Universality in the two matrix model with one quartic potential

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Two matrix model

We consider a pair of (M_1, M_2) of $n \times n$ Hermitian matrices taken randomly with respect to

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)} \mathrm{d}M_1 \mathrm{d}M_2$$

where

- $\mathrm{d}M = \prod_i \mathrm{d}M_{ii} \prod_{i>j} \mathrm{d}\,\mathrm{Re}\, M_{ij}\, \mathrm{d}\,\mathrm{Im}\, M_{ij}$
- V and W are two polynomials
- Z_n is the normalization constant
- τ is a constant (coupling constant)

In this talk we are interested in the asymptotic behavior of the eigenvalues as $n \to \infty$.

Two matrix model

- There is strong evidence that the two matrix model generates a bigger class of critical cases when compared to the one matrix models (e.g. (p, q) conformal minimal models)
- In this talk I will discuss recent results that show how to get the asymptotic behavior of the eigenvalues of M_1 in case

$$W(y) = \frac{1}{4}y^4 + \frac{t}{2}y^2 \qquad \text{and } V \text{ even.}$$

- Based on the recent work
  D-Kuijlaars, CPAM '09, Mo, CMP '09,
  D-Kuijlaars-Mo, arXiv '10, D-Geudens-Kuijlaars, arXiv '10
  and D-Geudens '11. See poster!

Eigenvalue density

- Denote the eigenvalues of M_1 by $x_1, \ldots, x_n$ and the eigenvalues of M_2 by $y_1, \ldots, y_n$.

  Then the joint probability distribution for the eigenvalues of M_1 and M_2 is given by

$$\frac{1}{\widetilde{Z}_n} \mathrm{e}^{-n\sum_{j=1}^{n}\left(V(x_j)+W(y_j)\right)} \det\exp\left(n\tau x_i y_j\right) \prod_{j>i}(x_j - x_i) \prod_{j>i}(y_j - y_i) \mathrm{d}^n x \mathrm{d}^n y.$$

  where $\widetilde{Z}_n$ is a new normalization constant.

Biorthogonal polynomials

- Consider two families of polynomials $\{p_{k,n}\}_k$ and $\{q_{j,n}\}_j$, where $p_{k,n}$ and $q_{j,n}$ are monic polynomials of degree k and j respectively and satisfy

$$\iint p_{k,n}(x) q_{j,n}(y) \mathrm{e}^{-n(V(x)+W(y)-\tau xy)} \, \mathrm{d}x\mathrm{d}y = 0, \qquad j \neq k$$

- The polynomials are well-defined and have real and simple zeros Ercolani-McLaughlin '01.
- The zeros of $p_{k+1,n}$ and $p_{k,n}$ interlace D-Geudens-Kuijlaars '10.

Relation with two matrix model

- The biorthogonal polynomials are the average characteristic polynomials

$$p_{n,n}(x) = \mathbb{E}(\det(x - M_1))$$
$$q_{n,n}(y) = \mathbb{E}(\det(y - M_2)).$$

- The eigenvalue form a determinantal process with kernel that is constructed out of these biorthogonal polynomials
Mehta-Shukla '94, Eynard-Mehta '98.

The tansformed functions $Q_{j,n}$ and $P_{k,n}$

- Introduce the transformed funtions

$$Q_{j,n}(x) = \mathrm{e}^{-nV(x)} \int q_{j,n}(y) \mathrm{e}^{-n\left(W(y)-\tau xy\right)} \mathrm{d}y$$

$$P_{k,n}(y) = \mathrm{e}^{-nW(y)} \int p_{k,n}(x) \mathrm{e}^{-n\left(V(x)-\tau xy\right)} \mathrm{d}x$$

- Note that we have the orthogonality relations

$$\int p_{k,n}(x) Q_{j,n}(x) \, \mathrm{d}x = 0, \qquad j \neq k$$

$$\int P_{k,n}(y) q_{j,n}(y) \, \mathrm{d}y = 0, \qquad j \neq k$$

- Let

$$h_{k,n}^2 = \iint p_{k,n}(x) q_{k,n}(y) \mathrm{e}^{-n\left(V(x)+W(y)-\tau xy\right)} \, \mathrm{d}x \mathrm{d}y$$

Four kernels

Define kernels by

$$K_{11}(x_1, x_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(x_1) Q_{k,n}(x_2),$$

$$K_{22}(y_1, y_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} P_{k,n}(y_1) q_{k,n}(y_2)$$

$$K_{12}(x, y) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(x) q_{k,n}(y)$$

$$K_{21}(y, x) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} P_{k,n}(y) Q_{k,n}(x) - \mathrm{e}^{-n\left(V(x)+W(y)-\tau xy\right)}$$

Eynard-Metha Theorem

- Denote the eigenvalues of M_1 by $x_1, \ldots, x_n$ and of M_2 by $y_1, \ldots, y_n$. The probability density function can be written as

$$\mathcal{P}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{n!^2} \det \begin{pmatrix} \left(K_{11}(x_i, x_j)\right)_{i,j=1}^{n} & \left(K_{12}(x_i, y_j)\right)_{i,j=1}^{n} \\ \left(K_{21}(y_i, x_j)\right)_{i,j=1}^{n} & \left(K_{22}(y_i, y_j)\right)_{i,j=1}^{n} \end{pmatrix}$$

  and the marginal densities are given by

$$\underbrace{\int \cdots \int}_{n-k+n-l \text{ times}} \mathcal{P}(x_1, \ldots, x_n, y_1, \ldots, y_n) \mathrm{d}x_{k+1} \cdots \mathrm{d}x_n \mathrm{d}y_{l+1} \cdots \mathrm{d}y_n$$

$$= \frac{(n-l)!(n-k)!}{n!^2} \det \begin{pmatrix} \left(K_{11}(x_i, x_j)\right)_{i,j=1}^{k} & \left(K_{12}(x_i, y_j)\right)_{i,j=1}^{k,l} \\ \left(K_{21}(y_i, x_j)\right)_{i,j=1}^{l,k} & \left(K_{22}(y_i, y_j)\right)_{i,j=1}^{l} \end{pmatrix}$$

- Concluding, the k-point correlation functions are determinants of a matrix involving the kernels K_{ij}

Averaging over M_2

- When averaged over M_2 we see that the eigenvalues of M_1 describe a determinantal point process with kernel K_{11}.

$$\underbrace{\int \cdots \int}_{n-k \text{ times}} \mathcal{P}(x_1, \ldots, x_n) \mathrm{d}x_{k+1} \cdots \mathrm{d}x_n = \frac{(n-k)!}{n!} \det \left(K_{11}(x_i, x_j) \right)_{i,j=1}^{k}$$

- This is a particular example of a so-called biorthogonal ensemble.

Asymptotic analysis

Question: Find a full asymptotic description of the biorthogonal polynomials and the associated kernels.

- There exist several Riemann-Hilbert characterizations of the biorthogonal polynomials
  Ercolani-Mclaughlin '01, Kapaev '03, Bertola-Eynard-Harnad '03, Kuijlaars-McLaughlin '05
- Except for the special in which both V and W are quadratic Ercolani-McLaughlin '01, a steepest descent analysis turns out to be complicated.

Multiple Orthogonality

- The main idea in Kuijlaars-McLaughlin '05 is to interpret the polynomials as multiple orthogonal polynomials.
- Define the weight function w_j by

$$w_j(x) = \mathrm{e}^{-nV(x)} \int_{\mathbb{R}} y^j \mathrm{e}^{-n(W(y)-\tau xy)}\, \mathrm{d}y, \qquad j = 0, 1, \ldots d-2.$$

  where $d = \mathrm{degree}(W)$.
- The polynomials $p_{k,n}$ are multiple orthogonal polynomials of type II with respect to the weights w_j on $\mathbb{R}$. For $p_{n,n}$ this means that

$$\int_{\mathbb{R}} p_{n,n}(x) x^l w_j(x)\, \mathrm{d}x = 0, \quad l = 0, \ldots, n_j - 1, \quad j = 0, 1, \ldots, d-2,$$

  where n_j is the integer part of $(n + d - 2 - j)/(d - 1)$.

The Riemann-Hilbert problem

- For multiple orthogonal polynomials a Riemann-Hilbert characterization is known Van Assche-Geronimo-Kuijlaars '01.
- We seek for a $d \times d$ matrix valued function Y such that

$$\begin{cases} Y \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_0(x) & \dots & w_2(x) \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, & x \in \mathbb{R} \\ Y(z) = (I + \mathcal{O}(1/z)) \operatorname{diag}(z^n, z^{-n_0}, \dots, z^{-n_{d-2}}), & z \to \infty \end{cases}$$

where n_j is the integer part of $(n + d - 2 - j)/(d - 1)$.

The solution of the Riemann-Hilbert problem

- The solution exists and is unique. Moreover

$$Y_{11}(z) = p_{n,n}(z)$$

Van Assche-Geronimo-Kuijlaars '01

- Also the kernel $K_{11}^{(n)}$ can be expressed in Y

$$K_{11}^{(n)}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_0(y) & \cdots & w_2(y) \end{pmatrix} Y_+(y)^{-1} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Daems-Kuijlaars '04

- A steepest descent analysis for the RH problem in the general situation is still an important open problem!

Equilibrium problem

- One of the obstacles is the lack of an equilibrium problem characterizing the limiting zero distribution.
- Recall that the one matrix unitary ensembles are given by

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM,$$

- The limiting eigenvalue distribution of converges now to the equilibrium measure μ that minimizes the energy functional

$$\iint_{\mathbb{R}^2} \frac{1}{\log|x-y|} d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x).$$

  This equilibrium measure is a key ingredient in the asymptotic analysis of the orthogonal polynomials.

Quartic potential

From now we consider the two matrix model for the special case

$$W(y) = \frac{y^4}{4} + t\frac{y^2}{2} \text{ and } V \text{ even}$$

and consider the asymptotics of the polynomials $p_{n,n}$ and the kernel K_{11} as $n \to \infty$.

We found an equilibrium problem in the cases

- $t = 0$. D-Kuijlaars '09 for the one-cut case and Mo '09 for the multicut case.
- $t \neq 0$ and $V(x) = \frac{x^2}{2}$ D-Geudens-Kuijlaars '10.
- $t \neq 0$ and general even V D-Mo-Kuijlaars '10

Using the equilibrium problem we are able to perform a steepest descent analysis on the RH problem and prove asymptotic results.

$t = 0$

The equilibrium problem for $t = 0$

We seek to minimize the energy functional

$$I(\nu_1, \nu_2, \nu_3) = \sum_{j=1}^{3} \iint \log |x - y|^{-1} \, d\nu_j(x) d\nu_j(y)$$

$$- \sum_{j=1}^{2} \iint \log |x - y|^{-1} \, d\nu_j(x) d\nu_{j+1}(y)$$

$$+ \int \left(V(x) - \frac{3}{4} \tau^{4/3} |x|^{4/3} \right) \, d\nu_1(x)$$

among all measures (ν_1, ν_2, ν_3) satisfying

1. ν_1 is a measure on $\mathbb{R}$ with $\nu_1(\mathbb{R}) = 1$
2. ν_2 is a measure on $i\mathbb{R}$ with $\nu_2(i\mathbb{R}) = 2/3$
3. ν_3 is a measure on $\mathbb{R}$ with $\nu_3(\mathbb{R}) = 1/3$
4. $\nu_2 \leq \sigma$ with

$$d\sigma(z) = \frac{\sqrt{3}\tau^{4/3} |z|^{1/3}}{2\pi} \, |dz|$$

The minimizer

Theorem (D-Kuijlaars, '09)

There is unique minimizer (μ_1, μ_2, μ_3) *of the energy functional* I. *Moreover, the measure* μ_1 *is the weak limit of the normalized zero distribution of the polynomial* $p_{n,n}$,

$$\frac{1}{n} \sum_{x \,:\, p_{n,n}(x)=0} \delta_x \to \mu_1$$

as $n \to \infty$.

The minimizer

Theorem (D-Kuijlaars '09)

The minimizer (μ_1, μ_2, μ_3) *has the following properties*

1. μ_1 *is supported on finitely many intervals* $\cup_{j=1}^{r}[a_j, b_j]$ *and there exists real analytic* h_j *such that*

$$\frac{\mathrm{d}\mu_1(x)}{\mathrm{d}x} = h_j(x)\sqrt{(b_j - x)(x - a_j)}, \quad x \in [a_j, b_j]$$

2. μ_2 *is supported on* $\mathrm{i}\mathbb{R}$ *and* $\mu_2 = \sigma$ *on* $\mathrm{i}[-c, c]$. *Moreover, there exists an analytic function* ψ_2 *such that*

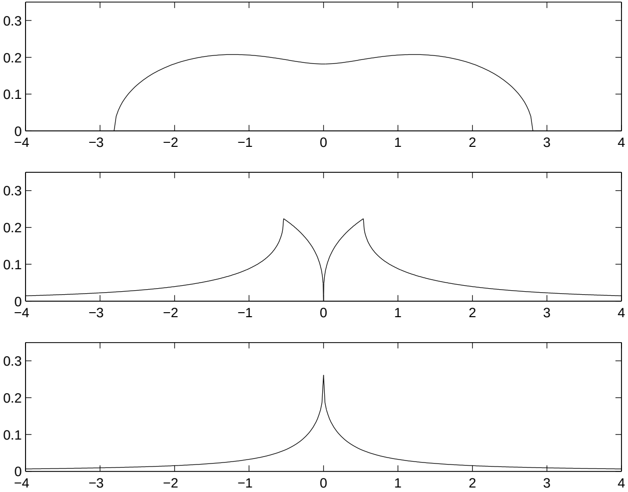
$$\mathrm{d}(\sigma - \mu_2)(y) = \rho_2(y)|\mathrm{d}y|$$

and ρ_2 *vanishes as as square root near* $y = \pm ic$.

3. μ_3 *is supported on* $\mathbb{R}$ *and there exists a function* ρ_3 *which is real analytic in* $\mathbb{R} \setminus \{0\}$ *and such that*

$$\mathrm{d}\mu_3(x) = \rho_3(x)\mathrm{d}x$$

Example: $V(x) = x^2/2$ *and* $\tau = 1$



Theorem

Let $W(y) = y^4/4$ and V even. Let μ_1 be the first component of the minimizer of I. Then

- *The measure μ_1 also describes the limiting mean eigenvalues density for the matrix M_1, i.e.*

$$\lim_{n\to\infty} \frac{1}{n} K_{11}^{(n)}(x,x) = \frac{d\mu_1(x)}{dx}$$

- *Universality:*
  For x^* in the bulk:

$$\lim_{n\to\infty} \frac{1}{cn} K_{11}\left(x^* + \frac{u}{cn}, x^* + \frac{v}{cn}\right) = \frac{\sin \pi(u-v)}{\pi(u-v)}$$

  For x^* at the endpoints: Airy kernel.

- *All critical cases from the one-matrix case can occur, but no other.*

The one-cut case is D-Kuijlaars '09 and the multicut case is Mo '09

$t \neq 0$ and V even

The equilibrium problem for $t \neq 0$ and V even

We seek to minimize the energy functional

$$I(\nu_1, \nu_2, \nu_3) = \sum_{j=1}^{3} \iint \log |x-y|^{-1} \, d\nu_j(x) d\nu_j(y) - \sum_{j=1}^{2} \iint \log |x-y|^{-1} \, d\nu_j(x) d\nu_{j+1}(y) + \int V_1(x) \, d\nu_1(x) + \int V_3(x) \, d\nu_3(x)$$

among all measures (ν_1, ν_2, ν_3) satisfying

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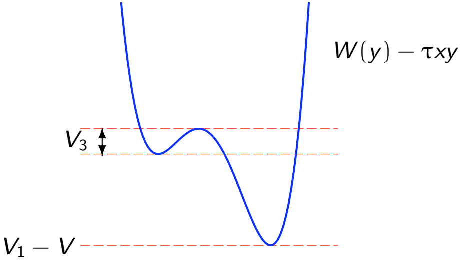
4. $\nu_2 \leq \sigma$

Definition of V_1

- The external field V_1 is defined by

$$V_1(x) = V(x) + \min_{y \in \mathbb{R}} \left(W(y) - \tau xy \right), \quad x \in \mathbb{R}$$

- The external field V_3 is the difference between the other two extreme values of $W(y) - \tau xy$ (viewed as a function in y).



$|x| > x^*$ and $t < 0$.

Definition of σ

Again we consider

$$W'(\omega) - \tau z = \omega^3 + t\omega - \tau z = 0,$$

but now for $z \in \mathrm{i}\mathbb{R}$. Then

$$\frac{\mathrm{d}\sigma(z)}{|\mathrm{d}z|} = \frac{\tau}{\pi} \operatorname{Re} \omega_1(z),$$

where ω_1 is the solution of the cubic equation with biggest real part.

$$\operatorname{supp}(\sigma) = (-\mathrm{i}\infty, -\mathrm{i}y^*] \cup [\mathrm{i}y^*, \mathrm{i}\infty)$$

where

$$y^* = \begin{cases} 0, & t < 0 \\ \frac{2t^{3/2}}{3\sqrt{3}\tau}, & t \geq 0 \end{cases}$$

The minimizer

Theorem (D-Geudens-Kuijlaars '10, D-Kuijlaars-Mo '10)

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as $n \to \infty$.

Supports of the measure

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2. $\mu_2 - \sigma$ *on* $\mathrm{i}\mathbb{R} \setminus \mathrm{i}[-c_2, c_2]$ *and there exsist exists an analytic function* ψ_2 *such that*

$$\mathrm{d}(\sigma - \mu_2)(y) = \rho_2(y)|\mathrm{d}y|.$$

and if $c_2 > 0$ *then* ρ_2 *vanishes as a square root at* $\pm\mathrm{i}c_2$*.*

3. μ_3 *is supported on* $\mathbb{R} \setminus (-c_3, c_3)$ *and there exists a function* ρ_3 *which is real analytic in* $\mathbb{R} \setminus [-c_3, c_3]$ *and such that*

$$\mathrm{d}\mu_3(x) = \rho_3(x)\mathrm{d}x$$

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Theorem (D-Kuijlaars-Mo '10)

Let $W(y) = y^4/4 + ty^2/2$ and V even. Let μ_1 be the first component of the minimizer of I. Then

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- *Universality:*
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  For x^* at regular endpoints: Airy kernel.

Supports of the measure

In the analysis we distinguish the cases

Case I: $0 \in S(\mu_1)$, $0 \notin S(\sigma - \mu_2)$ and $0 \in S(\mu_3)$

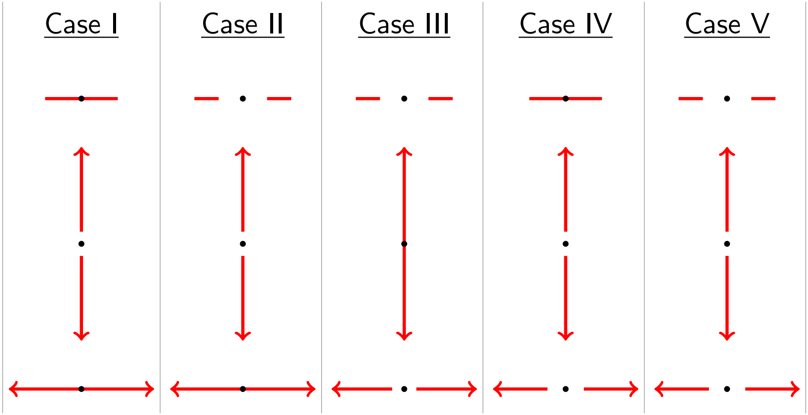
Case II: $0 \notin S(\mu_1)$, $0 \notin S(\sigma - \mu_2)$ and $0 \in S(\mu_3)$

Case III:$0 \notin S(\mu_1)$, $0 \in S(\sigma - \mu_2)$ and $0 \notin S(\mu_3)$

Case IV:$0 \in S(\mu_1)$, $0 \notin S(\sigma - \mu_2)$ and $0 \notin S(\mu_3)$

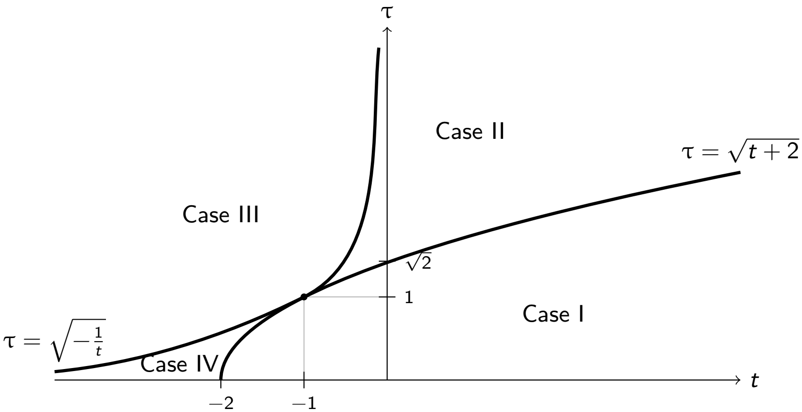
Case V: $0 \notin S(\mu_1)$, $0 \notin S(\sigma - \mu_2)$ and $0 \notin S(\mu_3)$

Critical phenomena occur when going from one case to the other.



On top of each the supports $S(\mu_1)$, $S(\sigma - \mu_2)$ and $S(\mu_3)$
(also the cuts of the corresponding Riemann surface)

Phase diagram for $V(x) = x^2/2$



- Case I $\longrightarrow$ Case II: Merging in $S(\mu_1) \longrightarrow$ Painlevé II
- Case IV $\longrightarrow$ Case III: $S(\mu_1)$ splits and $S(\sigma - \mu_2)$ merges $\longrightarrow$ Pearcey
- Intersection point: simulataneous transition in all three measures.
  A new kernel appears D-Geudens '11?.

Derivation of the equilibrium problem

- In D-Kuijlaars,D-Kuijlaars-Mo we use the equilibrium problem as an anzats and prove the results as a consequence of the RH analaysis.
- In D-Geudens-Kuijlaars we give a constructive approach how to obtain the equilibrium problem for $V(x) = x^2/2$ and $W(y) = y^4/4 + ty^2$.

Derivation of the equilibrium problem

- If $V(x) = x^2/2$ then $q_{k,n}$ are orthogonal polynomials on the real line. The asymptotics of these polynomials is well-known. In particular the asymptotics for the recurrence coefficients
- The polynomials $p_{k,n}$ satisfy a five term recurrence and the coefficients can be expressed in terms of the recurrence coefficients of the other family. So we know the asymptotic behavior of the recurrence coefficients.
- The zeros of the polynomials are the eigenvalues of the 'Jacobi' matrix. Which in the limit, locally has a banded Toeplitz structure.
- In D-Kuijlaars '08 we formulated an equilibrium problem for the eigenvalue distribution of banded Toeplitz matrices. The equilibrium problem for the biorthogonal family follows by integrating these equilibrium problems.

Banded Toeplitz matrices

Let $T_n(a)$ be a Toeplitz matrix

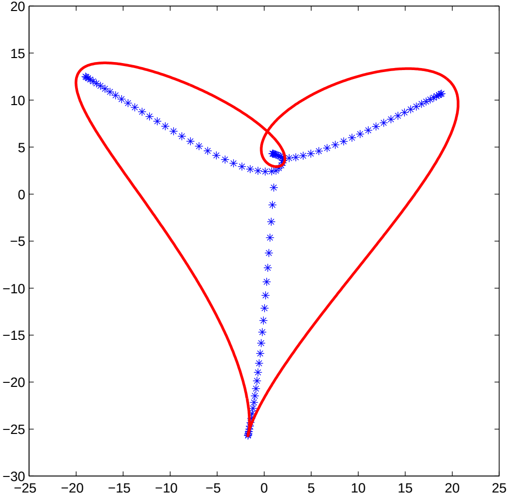
$$\big(T_n(a)\big)_{jk} = a_{j-k}, \qquad j,k = 1,\dots,n$$

for which the symbol a has only finitely many Fourier coefficients

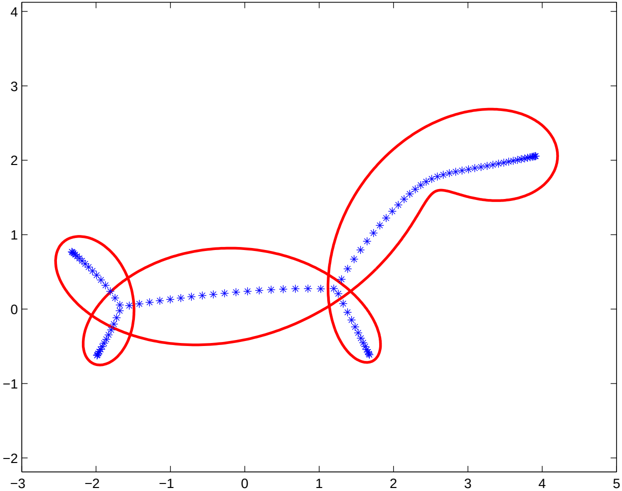
$$a(z) = \sum_{j=-q}^{p} a_j z^j, \qquad p,q > 0, \quad a_{-p}, a_q \neq 0$$

What is the limiting behavior of the spectrum $\sigma(T_n(a))$ as $n \to \infty$?

Example



Example



An associated Riemann surface

The central object to study is the algebraic equation

$$a(z) - \lambda = \sum_{j=-q}^{p} a_q z^j - \lambda = 0.$$

For each λ this equation has $p + q$ solutions which we order according to magnitude

$$0 < |z_1(\lambda)| \leq \ldots \leq |z_{p+q}(\lambda)|$$

Define

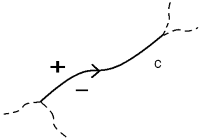
$$\Gamma_k = \{\lambda \mid |z_{q+k}(\lambda)| = |z_{q+k+1}(\lambda)|\},$$

for $k = -q + 1, \ldots, p - 1$.

(Assumption: $\quad \gcd\{k \mid a_k \neq 0\} = 1)$

The contours Γ_k and the measures μ_k

- The contour Γ_0 is bounded, the other are unbounded. All consist of finitely many analytic arcs



- Define the measure μ_k on Γ_k by

$$d\mu_k(\lambda) = \frac{1}{2\pi i} \sum_{j=1}^{q+k} \left(\frac{z'_{j+}(\lambda)}{z_{j+}(\lambda)} - \frac{z'_{j-}(\lambda)}{z_{j-}(\lambda)} \right) d\lambda,$$

- Each measure μ_k is positive measure on Γ_k with finite total mass given by

$$\mu_k(\Gamma_k) = \begin{cases} \frac{q+k}{q} & -q+1 \leq k \leq 0 \\ \frac{p-k}{p} & 0 \leq k \leq p-1 \end{cases}$$

Two result on the asymptotic behavior

Theorem (Schmidt-Spitzer '60)

The eigenvalues of $T_n(a)$ accumulate on the contour Γ_0

$$\sigma(T_n(a)) \longrightarrow \Gamma_0$$

as $n \longrightarrow \infty$.

Theorem (Hirschman '67)

The measure μ_0 describes the limiting distribution of the eigenvalues along Γ_0,

$$\frac{1}{n} \sum_{\lambda \in \sigma(T_n(a))} \delta_\lambda \longrightarrow \mu_0$$

as $n \longrightarrow \infty$.

Equilibrium problem for banded Toeplitz matrices

Theorem (D-Kuijlaars '08)

The vector of measures $(\mu_{-q+1}, \ldots, \mu_{p-1})$ *is the unique minimizer of the energy functional* E *defined by*

$$E(\nu_{-q+1}, \ldots \nu_{p-1}) = \sum_{k=-q+1}^{p-1} \iint \log \frac{1}{|x-y|} d\nu_k(x) d\nu_k(y) - \sum_{k=-q+1}^{p-2} \iint \log \frac{1}{|x-y|} d\nu_k(x) d\nu_{k+1}(y)$$

where each measure ν_k *is a measure on* Γ_k *with total mass*

$$\nu_k(\Gamma_k) = \begin{cases} \frac{q+k}{q}, & k \leq 0 \\ \frac{p-k}{p}, & k \geq 0 \end{cases}$$

Equilibrium problem for banded Toeplitz matrices

Theorem (D-Kuijlaars '08)

The vector of measures $(\mu_{-q+1}, \ldots, \mu_{p-1})$ *is the unique minimizer of the energy functional* E *defined by*

$$E(\nu_{-q+1}, \ldots \nu_{p-1}) = \sum_{k=-q+1}^{p-1} \iint \log \frac{1}{|x-y|} \mathrm{d}\nu_k(x) \mathrm{d}\nu_k(y) - \sum_{k=-q+1}^{p-2} \iint \log \frac{1}{|x-y|} \mathrm{d}\nu_k(x) \mathrm{d}\nu_{k+1}(y)$$

where each measure ν_k *is a measure on* Γ_k *with total mass*

$$\nu_k(\Gamma_k) = \begin{cases} \frac{q+k}{q}, & k \leq 0 \\ \frac{p-k}{p}, & k \geq 0 \end{cases}$$

Generalization to Toeplitz matrices with rational symbols (Delvaux-D '10)

Back to the biorthogonal polynomials

The biorthogonal polynomials were defined by the relation

$$\iint p_{k,n}(x) q_{j,n}(y) \mathrm{e}^{-n(V(x)+W(y)-\tau xy)}\, \mathrm{d}x \mathrm{d}y = 0, \qquad j \neq k$$

and we were interested in the case

$$W(y) = \frac{1}{4}y^4 + \frac{1}{2}tx^2 \text{ and } V(x) = \frac{1}{2}x^2$$

and asymptotics for $p_{n,n}$ and K_{11}.

$V(x) = x^2/2$ *implies* $q_{k,n}$ *are OP's*

Theorem

If $V(x) = x^2/2$ *then* $q_{k,n}$ *is the monic orthogonal polynomial of degree* k *wrt the weight*

$$e^{-n\left(y^4/4-\tau^2 y^2/2\right)}\,dy.$$

Proof
Recall that

$$P_{k,n}(y) = e^{-nW(y)} \int p_{k,n}(x) e^{-n\left(V(x)-\tau xy\right)}\,dx$$

and

$$\int P_{k,n}(y) q_{j,n}(y)\,dy = 0, \quad j \neq k.$$

Since $V(x) = x^2/2$ it follows that $P_{k,n}$ is a polynomial of degree k multiplied with a Gaussian and the statement follows.

Recurrence coefficients (1)

- The orthogonal polynomials $q_{k,n}$ for $\mathrm{e}^{-n\left(y^4/4-\tau^2y^2/2\right)}$ satisfy a recurrence relation

$$yq_{k,n}(y) = q_{k+1,n}(y) + a_{k,n}q_{k-1,n}(y)$$

- Bleher and Its proved that in the limit $k, n \to \infty$ and $k/n \to \xi$ we have

$$\lim_{n\to\infty,\ k/n\to\xi} a_{k,n} = \frac{\tau^2 + \sqrt{\tau^4 + 12\xi}}{6}, \qquad \xi > \tau^4/4$$

and

$$\lim_{n\to\infty,\ k/n\to\xi} a_{k,n} = \begin{cases} \frac{\tau^2-\sqrt{\tau^4-4\xi}}{2}, & k \text{ even} \\ \frac{\tau^2+\sqrt{\tau^4-4\xi}}{2}, & k \text{ odd} \end{cases}, \qquad \xi < \tau^4/4$$

Recurrence coefficients (2)

- Using integration by parts and the recurrence for $q_{k,n}$, we find that the polynomials $p_{k,n}$ satisfy a longer recursion

$$xp_{k,n}(x) = p_{k+1,n}(x) + b_{k,n}p_{k-1,n}(x) + c_{k,n}p_{k-3,n}(x)$$

- The recurrence coefficients $b_{k,n}$ and $c_{k,n}$ can be expressed in terms of the recurrence coefficients $a_{k,n}$

$$b_{k,n} = (a_{k+1,n} + a_{k,n} + a_{k-1,n})a_{k,n}$$
$$c_{k,n} = \tau^2 a_{k,n}a_{k-1,n}a_{k-2,n}$$

- Since we know the asymptotic behavior of $a_{k,n}$ we also know the asymptotic behavior of $b_{k,n}$ and $c_{k,n}$.

Recurrence coefficients (3)

- Put the recurrence coefficients in a matrix R_n

$$R_n = \begin{pmatrix} 0 & 1 & & & & & \\ b_{1,n} & 0 & 1 & & & & \\ 0 & b_{2,n} & 0 & 1 & & & \\ c_{3,n} & 0 & b_{3,n} & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 1 \\ & & & c_{n-1,n} & 0 & b_{n-1,n} & 0 \end{pmatrix}$$

  Then the zeros of $p_{n,n}$ coincide with the eigenvalues of R_n

- By the asymptotic behavior of the recurrence coefficients it follows that R_n locally has a banded Toeplitz structure

Recurrence coefficients (4)

- If we zoom in around the kk-entry, with

$$\frac{k}{n} \approx \xi$$

then R_n locally looks like a Toeplitz matrix with a certain symbol

- For each $\xi \in (0,1]$ we obtain three measures $\mu_{0,\xi}$, $\mu_{1,\xi}$ and $\mu_{2,\xi}$ defined on the contours $\Gamma_{0,\xi}$, $\Gamma_{1,\xi}$ and $\Gamma_{2,\xi}$.
- The limiting zero distribution μ_0 is now given by

$$\mu_0 = \int_0^1 \mu_{0,\xi} \, d\xi$$

- We can also integrate the other measures,

$$\mu_j = \int_0^1 \mu_{j,\xi} \, d\xi$$

and the energy problem to obtain the energy function for (μ_0, μ_1, μ_2).