - \sqrt{\zeta}\right)^2 + \frac{\alpha^2}{4}\log(\zeta U) - \alpha(\alpha+2)\log\left(\frac{\sqrt{\zeta}+\sqrt{U}}{2}\right)$$

with $\Delta = U - \zeta$

Large deviations to the left of $\lambda_{\min}$

- Coulomb Gas approach (as presented) not able to capture fluctuations to the left of $\lambda_{\min}$
- Reason: we only consider leading terms $\mathcal{O}(N^2)$, which capture bulk properties



Large deviations to the left of λ_{min}

- Energetic Argument (Majumdar & Vergassola)
- Expression the free energy $F(\boldsymbol{\lambda})$
- Energetic cost of moving the smallest eigenvalue to the left $t \ll \zeta_- N$ (this does not require a global rearrangement of the bulk)

$$
\begin{aligned}
\Delta E(t) &= F\left(t, \lambda_2, \ldots, \lambda_N\right) - F\left(\zeta_- N, \lambda_2, \ldots, \lambda_N\right) \\
&= t - \alpha N \log(t) - 2 \sum_k \log|t - \lambda_k| + C \\
&= t - \alpha N \log(t) - 2N \int d\lambda\, \rho_{\text{MP}}(\lambda) \log|t - \lambda| + C
\end{aligned}
$$

C so that $\Delta E(t = \zeta_- N) = 0$.

Large deviations to the left of λ_{min}

- Obtain

$$P_N^{(\text{min})}(t) \sim e^{-\beta N \Phi_-^{(\text{min})}\left(\frac{N\zeta_- - t}{N}\right)}, \qquad 0 \le t \le N\zeta_-$$

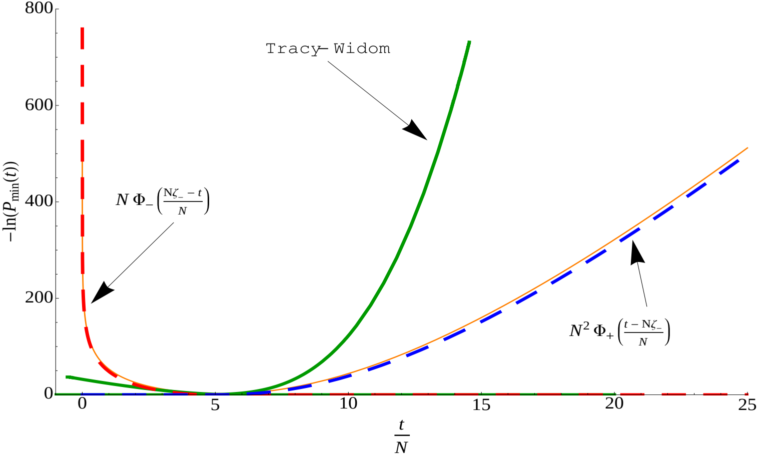
- Left rate function

$$\Phi_-^{(\text{min})}(x) = -\frac{\alpha}{2}\log\left(1 - \frac{x}{\zeta_-}\right) - \frac{1}{2}\sqrt{x(x+\Delta_-)}$$
$$+ 2\log\left(\frac{\sqrt{x+\Delta_-} - \sqrt{x}}{\sqrt{\Delta_-}}\right)$$
$$+ \alpha\log\left(1 + 2\frac{\sqrt{x(x+\Delta_-)} - x}{\Delta_-\sqrt{\zeta_-}}\right), \quad 0 \le x \le \zeta_-$$

with $\Delta_- = \zeta_+ - \zeta_- = 4\sqrt{1+\alpha}$.

Large deviations- Numerics

$N = 11$, $M = 110$. Comparison with Edelman's (91) for $\beta = 1$



Comparison with Tracy-Widom

$$P_N^{(\min)}(t) \equiv \lim_{N\to\infty} \mathbb{P}_{\beta,N}\left((\lambda_{\min} - z_N^{(\beta)})/s_N^{(\beta)} \le t\right)$$

To compare with Tracy-Widom, expand rate functions:

$$\Phi_-^{(\min)}(x) \underset{x\to 0}{\sim} \frac{2}{3\zeta_- c^{1/4}} x^{3/2}\,, \quad \Phi_+^{(\min)}(x) \underset{x\to 0}{\sim} \frac{1}{24\zeta_-^2\sqrt{c}} x^3$$

Then

$$P_N^{(\min)}(t) \sim \begin{cases} \exp\left(-\frac{2\beta}{3}\chi^{3/2}(t)\right)\,, & 0 \le t \le \zeta_- N \\ \exp\left(-\frac{\beta}{24}|\chi(t)|^3\right)\,, & t > \zeta_- N \end{cases}$$

with $\chi(t) = -\frac{N\zeta_- - t}{N^{1/3}\zeta_-^{2/3}c^{1/6}}$

Almost Square Matrices

- $M = N + a, \alpha = a/N \quad , a \to a(\beta) = a + (\beta - 2)/\beta$
- Look at the behaviour for $z = Nt$

$$P_N^{(\min)}(z) \sim \begin{cases} \exp\left(-\beta a \Psi_-^{(\min)}\left(\frac{4z}{a^2}\right)\right), & z \in [0, a^2/4] \\ \exp\left(-\beta a^2 \Psi_+^{(\min)}\left(\frac{4z}{a^2}\right)\right), & z \in [a^2/4, \infty) \end{cases}$$

with

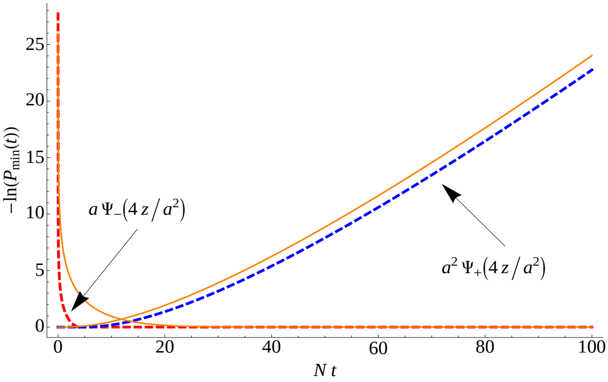
$$\Psi_+^{(\min)}(x) = \frac{1}{8}\left(x - 4\sqrt{x} + 3 + \ln x\right) ,$$
$$\Psi_-^{(\min)}(x) = \ln \frac{1+\sqrt{1-x}}{\sqrt{x}} - \sqrt{1-x}$$

- $a(\beta) = 0$ ($a = 1, \beta = 1$ or $a = 0, \beta = 2$)

$$P_N^{(\min)}(z) = e^{-\beta z/2}$$

Almost Square Matrices

Comparison with Edelman's exact result for $\beta = 1$ ($N = 200$, a=5)



Subleading contributions

- Entropic contribution: Saddle-point equation

$$\frac{1}{2}(x - \alpha \log x) + \frac{1}{\beta N} \log f(x) + D = \int_{\zeta}^{\infty} dy\, f(y) \log |x - y|$$

  Support of $f(x)$ is not compact $\Rightarrow$ fluctuations to the left of λ_{min}
- Non-linear integral equation (Hammerstein type)
- Standard perturbation is hopeless
- Non-standard perturbation (boundary layer theory ?) as difficult as the original equation

Subleading contributions

Two options:

- simplest analytical approach: $y \in \mathcal{R}_{\text{interior}}$, $x \in \mathcal{R}_{\text{exterior}}$, $V(x) = \frac{1}{2}(x - \alpha \log x)$

$$f(x) \sim \exp\left[-\beta N \left(V(x) - \int_{y \in \mathcal{R}_{\text{interior}}} dy\, f_{\text{MP}}(y) \log|x-y| \right)\right]$$

(instanton contribution as in Fyodorov 2004)

Subleading contributions

- Numerical solution (Abdou & Ismail 2002)