From Random Matrices to Geometry: the "topological recursion"

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Brunel workshop on random matrix theory

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Contents

1. Random Matrices, statistical properties of the spectrum.
Question: How to compute large size asymptotics ?
To all orders ?
$\longrightarrow$ answer: geometric invariants of a plane curve.

2. What Riemann surface ?
which plane curve is associated to a matrix model ?

3. Solution: "topological recursion".

4. General properties of the topological recursion
$\longrightarrow$ symplectic invariants of a plane curve.

5. Examples of applications

6. Conclusion and prospects

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1. Random Matrices

Random matrices

Introduction

- Introduced in maths by Wishart, Ginibre, ... etc, from the 1920's.

- Introduced in physics by Wigner, Dyson, Mehta, ... etc, from the 1950's.

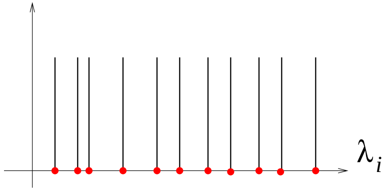
Many applications:

- Nuclear physics, quantum chaos, disordered systems, quantum gravity, string theory, ...etc
- Probabilities, orthogonal polynomials, integrable systems, combinatorics, algebraic geometry, ... etc
- Also finance, biology, cell phones, ... etc

Random matrices

Random matrix $N \times N$ (Hermitian) $\longrightarrow$ Random eigenvalues λ_i

$$M = \left(\quad M_{i,j} = \text{random} \quad \right)$$



Probability law = Boltzmann weight

$$\mathcal{P}(M) = \frac{1}{Z_N}\, \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)} \underbrace{\prod_{i,j} dM_{i,j}}_{dM}\, , \qquad Z_N = \int dM\, \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)}$$

Question: Large N asymptotic behavior of eigenvalues statistics ?

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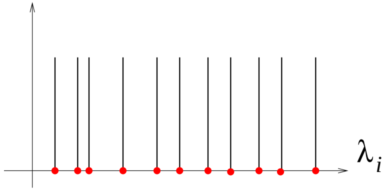

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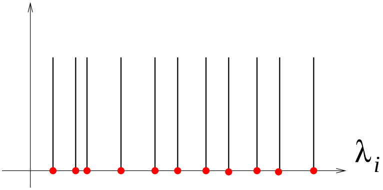

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Large size asymptotics

Large N asymptotics of the partition function:

$$Z_N \sim e^{N^2 F_0}$$

F_0 known for a long time, various expressions.
F_1 known since the 90s, $F_1 \sim \frac{1}{4} \ln \det \Delta$

Question: compute F_g for $g \geq 2$?

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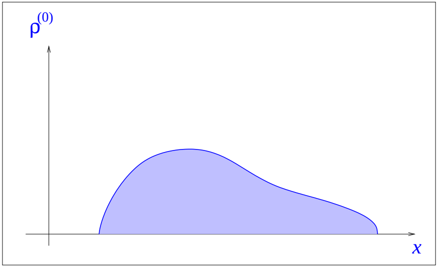
Eigenvalues statistics

Density of eigenvalues

$$\rho(x) = \sum_{i=1}^{N} \left\langle \delta(x-\lambda_i) \right\rangle$$



Large N asymptotics:

$$\rho(x) \sim N\, \rho^{(0)}(x)$$

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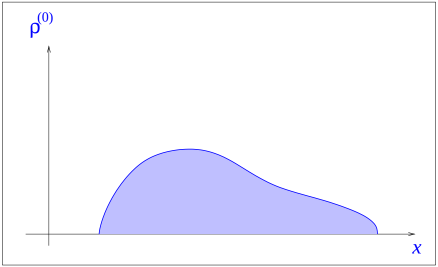

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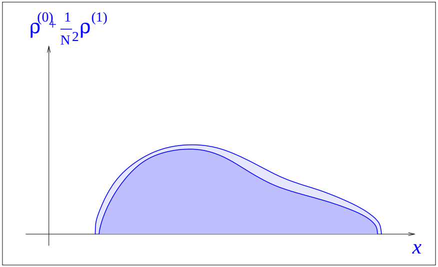

Large N asymptotics:

$$\rho(x) \sim N\, \rho^{(0)}(x) + N^{-1}\, \rho^{(1)}(x)$$

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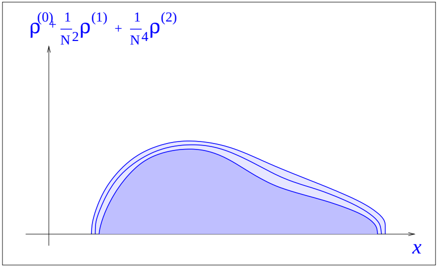

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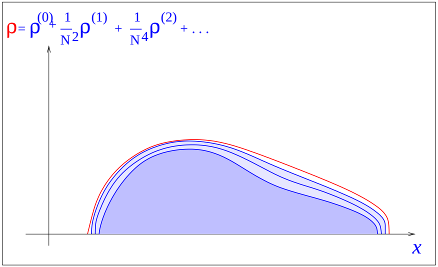

Large N asymptotics:

$$\rho(x) \sim N\,\rho^{(0)}(x) + \sum_{g=1}^{\infty} N^{1-2g}\,\rho^{(g)}(x)$$

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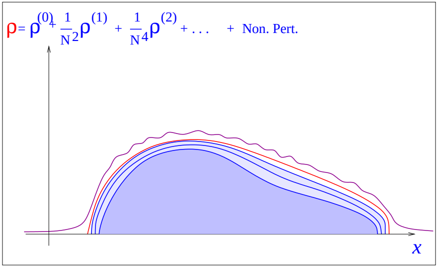

$$\rho(x) \sim N\,\rho^{(0)}(x) + \sum_{g=1}^{\infty} N^{1-2g}\,\rho^{(g)}(x) + \text{Non.perturbative}$$

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Question: compute $\rho^{(g)}(x)$ for $g \geq 1$? Compute Non. pert ?

Asymptotics

Answer: all orders in the asymptotics expansion can be computed by a recursion formula.

$$\ln Z = \sum_g N^{2-2g} F_g + \mathrm{Non.\,Pert}$$

$$\rho(x) = \sum_g N^{1-2g} \rho^{(g)}(x) + \mathrm{Non.\,Pert}$$

Each term has a geometric interpretation:
F_g = "symplectic invariant" of some embedded Riemann surface.

Ex: F_0 ="prepotential", $F_1 = \frac{1}{4} \ln \det \Delta, \cdots$

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2. Riemann surface

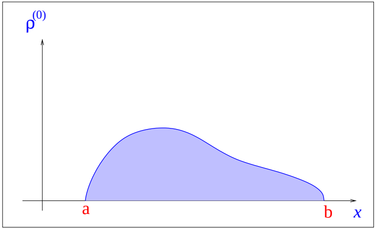
How to Obtain a plane curve (embedded Riemann surface) from the large N density

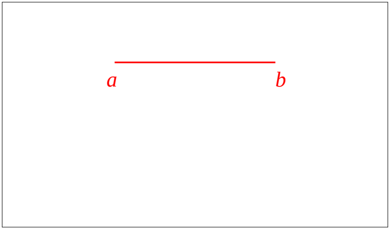
$$\rho^{(0)}(x)$$

Density → Riemann surface

Consider the "resolvent" =Stieljes transform of $\rho^{(0)}$:

$$W^{(0)}(x) = \int_{\text{supp}\,\rho^{(0)}} \frac{\rho^{(0)}(x')\,dx'}{x-x'} \qquad \text{defined for } x \notin \text{supp}\,\rho^{(0)}$$





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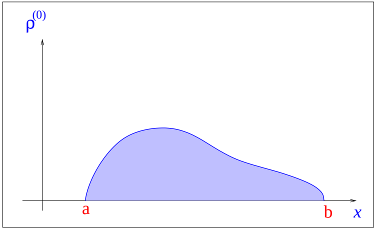

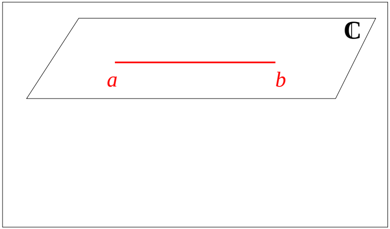

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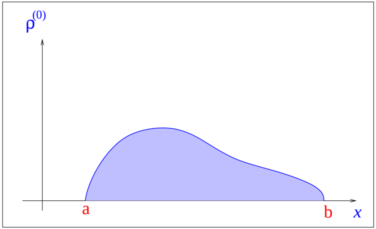

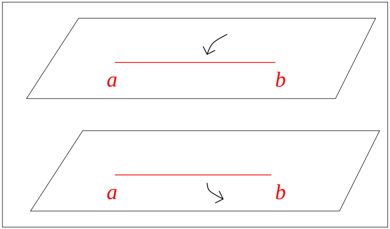

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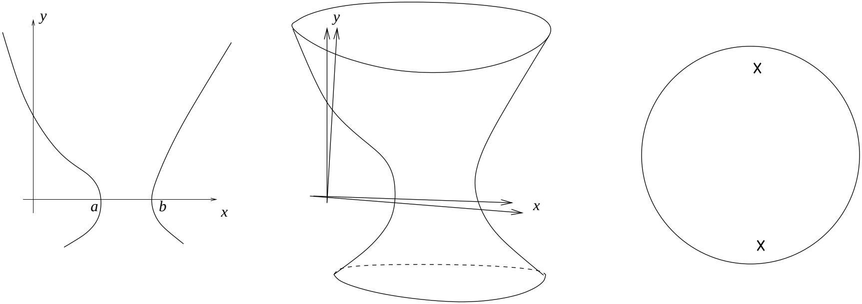
A Riemann surface

Plot of the resolvent $y = W^{(0)}(x) = \int_a^b \frac{\rho^{(0)}(x')\,dx'}{x-x'}$, in $\mathbb{C} \times \mathbb{C}$

1-cut case:



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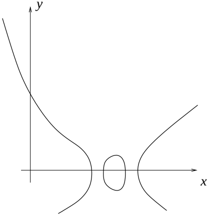
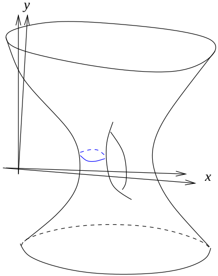
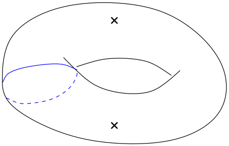
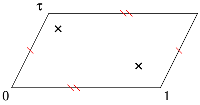
Plot of the resolvent $y = W^{(0)}(x) = \int_a^b \frac{\rho^{(0)}(x')\,dx'}{x-x'} + \int_c^d \frac{\rho^{(0)}(x')\,dx'}{x-x'}$, in $\mathbb{C}\times\mathbb{C}$

2-cuts case:









Riemann surface

Riemann surface summary:

• starting point: probability law $e^{-N \operatorname{tr} V(M)} dM \to$ potential V
• $\to$ leading density $\rho^{(0)}(x)$ (well known),
• Stieljes transform
$W^{(0)}(x) = \int \rho^{(0)}(x')\, dx'/(x-x')$ =resolvent,
• The resolvent $y = W^{(0)}(x)$ is a Riemann surface embedded into $\mathbb{C} \times \mathbb{C}$ (also called "plane curve", or "spectral curve").
• It has 2 natural metrics:

– an extrinsic metrics coming from the embedding $(x, y) \in \mathbb{C} \times \mathbb{C}$
(not universal, it depends on the potential V).

– an intrinsic metrics depending only on the conformal structure of the Riemann surface, by the embedding of the Riemann surface into its Jacobian $\mathbb{C}^{\text{genus}}/\text{lattice}$, (universal)

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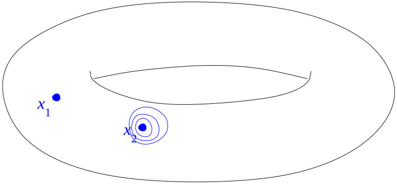
The Bergman kernel

Green function on the Riemann surface (intrinsic metrics):

$$\Delta_{x_1} G(x_1; x_2) = 2i\pi\, \delta(x_1, x_2)$$



The fundamental 2-form (= Bergman kernel) is defined as:

$$B(x_1, x_2) = d_{x_1}\, d_{x_2}\, G(x_1; x_2)$$

Properties: $B(x_1, x_2) = B(x_2, x_1)$, double pole at $x_1 = x_2$.
Depends only on the conformal structure = Universal.

3. Results: large N expansion and Geometry

The 2-point function

2-point function:

$$\rho_2(x_1,x_2) = \sum_{i=1}^{N}\sum_{j=1}^{N} \left\langle \delta(x_1-\lambda_i)\ \delta(x_2-\lambda_j) \right\rangle \quad - \rho(x_1)\rho(x_2)$$

Large N expansion : $\rho_2(x_1,x_2) = \rho_2^{(0)}(x_1,x_2) + N^{-2}\rho_2^{(1)}(x_1,x_2) + \dots$

Stieljes transform : $W_2^{(0)}(x_1,x_2) = \int\int \frac{\rho_2^{(0)}(x_1',x_2')\ dx_1'\ dx_2'}{(x_1-x_1')(x_2-x_2')}$

Theorem (many authors in 90s)

Leading order 2-point function

$$W_2^{(0)}(x_1,x_2)dx_1dx_2 + \frac{dx_1dx_2}{(x_1-x_2)^2} = B(x_1,x_2)$$

= universal (depends only on the conformal structure, not on the embedding (x,y) in $\mathbb{C}^2$)

Higher correlation functions:

n-point function:

$$\rho_n(x_1,x_2,\dots,x_n) = \sum_{i_1,\dots,i_n} \left\langle \delta(x_1-\lambda_{i_1})\ \delta(x_2-\lambda_{i_2})\dots\delta(x_n-\lambda_{i_n}) \right\rangle_{\text{conn.}}$$

Large N asymptotic expansion:

$$\rho_n(x_1,\dots,x_n) = N^{2-n}\,\rho_n^{(0)}(x_1,\dots,x_n) + N^{-n}\rho_n^{(1)}(x_1,\dots,x_n) + \dots$$

Stieljes transform:

$$W_n^{(g)}(x_1,\dots,x_n) = \int\dots\int \frac{\rho_n^{(g)}(x'_1,\dots,x'_n)\ dx'_1\dots dx'_n}{(x_1-x'_1)\dots(x_n-x'_n)}$$

Lemma

all $W_n^{(g)}$ have the same conformal structure.

Summary

$W_n^{(g)}$ $=g^{\text{th}}$ order term of the n point function

and by notation:

$F_g = W_0^{(g)}$ $=g^{\text{th}}$ order term of $\ln Z$

So far, we already know:

	$2-2g-n$
• $y = W_1^{(0)}(x)$ known = Riemann surface $\subset \mathbb{C}\times\mathbb{C}$	1
• $W_2^{(0)}(x_1,x_2) = B(x_1,x_2)$ known = Bergman kernel	0
• F_0 known = "prepotential" of the plane curve,	2
• F_1 known = $\frac{1}{4}\ln\det\Delta$,	0

• Question: $W_n^{(g)}$ for $2-2g-n<0$?

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• Question: $W_n^{(g)}$ for $2-2g-n<0$?

Higher correlation functions:

Summary

$W_n^{(g)}$ $=g^{\text{th}}$ order term of the n point function

and by notation:

$F_g = W_0^{(g)}$=g^{th} order term of $\ln Z$

So far, we already know:

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• $y = W_1^{(0)}(x)$ known = Riemann surface $\subset \mathbb{C}\times\mathbb{C}$	1
• $W_2^{(0)}(x_1,x_2) = B(x_1,x_2)$ known = Bergman kernel	0
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Recursion

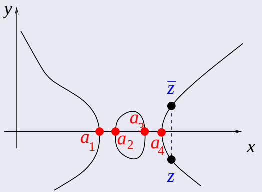
Theorem ([E., 2004])

Higher $W_n^{(g)}$ computed by the "topological recursion":

$$W_{n+1}^{(g)}(x_0,\overbrace{x_1,\dots,x_n}^{J}) = \sum_{\text{branch points } a_i} \operatorname*{Res}_{z\to a_i} K(x_0,z)\Big[W_{n+2}^{(g-1)}(z,\bar z,J) + \sum_{h=0}^{g}\sum_{I\subset J}^{\prime} W_{1+\#I}^{(h)}(z,I)\,W_{1+n-\#I}^{(g-h)}(\bar z,J\setminus I)\Big]$$

where $\bar z = z$ in the other sheet,
and K is the "recursion kernel":

$$K(x_0,z) = \frac{\int_{\bar z}^{z} W_2^{(0)}(x_0,z')\,dz'}{2\,(W_1^{(0)}(z) - W_1^{(0)}(\bar z))}$$



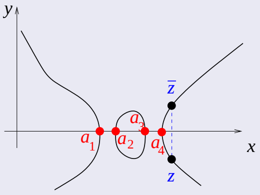
Theorem ([Chekhov, E., Orantin, 2005])

F_g, for $g \geq 2$ computed by:

$$F_g = W_0^{(g)} = \frac{1}{2-2g} \sum_{\text{branch points } a_i} \operatorname*{Res}_{z \to a_i} \Phi(z)\, W_1^{(g)}(z)$$

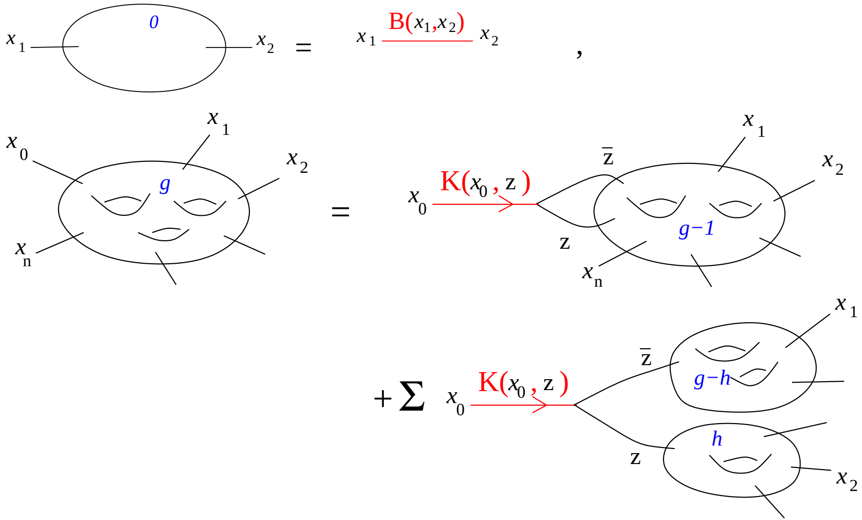
where

$$\Phi(z) = \int_*^z W_1^{(0)}(z')\, dz'$$



Reminder: $F_0 =$"area"(ext), and $F_1 = \frac{1}{2} \ln \det \Delta_{\text{ext}}$ already known.

Geometric representation



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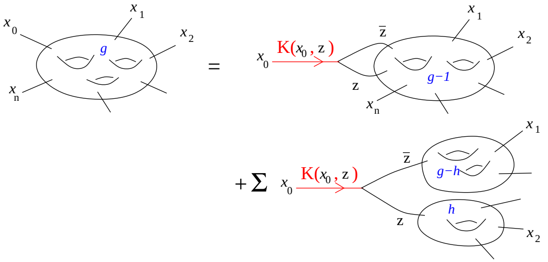
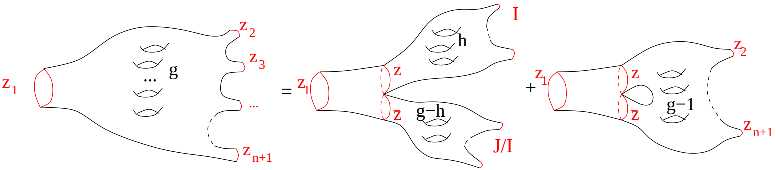
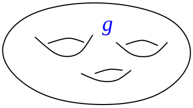


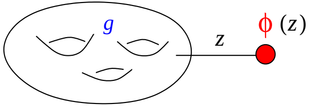


Geometric representation

Computation of F_g:



$$= \frac{1}{2-2g}$$



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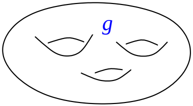

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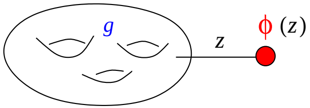
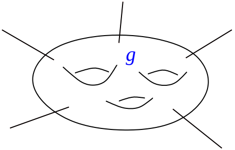
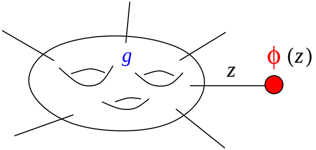
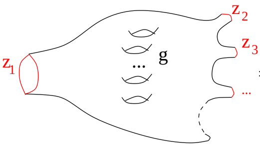


Notice that in general:



$$= \frac{1}{2-2g-n}$$





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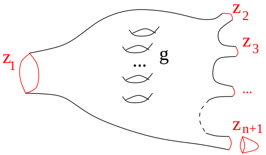
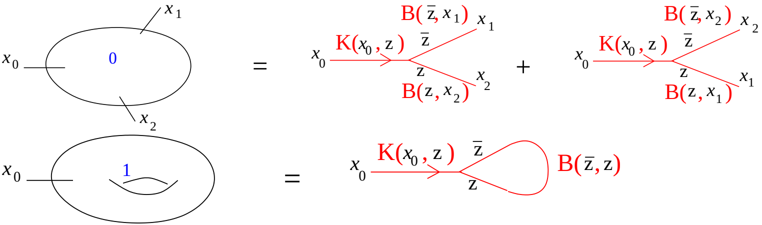


Examples:


x0
x1
0
x2
=
K(x0 , z)
z̄
z
B(z̄, x1)
x1
B(z, x2)
x2
+
K(x0 , z)
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x0
1
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K(x0 , z)
B(z̄, z)

Geometric representation

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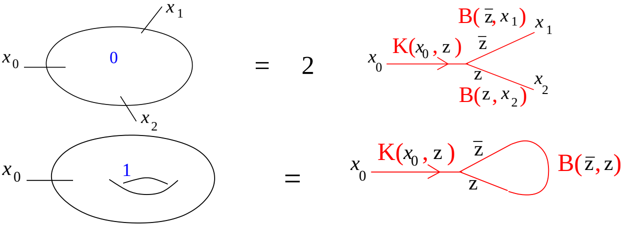

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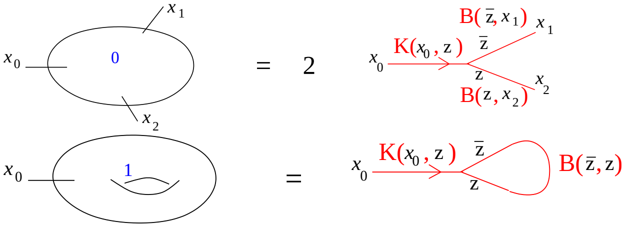

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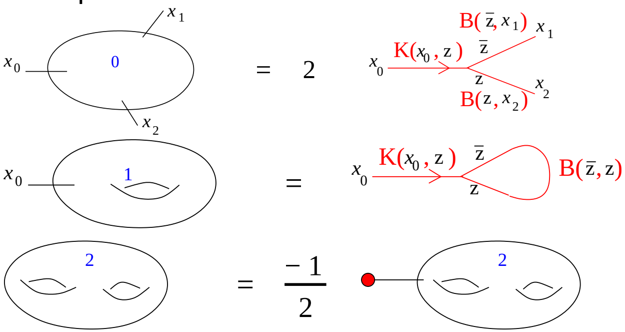

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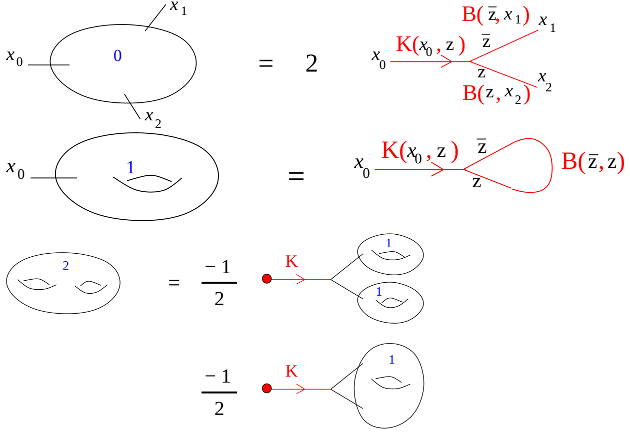

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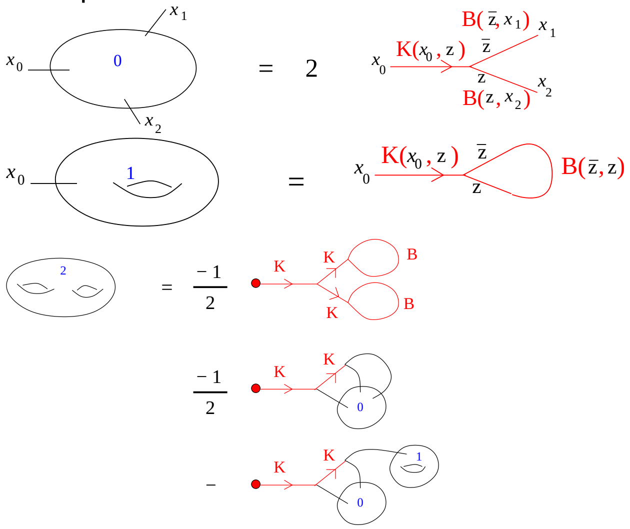

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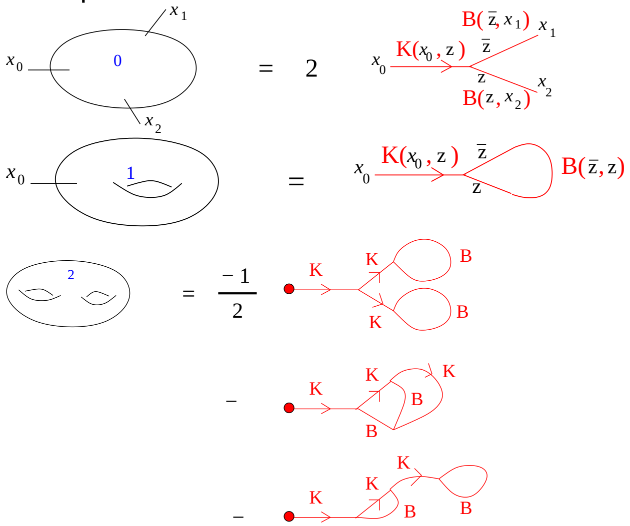

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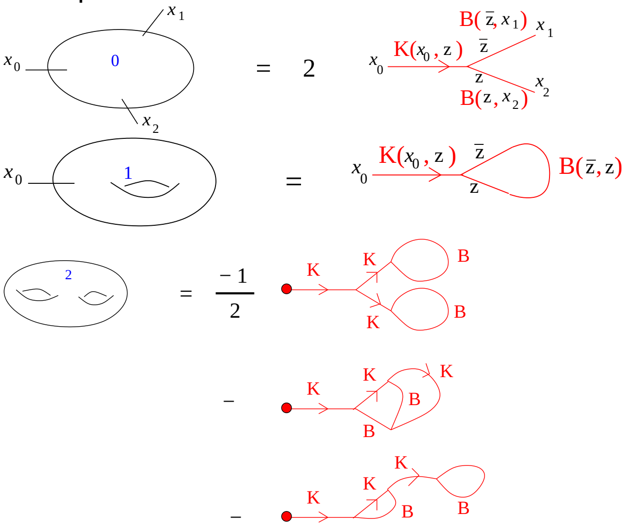

4. General properties of the recursion

Remark: we can apply this "topological recursion" algorithm to any plane curve $y = W_1^{(0)}(x)$ (spectral curve), (*related to a matrix model or not*).

This defines some F_g and $W_n^{(g)}$ for any plane curve:

Definition

$F_g =$"Symplectic Invariants" of a plane curve.

Properties

General properties (valid for any embedded Riemann surface):

- F_g = symplectic invariant,
- F_g = (almost) modular form,
- Integrability: $Z_N = \exp\left(\sum N^{2-2g} F_g\right)(1 + \mathrm{Non.\,Pert.}) =$ Tau-function
- Limits: F_g commute with limits: $\lim F_g(\mathcal{S})" = "F_g(\lim \mathcal{S})$.
This allows to study microscopic critical scaling regimes with the same method.
Ex: easily recover Tracy-Widom universal law near boundaries ($y \sim \sqrt{x}$).
Ex: recover KdV $(p, 2)$ reductions near critical points of order p (i.e. $y \sim x^{p/2}$), i.e. Painlevé I hierarchy.
- Many other nice properties, like special geometry deformations (form-cycle duality), Virasoro or W algebra, ...etc.

Properties

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• $F_g =$ symplectic invariant,
Theorem: if two spectral curves (x, y) and $(\tilde{x}, \tilde{y})$ are such that $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$, then $F_g = \tilde{F}_g$.

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$F_g + \text{polynomial}((\mathrm{Im}\,\tau)^{-1})$ is modular invariant, but not analytical.
Satisfies BCOV holomorphic anomaly equation.
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Z_N satisfies formal Hirota equations. Correlators $W_n^{(g)}$ are obtained as determinants of some integrable kernel.

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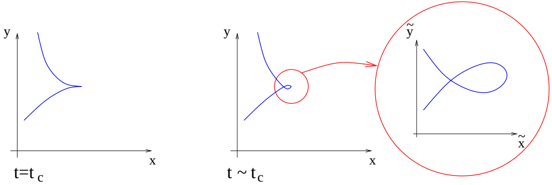
$\lim_{t\to t_c} (t-t_c)^{(2-2g)(\mu_x+\mu_y)} F_g(\mathcal{S}_t) = F_g(\tilde{\mathcal{S}}),\ \ \tilde{\mathcal{S}} =$ resolved curve.



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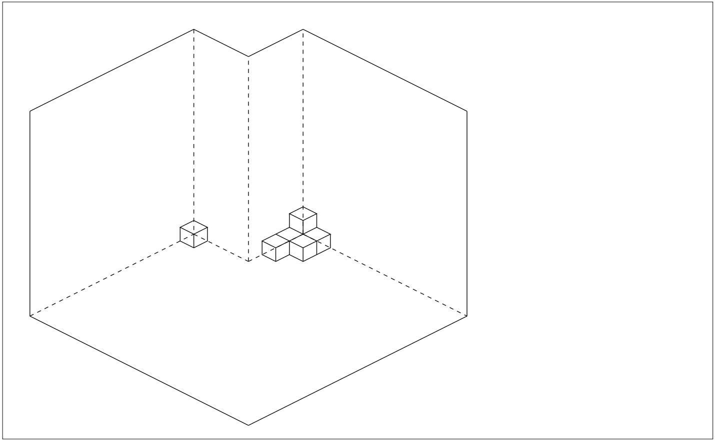
5. Some applications
Beyond Random Matrix Theory

Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g} \mathcal{F}_g.$



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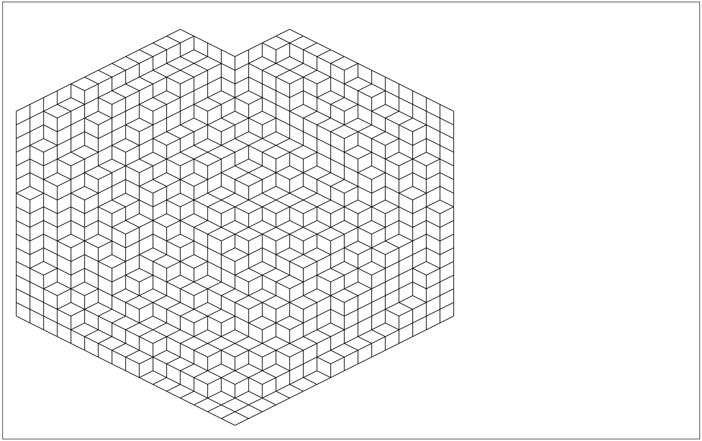

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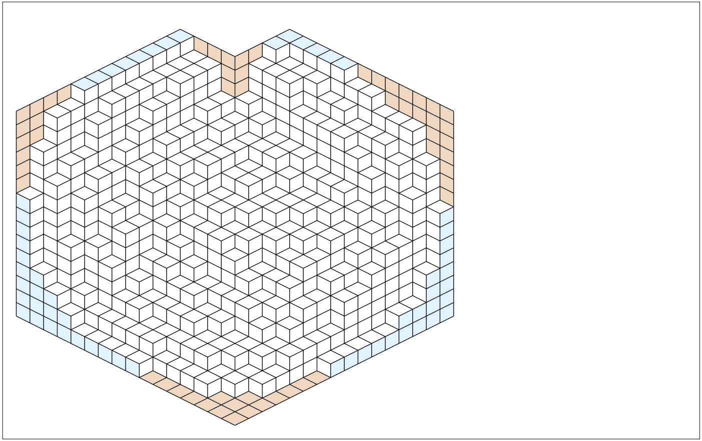

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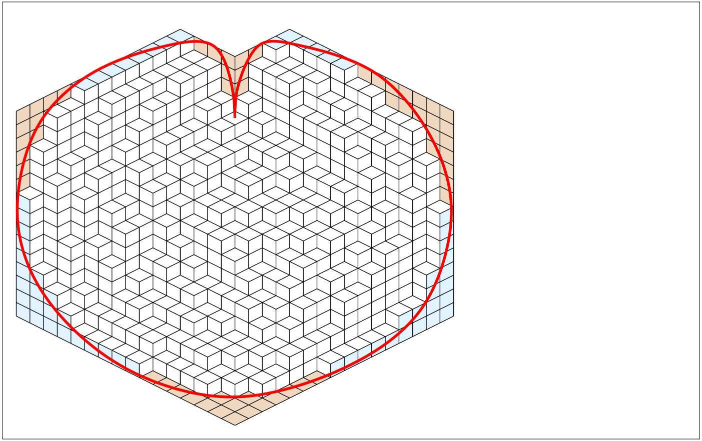

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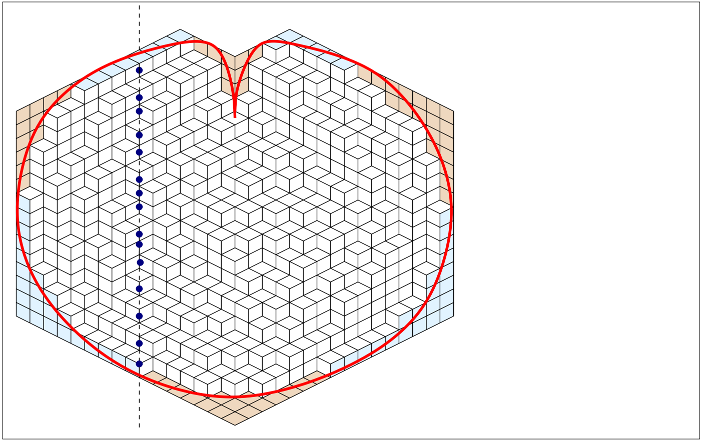

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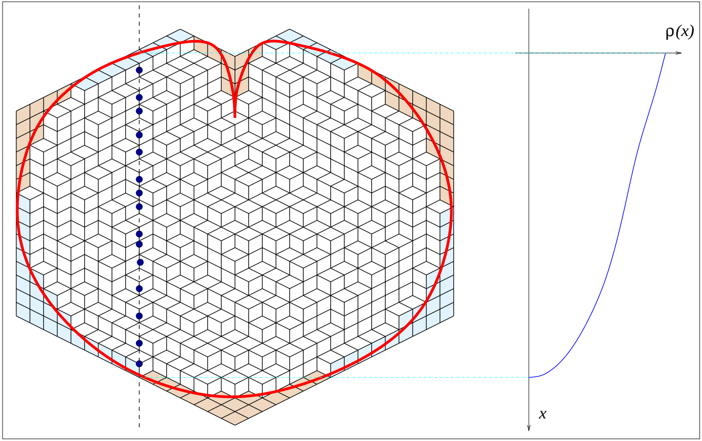

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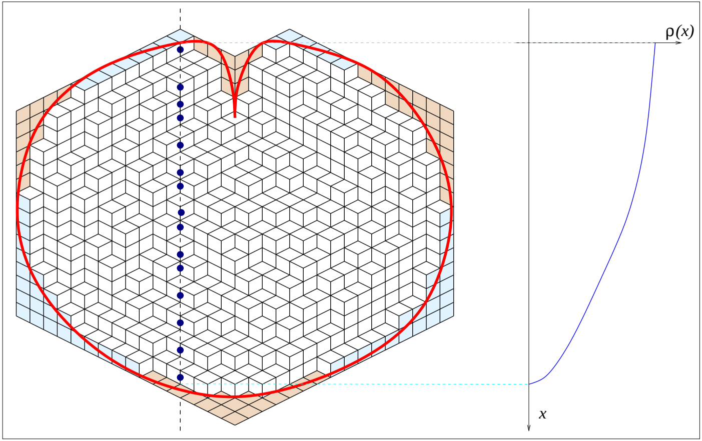

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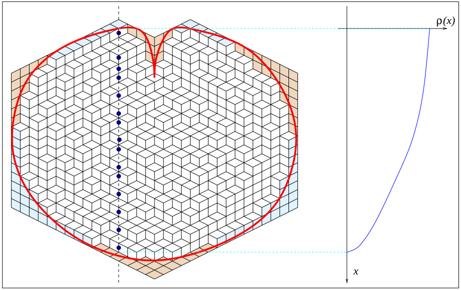

Conjecture:
$\mathcal{F}_g = F_g$(Stieljes transf. of limit density along a vertical line)
Idea of a proof: Z=matrix integral, which implies that it satisfies the topological recursion. Problem: show that
$W_1^{(0)}$ =Kenyon-Okounkov-Sheffield curve (limit shape) ?

Topological strings - Gromov-Witten

- Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry

- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "# of conformal mappings of a Riemann surface of genus g into $\mathfrak{X}$, with homology class d, and passing through given points".

- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.

- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus g in target space $\mathfrak{X}$.

- Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\, \mathfrak{X})$$

Few cases proved so far:
many low genus examples,
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- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "# of conformal mappings of a Riemann surface of genus g into $\mathfrak{X}$, with homology class d, and passing through given points".
- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.
- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus g in target space $\mathfrak{X}$.
- Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Few cases proved so far:
many low genus examples,
and to all genus for $\mathfrak{X} = \mathbb{C}^3$.

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6. Conclusion and prospects

Conclusion

• Topological recursion=explicit algorithm to compute the partition function to all orders $\ln Z = \sum N^{2-2g} F_g$, and n point function to all orders $W_n = \sum N^{2-2g-n} W_n^{(g)}$. Can be extended to compute the non-perturbative/oscillating part as well.

• The **same** recursion applies to many matrix models: 2-matrix model, matrix model with external field, chain of matrices, $O(n)$ model,... and can be generalized to non-hermitian matrices (β-ensembles).

• It leads to the definition of symplectic invariants, which are very interesting geometric objects.

• Beyond matrix models: it was found recently that the **same** recursion is also satisfied by topological strings (Gromov-Witten invariants), by random 2D and 3D partitions, ...etc.