# Orthogonal polynomials, random matrices and wireless communications

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**Matthew McKay**
**Hong Kong University of Science and Technology**
**Clear Water Bay, Hong Kong**

**Yang Chen**
**Imperial College London**
**United Kingdom**
**ychen@ic.ac.uk**

- Outline
- Introduction
- Summary of Results
- Ladder operators and Painlevé
- Coulomb Fluid approximation for large $n$
- Deviation from Gaussian.

• "It is not clear what we mean when we say Painlevé equations are integrable"

——Dinner conversation with a colleague.

$$y''(t) = R(y'(t), y(t), t)$$

**Painlevé property: The only movable singularities of its solution are poles**

- Information-theoretic quantities in singer and multi-user multiple-input multiple-output (MIMO) antenna wireless communication systems.

- Generating function of Shannon capacity of multi-antenna Gaussian channels —ultimate limits of communications achievable by any transmission scheme.

• The mathematical problem is the computation of the Hankel determinant generated by a deformation of classical weights:

• $w(x) = x^{\alpha}\, \mathrm{e}^{-x}\, (x+t)^{\lambda}$,
$0 \le x < \infty,\ \alpha > 0,\ t > 0$,
deformed Laguerre, single-user

• $w(x) = x^{\alpha_1}(1-x)^{\alpha_2} \left(\frac{x+t}{1-x}\right)^{\lambda}$,
$x \in (0,1),\ \alpha_1 > 0,\ \alpha_2 > 0,\ t > 0$,
deformed Jacobi, multi-users

• Main results. Ladder operators approach to orthogonal polynomials shows that the Hankel determinants in the single user case a particular Painlevé V, and in the multi-users case a particular Painlevé VI.

• Multiple-input multiple-output (MIMO) systems have been at the forefront of wireless communications research and development, e.g., next-generation wireless local area networks (WLAN) and cellular mobile networks. The main reason for this explosion of interest is mainly due to the independent discoveries of Telatar (1990, 1999, 2008, 7157) and Foschini (1990), demonstrated that the fundamental information-theoretic capacity of MIMO systems grows *linearly* with the number of antennas.

• Traditional methods give logarithmic capacity increase (in $P$). MIMO is a key technology for meeting the ever-increasing demands for higher-rate data-oriented wireless communications applications and services.

•*Ergodic Capacity*, specifies the maximum achievable average mutual information between the transmitter and receiver, assumes that a user's codeword span a large number of "independent channels". Mathematically the expectation value of a certain random variable.

• *Outage Capacity*. Characterizing the communication limits of systems which are not highly dynamic (for example, WLANs), requires the *distribution* of the mutual information between the transmitter and receiver; *CDF* or *PDF*.

Methods for studying the outage capacity, which boils down to the Hankel determinant

$$D_n = \text{det}\left(\mu_{i+j}\right)_{i,j=0}^{n-1}$$

generated from the moments of a certain weight function $w(x)$,

$$\mu_k := \int_a^b x^k w(x) dx \ , \ \ k = 0, 1, 2, \ldots$$

with $w(x)$ supported on $[a, b]$. Weights depend on the MIMO configuration.

• single-user and multi-user MIMO systems.

**Single-user:** deformed Laguerre weight
$w(x) = x^{\alpha}\, \mathrm{e}^{-x}\, (x + t)^{\lambda}, \quad t = \frac{n}{P}$
$0 \leq x < \infty,\ \alpha > 0,\ t > 0$ ,

**multi-user:** deformed Jacobi weight
$w(x) = x^{\alpha_1}(1 - x)^{\alpha_2} \left(\frac{x+t}{1-x}\right)^{\lambda}, x \in [0, 1], \alpha_1 > 0, \alpha_2 > 0,\ t > 0.$

• Two different methods from random matrix theory to compute Hankel determinants.

**1.** Exact expressions employing orthogonal polynomials and ladder operators
(Belmehdi,Bonan,Clark,Lubinsky,Magnus...)
Logarithmic derivative of Hankel determinants are the $\sigma-$function of certain $P_V$ (Single User) and $P_{VI}$ (Multi User).

**2.** Large $n$ approximations for these determinants by employing the general linear statistics theorems.
(Essentially Szegö limit theorem.)

•Closed-form expressions for the distribution function. Valid for large dimension; approximations are remarkably accurate for even very small matrix dimensions ($2 \times 2$).

• Comparison of large $n$ with Painlevé

•Single-user MIMO (deformed Laguerre )
$n_r = n_t$, or $\alpha = 0$. Coulomb fluid gives the distribution of the mutual information, **a Gaussian**, to leading order in $n$.

• Use $P_V$ to compute the large-$n$ correction terms for the mean, variance, and third cumulant.

• Deviations from Gaussian as $P$ (SNR) increases. Sensitivities of mean, variance, and third moment, with respect to $P$.

•MIMO communication system has $n_t$ transmit and $n_r$ receive antennas.
• linear model:
Transmitted vector $\mathbf{x}_{n_t \times 1} \in \mathbb{C}^{n_t}$
Received vector $\mathbf{y}_{n_r \times 1} \in \mathbb{C}^{n_r}$,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \; .$$

• $\mathbf{n}_{n_r \times 1} \in \mathbb{C}^{n_r}$ is a complex Gaussian vector
• $E(\mathbf{n}) = 0$, $E(\mathbf{n}\mathbf{n}^\dagger) = \mathbf{Q_n}$.
• Covariance matrix account for receiver noise and multi-user interference; choice of $\mathbf{Q_n}$ distinguish between single-user and multi-user MIMO models.
• $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$,*channel matrix*, is stochastic, known to the receiver not to the transmitter.
• $\mathbf{H}$ complex Gaussian, i.i.d. elements, zero mean and unit variance.(Simplest choice).
• $\mathbf{x}$ subject to a power constraint:

$$E(\mathbf{x}^\dagger \mathbf{x}) \leq P \; .$$

•Shannon capacity gives highest data rate achievable with negligible errors by any transmission scheme.

*Mutual Information* between the input and output signals,

$$\begin{aligned} I(\mathbf{x};\mathbf{y}|\mathbf{H}) &:= \mathcal{H}(\mathbf{y}|\mathbf{H}) - \mathcal{H}(\mathbf{y}|\mathbf{x},\mathbf{H}) \\ &= \mathcal{H}(\mathbf{y}|\mathbf{H}) - \mathcal{H}(\mathbf{n}) \end{aligned}$$

• $\mathcal{H}(\mathbf{y}|\mathbf{H})$ conditional entropy of $\mathbf{y}$, defined by its density $p(\mathbf{y}|\mathbf{H})$:

$$\mathcal{H}(\mathbf{y}|\mathbf{H}) = E(-\log p) := -\int_{\mathbb{C}^{nr}} p(\mathbf{y}|\mathbf{H}) \log p(\mathbf{y}|\mathbf{H}) d\mathbf{y}.$$

• Ergodic capacity ($C$) relevant for highly dynamic channels; high-mobility wireless applications; $\mathbf{H}$ varies quickly over time, each transmission codeword sees a large number of "independent" channel realizations.

$$C = \max_{p(\mathbf{x})} E_{\mathbf{H}} \left( I(\mathbf{x};\mathbf{y}|\mathbf{H}) \right)$$

where the maximum is taken over all densities $p(\mathbf{x})$ of the input vector $\mathbf{x}$, subject to the power constraint.

• Telatar proved that the optimal input density $p^*(\mathbf{x})$ is multi-variate complex Gaussian with zero mean.

$$I(\mathbf{x};\mathbf{y}|\mathbf{H}) = \log\det\left(\mathbf{I}_{n_r} + \mathbf{H}\mathbf{Q}\mathbf{H}^\dagger\mathbf{Q}_{\mathbf{n}}^{-1}\right)$$

where $\mathbf{Q} = E(\mathbf{x}\mathbf{x}^\dagger)$ is the input signal covariance.

• **Capacity**

$$C = \max_{\mathbf{Q}\geq 0} E_{\mathbf{H}}\left(I(\mathbf{x};\mathbf{y}|\mathbf{H})\right)$$

subject to $\text{tr}(\mathbf{Q}) \leq P$.

• For this model,

$$\mathbf{Q}^* = \frac{P}{n_t}\mathbf{I}_{n_t} \; .$$

In other words, the capacity is achieved by sending independent Gaussian signals from each of the transmit antennas with equal power.

- Outage probability $P_{\text{out}}$

$$
\begin{aligned}
P_{\text{out}}(C_{\text{out}}) &= \Pr\left(I(\mathbf{x};\mathbf{y}) < C_{\text{out}}\right) \\
&= \Pr\left(\log\det\left(\mathbf{I}_{n_r} + \frac{P}{n_t}\mathbf{H}\mathbf{H}^{\dagger}\mathbf{Q}_{\mathbf{n}}^{-1}\right) < C_{\text{out}}\right)
\end{aligned}
$$

with $\mathbf{Q}^*$ denoting the input covariance which maximizes the mutual information. The outage probability can be calculated via

$$
P_{\text{out}}(C_{\text{out}}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{M}(i\omega)\frac{1 - e^{-i\omega C_{\text{out}}}}{i\omega} d\omega \, ,
$$

where $\mathcal{M}(\cdot)$ the moment generating function of the mutual information

$$
\begin{aligned}
\mathcal{M}(\lambda) &:= E_{\mathbf{H}}\left(\exp\left(\lambda I(\mathbf{x};\mathbf{y}|\mathbf{H})\right)\right) \\
&= E_{\mathbf{H}}\left(\det\left(\mathbf{I}_{n_r} + \frac{P}{n_t}\mathbf{H}\mathbf{H}^{\dagger}\mathbf{Q}_{\mathbf{n}}^{-1}\right)^{\lambda}\right) \, ,
\end{aligned}
$$

and $i := \sqrt{-1}$.

• Single-User MIMO and the Deformed Laguerre Weight
• Without loss of generality

$$\mathbf{Q_n} = \mathbf{I}_{n_r} \; .$$

Due to the normalization of the trace of $\mathbf{Q}_n$, the transmit power $P$ also represents the SNR.
• Moment generating function,

$$\mathcal{M}(\lambda) = E_{\mathbf{H}}\left[\det\left(\mathbf{I}_{n_r} + \frac{1}{t}\mathbf{H}\mathbf{H}^{\dagger}\right)^{\lambda}\right], \quad t := \frac{n_t}{P} \; .$$

Let

$$m := \max\{n_r, n_t\}, \quad n := \min\{n_r, n_t\}, \quad \alpha := m - n$$

and define

$$\mathbf{W} \;\; := \;\; \begin{cases} \mathbf{H}\mathbf{H}^{\dagger}, & n_r < n_t \\ \mathbf{H}^{\dagger}\mathbf{H}, & n_r \geq n_t \end{cases} .$$

• $\mathbf{W}$ is a complex Wishart random matrix with **positive eigenvalues** denoted by $\{x_i\}_{i=1}^{n}$ with j.p.d.f.

$$p(x_1, x_2, ..., x_n) \propto \prod_{i=1}^{n} w_{\mathsf{Lag}}(x_i) \prod_{1\leq j<k\leq n} (x_j - x_k)^2,$$

where $w_{\mathsf{Lag}}(x) = x^{\alpha}\mathrm{e}^{-x}$ is the classical Laguerre weight.
• MGF

$$\begin{aligned} \mathcal{M}(\lambda) &= E\left[\det\left(\mathbf{I}_n + \frac{1}{t}\mathbf{W}\right)^{\lambda}\right] = E\left[\prod_{k=1}^{n}\left(1 + \frac{x_k}{t}\right)^{\lambda}\right] \\ &= \frac{\int_{\mathbb{R}_+^n} \prod_{i<j}(x_i - x_j)^2 \prod_{k=1}^{n} \left(1 + \frac{x_k}{t}\right)^{\lambda} w_{\mathsf{Lag}}(x_k) dx_k}{\int_{\mathbb{R}_+^n} \prod_{i<j}(x_i - x_j)^2 \prod_{k=1}^{n} w_{\mathsf{Lag}}(x_k) dx_k}. \end{aligned}$$

•Andreief-Heine identity:

$$\begin{aligned} D_n[w] &= \det(\mu_{i+j})_{i,j=0}^{n-1} \\ &= \frac{1}{n!}\int_{(a,b)^n} \prod_{1\le i<j\le n}(x_j-x_i)^2 \prod_{k=1}^{n} w(x_k)dx_k, \end{aligned}$$

where

$$\mu_i := \int_a^b x^i w(x)dx, \quad i=0,1,2,...$$

are moments of the weight $w$.

• Obviously the moments would depend on the parameters which may appear in the weight. With this identity, we immediately obtain

$$\mathcal{M}(\lambda) = t^{-n\lambda}\frac{D_n(t,\lambda)}{D_n(t,0)}$$

where

$$D_n(t,\lambda) = \det\left(\mu_{i+j}(t,\lambda)\right)_{i,j=0}^{n-1},$$

is the Hankel determinant generated from the deformed Laguerre weight

$$w_{\mathsf{dLag}}(x) = w_{\mathsf{dLag}}(x,t,\lambda) := (x+t)^{\lambda}\, x^{\alpha}\, \mathrm{e}^{-x}, \quad t>0$$

with moments ($k=0,1,2,...$)

$$\begin{aligned} &\mu_k(t,\lambda) := \int_0^{\infty} x^k w_{\mathsf{dLag}}(x)dx \\ = \;& t^{\alpha+\lambda+k+1}\Gamma(\alpha+k+1)\, U(\alpha+k+1;\alpha+\lambda+k+2;t), \end{aligned}$$

• Orthogonal polynomials and ladder operators

$$p(y_1, \ldots, y_n) \propto \prod_{k=1}^{n} w(y_k) \prod_{1\le i<j\le n} (y_j - y_i)^2, \quad y_i \in (a,b)$$

with $w$ is the weight.

$$\begin{aligned} D_n &= \frac{1}{n!} \int_{(a,b)^n} \prod_{1\le i<j\le n} (y_j - y_i)^2 \prod_{k=1}^{n} w(y_k) dy_k \\ &= \det \left( \int_a^b y^{i+j} w(y) dy \right)_{i,j=0}^{n-1} \end{aligned}$$

where $w(y) = w_0(y)\, f(y)$

$$D_n = \det \left( \int_a^b P_i(y) P_j(y) w(y) dy \right)_{i,j=0}^{n-1}$$

where $P_j(\cdot)$ represents any monic polynomial of degree $j$, written as

$$P_j(z) = z^j + \mathsf{p}_1(j)\, z^{j-1} + \ldots$$

• Orthogonalisation

$$\int_a^b P_i(y) P_j(y) w(y) dy = h_i \delta_{i,j}, \quad i, j = 0, 1, 2, \ldots$$

$$D_n = \prod_{k=0}^{n-1} h_k\ .$$

• Need to understand $P_j(x)$ and $h_j$.

• Facts about OP: Recurrence relations.

$$zP_n(z) = P_{n+1}(z) + \alpha_n\, P_n(z) + \beta_n\, P_{n-1}(z).$$

$$P_0(z) = 1, \;\; \beta_0 P_{-1}(z) = 0.$$

• $\mathsf{p}_1(n)$ plays an important role in later developments.

**Lemma 1** Suppose $\mathsf{v} = -\log w$ has a derivative in some Lipshitz class with positive exponent. The lowering and raising operators satisfy the following:

$$\begin{aligned} P_n'(z) &= -B_n(z)P_n(z) + \beta_n\, A_n(z)P_{n-1}(z) \\ P_{n-1}'(z) &= [B_n(z) + \mathsf{v}'(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z), \end{aligned}$$

where

$$\begin{aligned} A_n(z) &:= \frac{1}{h_n}\int_a^b \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y}\, P_n^2(y)w(y)dy \\ B_n(z) &:= \frac{1}{h_{n-1}}\int_a^b \frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} P_n(y)P_{n-1}(y)w(y)dy. \end{aligned}$$

**Lemma 2** The functions $A_n(z)$ and $B_n(z)$ satisfy the conditions:

$$B_{n+1}(z) + B_n(z) = (z-\alpha_n)A_n(z) - \mathsf{v}'(z) \quad (S_1)$$

$$1 + (z-\alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1}A_{n+1} - \beta_n A_{n-1}(z) \quad (S_2).$$

**Lemma 3** The functions $A_n(z)$, $B_n(z)$, and the sum $\sum_{j=0}^{n-1} A_j(z)$, satisfy the conditions:

$$B_n^2(z)+\mathsf{v}'(z)B_n(z)+\sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z). \quad (S_2')$$

- If $\mathsf{v}'(z)$ rational then

$$\frac{\mathsf{v}'(z)-\mathsf{v}'(y)}{z-y} = \frac{\alpha}{zy} + \frac{\lambda}{(z+t)(y+t)}$$

$A_n(z)$ and $B_n(z)$ rational in $z$.

- $\sigma$–Form of $P_V$

**Theorem 1:** The Hankel determinant of the deformed Laguerre weight $w(x)$ in admits the following representation:

$$D_n(t,\lambda) = t^{n\lambda} \exp\left(\int_{\infty}^{t} \frac{H_n(x)-n\lambda}{x} dx\right)$$

where $H_n(t)$ satisfies the Jimbo-Miwa-Okamoto $\sigma$–form of $P_V$:

$$\begin{aligned}(tH_n'')^2 &= \left[tH_n' - H_n + H_n'(2n+\alpha+\lambda)+n\lambda\right]^2 \\ &- 4[tH_n' - H_n + n(n+\alpha+\lambda)][(H_n')^2 + \lambda H_n']\end{aligned}$$

$$H_n = t\frac{d}{dt}\log D_n = n(n+\alpha+\lambda)+\mathsf{p}_1(n,t).$$

- Derivation: A Quick Sketch

$$A_n(z) = \frac{1 - R_n(t)}{z} + \frac{R_n(t)}{z+t}$$
$$B_n(z) = -\frac{n + r_n(t)}{z} + \frac{r_n(t)}{z+t}$$
$$R_n(t) = \frac{\lambda}{h_n}\int_0^\infty \frac{[P_n(y)]^2}{y+t} w(y,t)dy$$
$$r_n(t) = \frac{\lambda}{h_{n-1}}\int_0^\infty \frac{P_n(y)P_{n-1}(y)}{y+t} w(y,t)dy.$$

- Sub. into compatibility conditions,

$$\alpha_n = 2n + 1 + \alpha + \lambda - tR_n \qquad (\alpha)$$

$$\beta_n = \frac{1}{1-R_n}\left[(2n+\alpha+\lambda)r_n + \frac{r_n^2 - \lambda\, r_n}{R_n} + n(n+\alpha)\right] \qquad (\beta)$$

$$t\sum_{j=0}^{n-1} R_j = n(n+\alpha+\lambda) + \mathsf{p}_1(n,t) \qquad \text{(sum)}$$

- Difference equations in $n$

$$r_{n+1} + r_n = \lambda - R_n(t + 2n + 1 + \alpha + \lambda - tR_n) \qquad (d_1)$$

$$r_n^2 - \lambda\, r_n = \beta_n R_n R_{n-1} \qquad (d_2)$$

with "initial conditions" $r_0(t) = 0$, $R_0(t) = $ given.

• $t-$ or Toda evolution

•A pair of Riccati Equations

$$\begin{aligned}
2r_n &= tR_n' + \lambda - R_n(t + 2n + \alpha + \lambda - tR_n) \\
tr_n' &= \frac{r_n^2 - \lambda\, r_n}{R_n} \\
&- \frac{R_n}{1 - R_n}\left[(2n + \alpha + \lambda)r_n + \frac{r_n^2 - \lambda\, r_n}{R_n} + n(n + \alpha)\right].
\end{aligned}$$

• $R_n = y/(y-1)$.

$$\begin{aligned}
y'' &= \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\frac{\alpha^2}{2}y - \frac{\lambda^2}{2y}\right) \\
&+ \frac{(2n+1+\alpha+\lambda)y}{t} - \frac{y(y+1)}{2(y-1)},
\end{aligned}$$

•Using the sum,

$$H_n := t\frac{d}{dt}\log D_n = t\sum_{j=0}^{n-1} R_j = n(n+\alpha+\lambda) + \mathsf{p}_1(n,t).$$

•Representation of $R_n$ in terms of $H_n$, $H_n'$, $H''$

$$2R_n = 1 + \frac{tH_n'' + (2n+\alpha+\lambda)H_n' - n(n+\alpha)}{tH_n' - H_n + n(n+\alpha+\lambda)}$$

$$\frac{2}{R_n} = \frac{-tH_n'' + (2n+\alpha+\lambda+t)H_n' - H_n + n\lambda}{(H_n')^2 + \lambda\, H_n'}.$$

- Large $n$ Coulomb Fluid Method

- large $n$, the ratio approximated by

$$\frac{D_n(\lambda)}{D_n(0)} \approx \exp\left[-\lambda^2 \frac{S_1(T)}{2} - \lambda S_2(T)\right]$$

where

$$S_1(T) = -2\log\left[\frac{1}{2}\left(\frac{T+a}{T+b}\right)^{1/4} + \frac{1}{2}\left(\frac{T+b}{T+a}\right)^{1/4}\right].$$

$$S_2(T) = -\frac{n}{2}\Bigg[(a+b)\log\left(\frac{\sqrt{T+a}+\sqrt{T+b}}{2}\right)$$

$$- \quad \frac{(\sqrt{T+a}-\sqrt{T+b})^2}{2}$$

$$- \quad \sqrt{ab}\log\left(\frac{(\sqrt{ab}+\sqrt{(T+a)(T+b)}\,)^2 - T^2}{(\sqrt{a}+\sqrt{b}\,)^2}\right)\Bigg]$$

$$T := \frac{1}{P}$$

- $$\mathcal{M}(\lambda) \approx \exp\left[-\lambda^2 \frac{S_1(T)}{2} - \lambda(S_2(T) + n\log T)\right]$$

- Gaussian distribution with mean and variance given by

$$\begin{aligned} \mu_{\text{Coul.}} &= -S_2(T) - n \log T \\ \sigma^2_{\text{Coul.}} &= -S_1(T) \end{aligned}$$

- Outage probability

$$P_{\text{out}}(C_{\text{out}}) \approx \frac{1}{2}\left[1 + \text{erf}\left(\frac{C_{\text{out}} - \mu}{\sqrt{2\sigma^2}}\right)\right] .$$

$$a = 2 + \beta - 2\sqrt{1+\beta}, \quad b = 2 + \beta + 2\sqrt{1+\beta}$$

- In general

$$\log \mathcal{M}(\lambda) = \sum_{l=1}^{\infty} \kappa_l \, \frac{\lambda^l}{l!}.$$

• Beyond the Coulomb Fluid Approximation

$$H_n(t) =: n\lambda + G_n(t)$$

such that

$$\mathcal{M}(\lambda) = \exp\left(\int_\infty^t \frac{G_n(x)}{x} dx\right) \ .$$

Note also that

$$t\frac{d}{dt} \log \mathcal{M}(\lambda) = G_n(t).$$

$$\begin{aligned}(tG_n'')^2 &= \left(G_n'(t + 2n + \lambda) - G_n\right)^2 \\ &- 4(tG_n' - G_n + n^2)(G_n')^2 + \lambda G_n')\end{aligned}$$

where the derivatives are with respect to $t$.

• The cumulants: Assuming $G_n(t)$ is analytic in $\lambda$

$$G_n(t) = \lambda g_1(t) + \lambda^2 g_2(t) + \lambda^3 g_3(t) \cdots$$

♠ **Note** $t = n/P := nT$.

$\kappa_1 = n(P - P^2 + (5/3)P - (7/2)P^4 + ..)$
$+\frac{1}{n}(P^3/3 - (5/2)P^4 + 14P^5 + ..) + O(1/n^3)$

$\kappa_2 = P^2 - 4P^3 + \frac{29}{2}P^4 - 52P^5 + ..$
$\frac{1}{n^2}\left((5/2)P^4 - 32P^5 + (806/3)P^6 + ..\right) + O(1/n^4)$

$\kappa_3 = \frac{1}{n}\left(2P^3 - 18P^4 + 114P^5 + ..\right) + \frac{1}{n^3}\left(18P^5 - 350P^6 + ..\right) +$
$O(1/n^5)$ and so on.

•Summation of the series in $P$. ($t = n/P =: nT$)
•Look at $g_1$ in detail:

$$(g_1)^2 - 4n^2 g_1' - 2(t+2n) g_1 g_1' + (t^2 + 4nt)(g_1')^2 - t^2 (g_1'')^2 = 0$$

$$(g_1)^2 - 4n g_1' - 2(T+2) g_1 g_1' + (T^2 + 4T)(g_1')^2 - \frac{T^2}{n^2}(g_1'')^2 = 0.$$

$$g_1 = nY_0(T) + \frac{Y_1(T)}{n} + ...$$

$$Y_0^2 - 4Y_0' - 2(T+2)Y_0 Y_0' + (T^2+4T)(Y_0')^2 = 0, \quad (*)$$

$$\begin{aligned} & 2Y_0Y_1 - 2(2+T)Y_1Y_0' - 4Y_1' - 2(2+T)Y_0Y_1' \\ + \quad & 2T(4+T)Y_0'Y_1' - T^2 Y_0'' = 0, \quad (**) \end{aligned}$$

• Solutions,

$$Y_0(T) = -\frac{4+T-\sqrt{T(4+T)}}{4+T+\sqrt{T(4+T)}} \sim -1$$

$$Y_1(T) = -\frac{1}{\sqrt{T}(4+T)^{5/2}} \sim -\frac{1}{32\sqrt{T}}.$$

• Note from Coulomb Fluid

$$Y_0(T) = -T\frac{d}{dT}(S_2/n - \ln T),$$

satisfies (*) and when sub. into (**) it becomes a *linear* equation in $Y_1$.

Similarly

$$g_2 = Z_0(T) + \frac{Z_1(T)}{n^2} + ....$$

where

$$Z_0(T) = -T\frac{d}{dT}S_1(T)/2 = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{T}{4+T}} + \frac{1}{4+T} \sim -\frac{1}{4}$$

$$Z_1(T) = \text{large expression} \sim -\frac{1}{64T},$$

and

$$g_3 = \frac{X_0(T)}{n} + \frac{X_1(T)}{n^3} + ...$$

$$X_0(T) = \frac{1}{2}\frac{\sqrt{T}}{(4+T)^{3/2}}\left[\frac{2}{4+T} + \sqrt{\frac{T}{4+T}} - 1\right] \sim \frac{\sqrt{T}}{32}$$

$$X_1(T) = \text{larger expression} \sim -\frac{1}{128T^{3/2}}$$

• Summary

•Cumulants

**Note** $\mu_{\text{Coulomb}}$ is LINEAR in $n$

$$\text{Cumulants} \quad \kappa_l = -l! \int_0^P g_l(n/y) \frac{dy}{y}$$

$$\begin{aligned}
\kappa_1 &= \mu_{\text{Coulomb}} + \frac{1}{n}\,\mu_{\text{Corr.}} + \mathrm{O}\left(\frac{1}{n^3}\right) \\
\kappa_2 &= \sigma^2_{\text{Coulomb}} + \frac{1}{n^2}\,\sigma^2_{\text{Corr.}} + \mathrm{O}\left(\frac{1}{n^4}\right) \\
\kappa_3 &= \frac{1}{n}\kappa_{3,A} + \frac{1}{n^3}\kappa_{3,B} + \mathrm{O}\left(\frac{1}{n^5}\right).
\end{aligned}$$

• Analysis at large P. Large deviation(?)

$$\begin{aligned}
\mu &\sim a\,n \log P + b\,\frac{\sqrt{P}}{n}, \quad P = \mathrm{O}(n^4) \\
\sigma^2 &\sim a\,\log P + b\,\frac{P}{n^2}, \quad P = \mathrm{O}(n^2) \\
\kappa_3 &\sim \frac{a}{n} + b\,\frac{P^2}{n^3}, \quad P = \mathrm{O}(n)
\end{aligned}$$

**See slides**

**End**

- A selection of Integrals: Schwinger (1918–1994)

$$\log(a+b) = \log\ a + \int_0^1 \frac{bdx}{a+bx}$$

$$\int_a^b \frac{\log(x+t)}{\sqrt{(b-x)(x-a)}}dx = 2\pi \log\left(\frac{\sqrt{t+a}+\sqrt{t+b}}{2}\right)$$

$$\int_a^b \frac{\log(x+t)}{\sqrt{(b-x)(x-a)}(x+t)}dx$$

$$= -\frac{2\pi}{\sqrt{(t+a)(t+b)}} \log\left(\frac{1}{2\sqrt{t+a}} + \frac{1}{2\sqrt{t+b}}\right)$$

$$\int_a^b \frac{\log(x+t)}{\sqrt{(b-x)(x-a)}\,x}dx = \frac{\pi}{\sqrt{ab}} \log\left(\frac{(\sqrt{ab}+\sqrt{(t+a)(t+b)})^2-t^2}{(\sqrt{a}+\sqrt{b})^2}\right)$$

$$\int_a^b \frac{\log(x+t)}{\sqrt{(b-x)(x-a)}(x-1)}dx$$

$$= \frac{\pi}{\sqrt{(1-a)(1-b)}} \log \frac{[\sqrt{1-a}+\sqrt{1-b}]^2}{(t+1)^2-[\sqrt{(t+a)(t+b)}-\sqrt{(1-a)(1-b)}]^2}.$$

**The End**

**Thank you for your attention**