TRANSPORT MOMENTS

beyond the leading order

arXiv:1012.3526

Jack Kuipers

University of Regensburg

with

Gregory Berkolaiko

Texas A&M

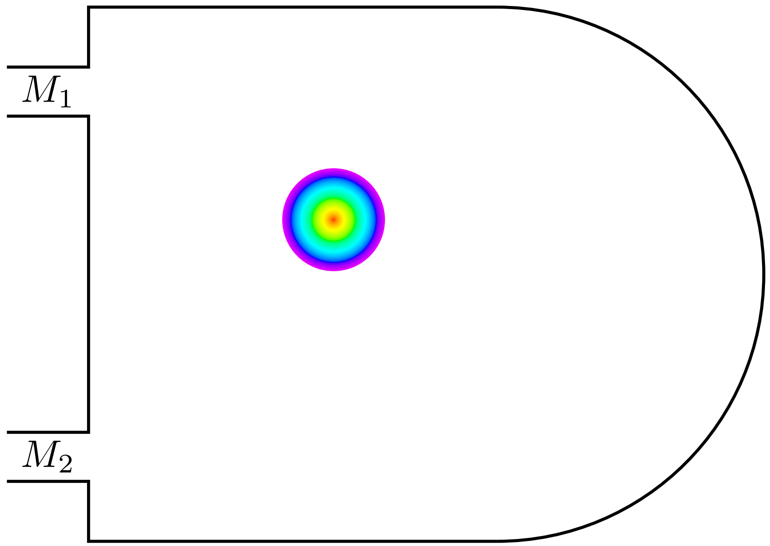
OUTLINE

- Open systems
  - moments
  - correlation functions
  - semiclassical expression
- Leading order
  - first moments
  - tree recursions
- Subleading orders
  - first correction
  - second correction
- Andreev billiards
- Nonlinear statistics

Open systems

- Chaotic cavity with $M_1 + M_2 = M$ open channels
- Scattering matrix $S(E)$

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$



- 'Transmission' matrix

$$T = t^\dagger t$$

Eigenvalues are transmission probabilities,
moments: $\tilde{m}_n = \mathrm{Tr}\left[T^n\right]$

- Wigner-Smith matrix

$$N = \frac{\hbar}{\mathrm{i}} S^\dagger(E) \frac{\mathrm{d}S(E)}{\mathrm{d}E}$$

Eigenvalues are delay times,
moments: $m_n = \frac{1}{M}\mathrm{Tr}\left[N^n\right]$

ENERGY DEPENDENT CORRELATION FUNCTIONS

- Define following correlation functions

$$C(\epsilon, n) = \frac{1}{M}\mathrm{Tr}\left[S^\dagger\left(E - \frac{\epsilon\hbar\mu}{2}\right)S\left(E + \frac{\epsilon\hbar\mu}{2}\right)\right]^n$$

μ is the classical escape rate

- Related to time delay, for example for n=1

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}C(\epsilon, 1)\bigg|_{\epsilon=0} = \frac{\hbar\mu}{2M}\mathrm{Tr}\left[S^\dagger(E)\frac{\mathrm{d}S(E)}{\mathrm{d}E} - \frac{\mathrm{d}S^\dagger(E)}{\mathrm{d}E}S(E)\right]$$

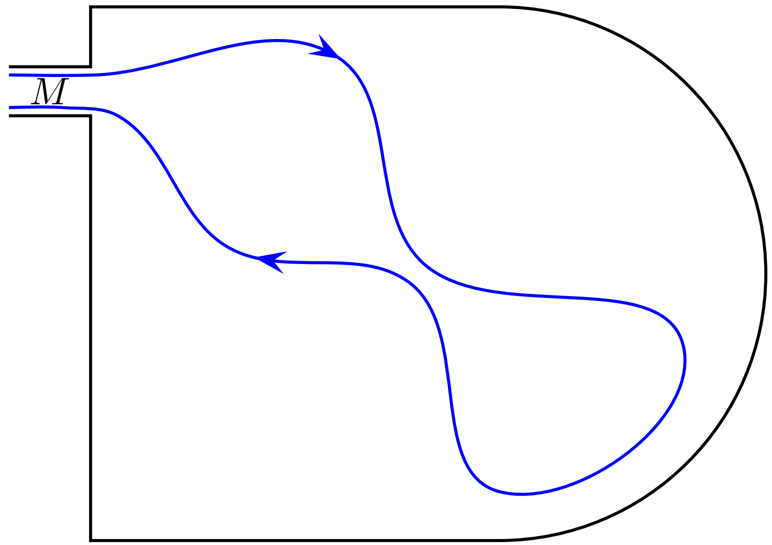
- Using the unitarity of the scattering matrix

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}C(\epsilon, 1)\bigg|_{\epsilon=0} = \frac{\mathrm{i}\mu}{M}\mathrm{Tr}\left[N\right] = (\mathrm{i}\mu)m_1$$

Correlation functions
Semiclassical expression

- Correlation functions

$$C(\epsilon, n) = \frac{1}{M} \mathrm{Tr} \left[S_-^\dagger S_+ \right]^n$$



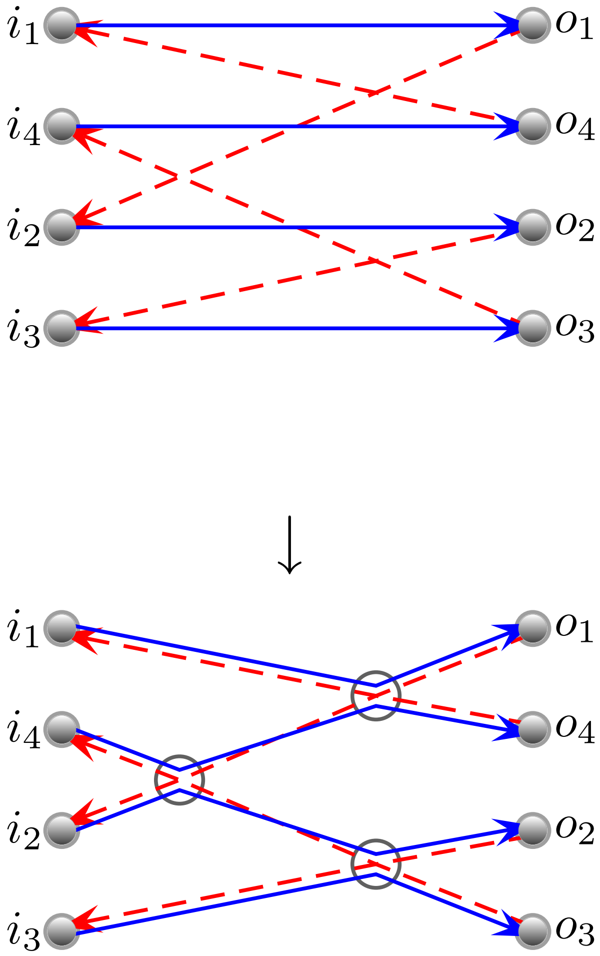
- Scattering matrix

$$S_{oi}(E) \approx \sqrt{\frac{\mu}{M}} \sum_{\zeta(i \to o)} A_\zeta \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_\zeta}.$$

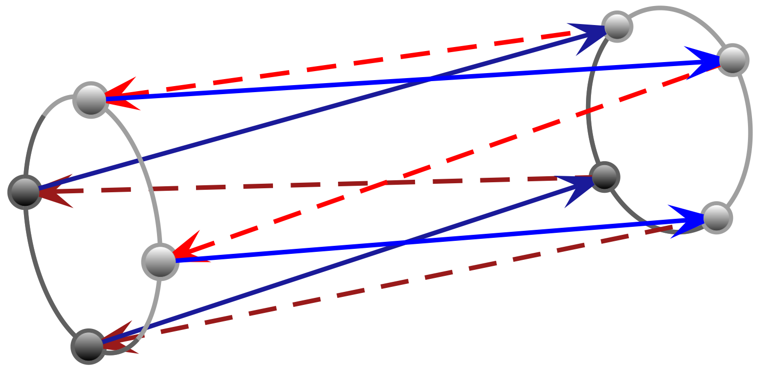
given semiclassically in terms of classical trajectories ζ connecting the corresponding channels

- Action S_ζ, amplitude A_ζ
- μ is the classical escape rate
- M open channels

Correlation functions
Semiclassical expression



- Correlation function $C(\epsilon, n)$ involves $2n$ trajectories
  - n from $i_j \to o_j$
  - n from $i_{j+1} \to o_j$



- Need total action to be small
  - Collapse trajectories onto each other
  - Links connected by encounters

LEADING ORDER $C(\epsilon, 1)$

- For $n = 1$, diagonal term

$$C(\epsilon, 1) = \frac{\mu}{M^2} \sum_{i,o} \sum_{\zeta(i\to o)} |A_\zeta|^2 \mathrm{e}^{\mathrm{i}\epsilon\mu T_\zeta}$$

- Open sum rule Richter and Sieber, PRL 2002

$$\sum_{\zeta(i\to o)} |A_\zeta|^2 \ldots \approx \int_0^\infty \mathrm{d}T \, \mathrm{e}^{-\mu T} \ldots$$

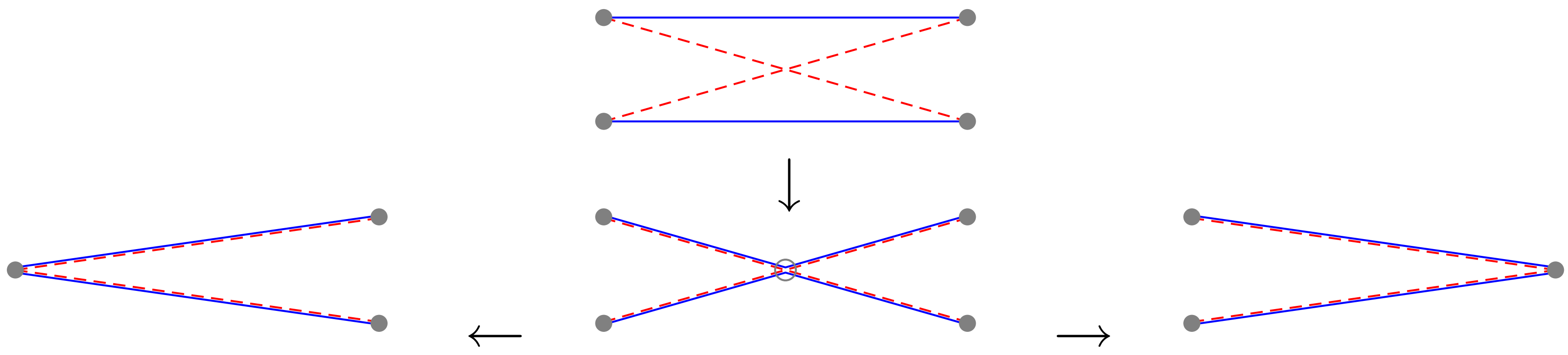
- Channel sum gives factor M^2

$$C(\epsilon, 1) = \frac{1}{(1-a)}, \qquad a = \mathrm{i}\epsilon$$

In fact we obtain this factor for each link in a structure

Leading order
$C(\epsilon, 2)$

- For $n = 2$, can have an encounter

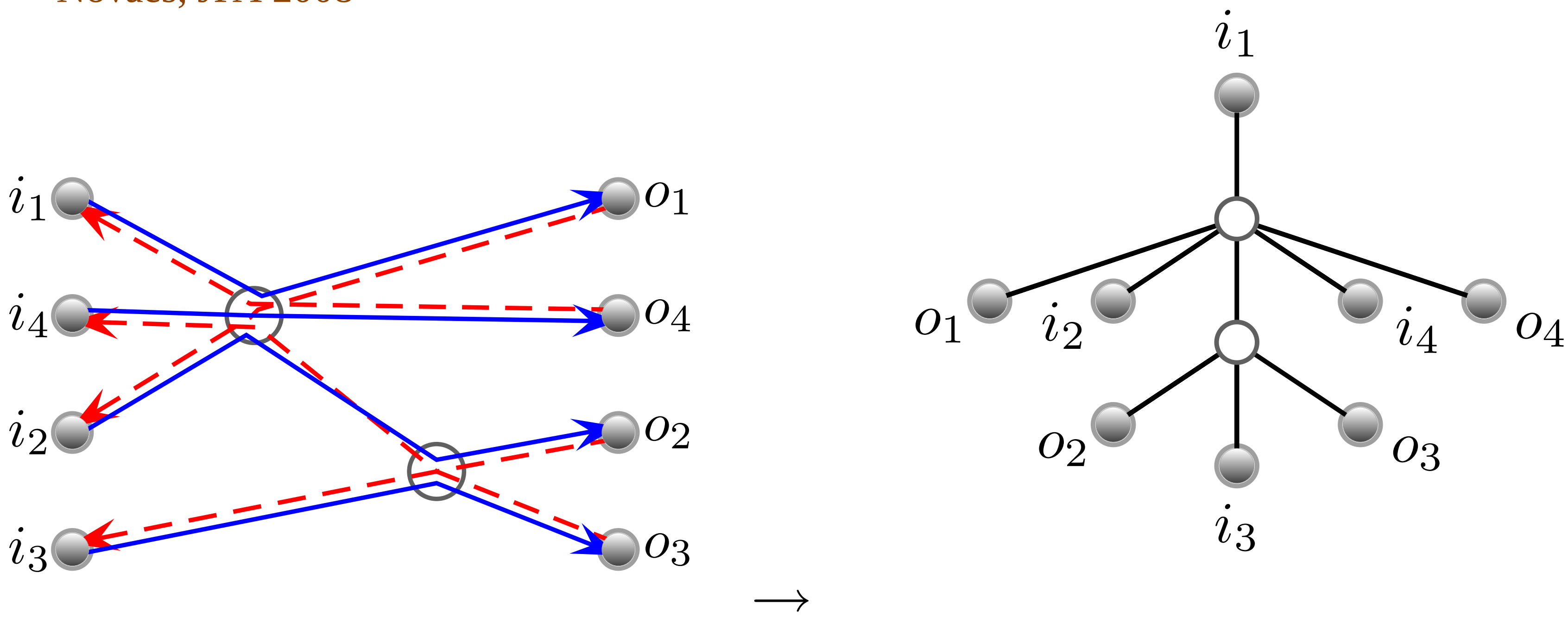


- Can also move encounter into the leads
- For each structure cf Heusler et al, PRL 2006
  - each link gives factor $(1-a)^{-1}$
  - each l-encounter gives factor $-(1-la)$

$$C(\epsilon, 2) = \frac{1-2a+2a^2}{(1-a)^4}$$

LEADING ORDER TREES

- Structures are related to planar rooted trees Berkolaiko, Harrison and Novaes, JPA 2008

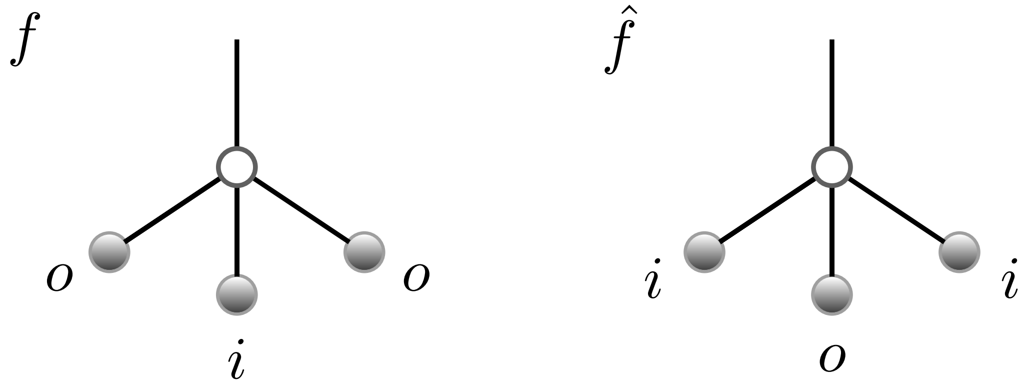


- l-encounter can enter the lead if attached directly to l i or o channels

Leading order

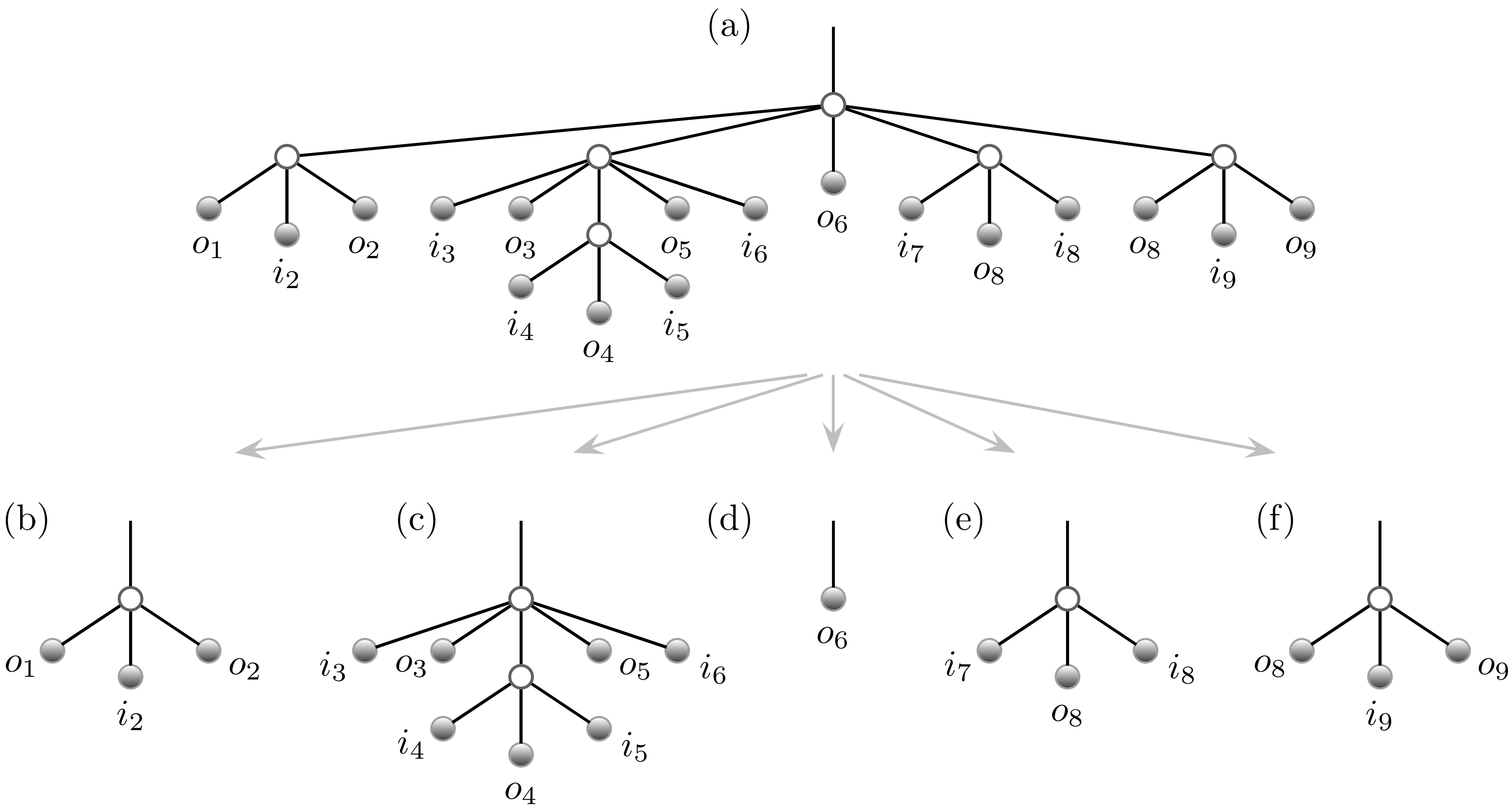
Subtree recursions

- Define generating function $f(\boldsymbol{x}, \boldsymbol{z}_i, \boldsymbol{z}_o)$ where the powers of
  - x_l enumerate the number of l-encounters,
  - $z_{i,l}$ enumerate the number of l-encounters that i-touch the lead,
  - $z_{o,l}$ enumerate the number of l-encounters that o-touch the lead.
- and generating function $\hat{f}(\boldsymbol{x}, \boldsymbol{z}_o, \boldsymbol{z}_i) = f(\boldsymbol{x}, \boldsymbol{z}_i, \boldsymbol{z}_o)$



Semiclassical evaluation

Subtree recursions



$$f = yc_o + y\sum_{l=2}^{\infty}\left[x_l f^l \hat{f}^{l-1} + z_{o,l}\hat{f}^{l-1}\right],$$

c : channel

y : link

SEMICLASSICAL EVALUATION
GENERATING FUNCTION

- Semiclassical contribution

$$\frac{1}{(1-a)^n}\prod_{\alpha=1}^{V}\frac{-(1-l_\alpha a)}{(1-a)^{l_\alpha}}$$

- Generating function

$$f-\frac{r}{1-a}=\frac{f^2}{1-a}\left[f-r-ra\right]$$

- Correlation functions

$$F(\epsilon,r)=\sum_{n=1}r^{2n}C(\epsilon,n)=\frac{rf}{1-rf}$$

- Set

$$\begin{aligned} y &= (1-a)^{-1} \\ x_l &= -(1-la) \\ z_{i,l} &= z_{o,l}=r^l \\ c_i &= c_o=r \end{aligned}$$

- Power of r counts
  # of channels

SEMICLASSICAL EVALUATION
MOMENTS OF THE DELAY TIMES

- Define related correlation functions

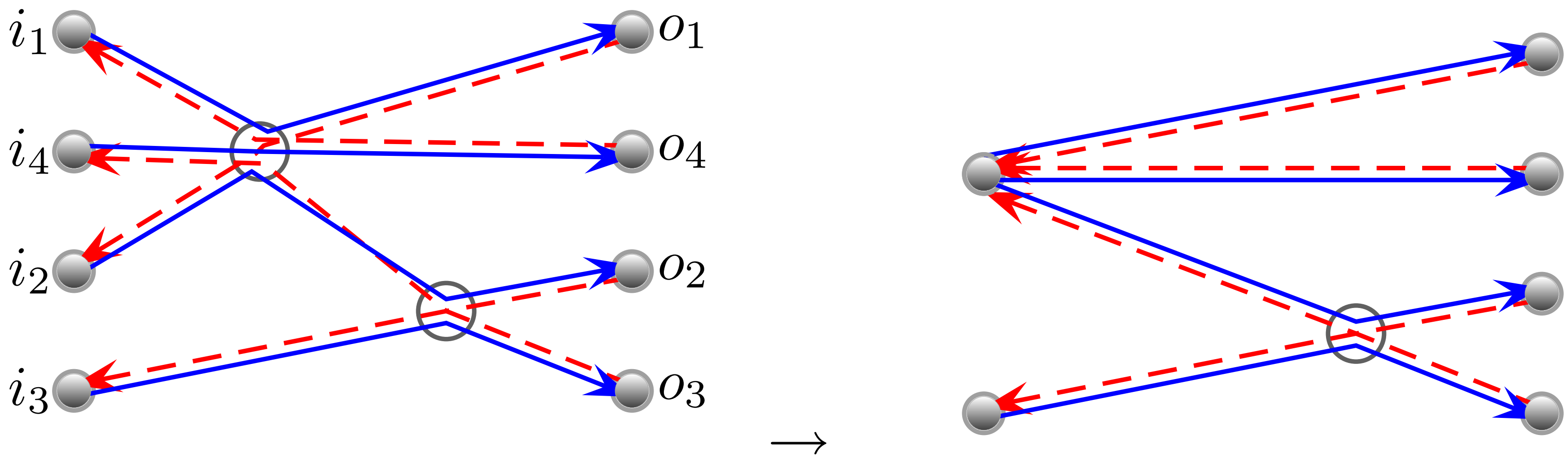
$$D(\epsilon, n) = \frac{1}{M}\mathrm{Tr}\left[S^\dagger\left(E - \frac{\epsilon\hbar\mu}{2}\right)S\left(E + \frac{\epsilon\hbar\mu}{2}\right) - I\right]^n$$

- So the moments are

$$m_n = \frac{1}{(\mathrm{i}\mu)^n n!}\frac{\mathrm{d}^n}{\mathrm{d}\epsilon^n}D(\epsilon, n)\bigg|_{\epsilon=0}$$

- Treat I as a diagonal pair
  - without energy difference

Semiclassical evaluation
Moments of the delay times



- New diagonal links:

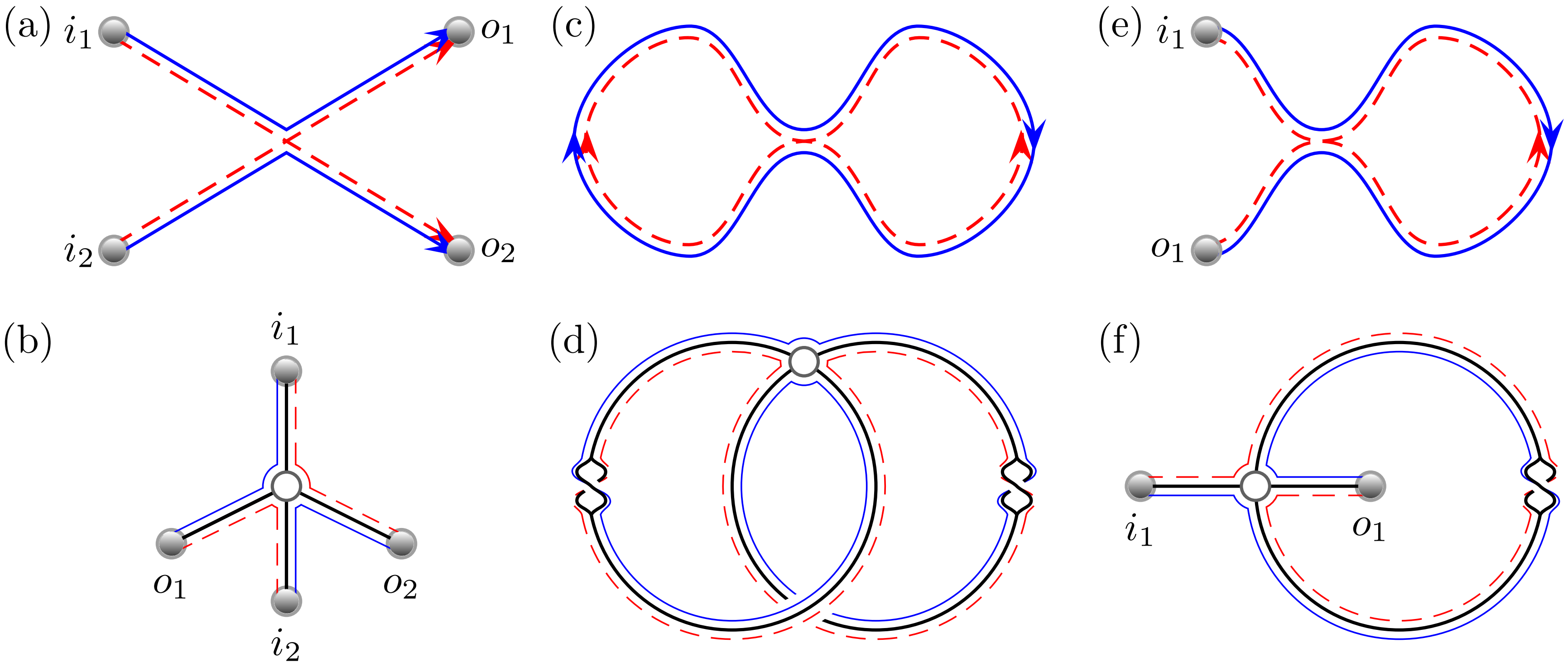
$$\frac{1}{1-a} \rightarrow \frac{1}{1-a} - 1 = \frac{a}{1-a}$$

- Obtain all moments JPA 2010
  - cf RMT result Brouwer et al, Waves in Random Media 1999

$$F(s) = \sum_{n=1} m_n \mu^n s^n, \qquad F_0(s) = \frac{1 - s - \sqrt{1 - 6s + s^2}}{2}$$

Subleading order

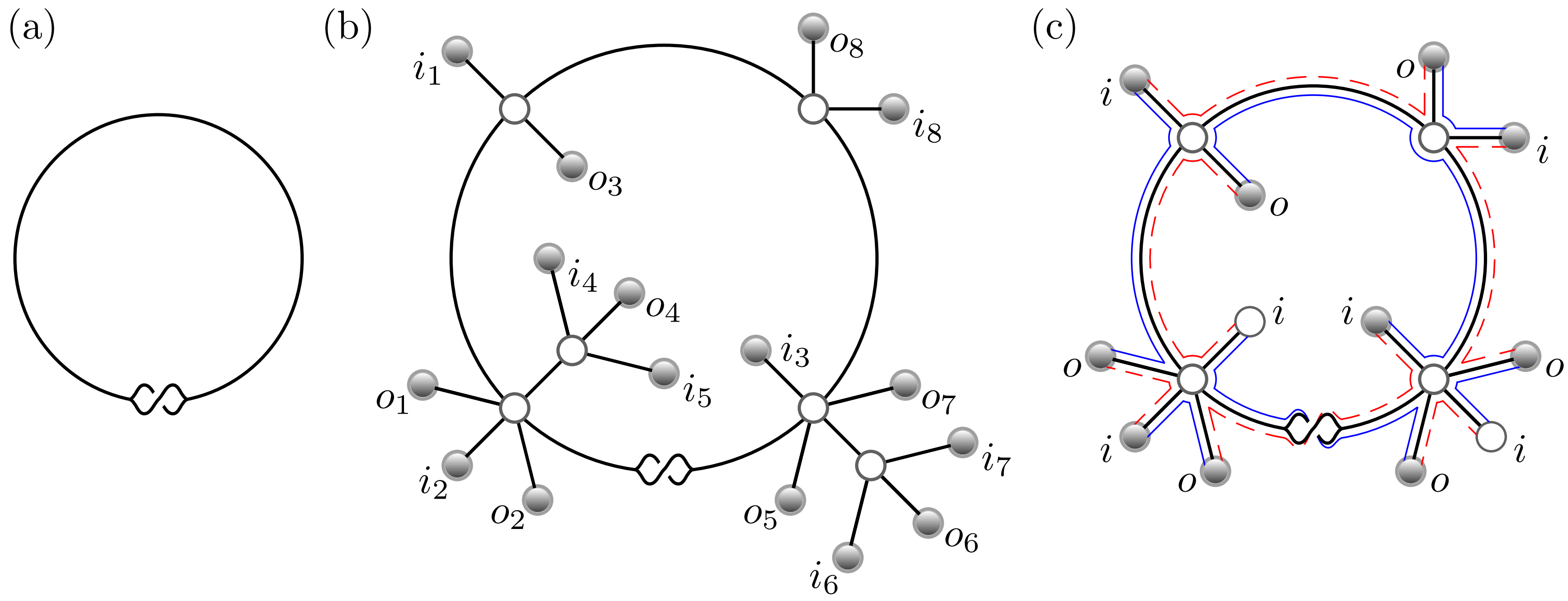
Boundary walks



- Turn semiclassical diagrams into graphs
- Boundary walks recreate semiclassical diagrams

SUBLEADING ORDER

GRAPHS



- Möbius strip 'dressed' with encounter nodes
- Odd # of 'odd' nodes
- Odd nodes may touch lead if i and o channels in same lead

SUBLEADING ORDER
GENERATING FUNCTION

- l-encounter node has $(l-2)$ subtrees attached
- Even node

$$A(p) = \sum_{l=2}^{\infty} l x_l (f\hat{f})^{l-1}$$

- Odd node

$$A'(p) = p\sum_{l=2}^{\infty} (l-1) x_l (f\hat{f})^{l-1}$$

- If can touch lead

$$B(p) = p\sum_{l=2}^{\infty} z_l \frac{f^{l-1} - \hat{f}^{l-1}}{f - \hat{f}}$$

- Power of p counts # of odd nodes

- Arbitrary number of nodes

$$\tilde{K}_1 = \frac{1}{2}\sum_{k=1}^{\infty} \frac{[y(A + A' + B)]^k}{k} = -\frac{1}{2}\ln[1 - y(A + B)]$$

SUBLEADING ORDER
GENERATING FUNCTION

- Odd # of odd nodes, F allows any channel to be the first

$$K_1 = \frac{\tilde{K}_1(p=1) - \tilde{K}_1(p=-1)}{2}, \qquad F = r\frac{\mathrm{d}K}{\mathrm{d}r}$$

- Transmission eigenvalues:

$$F(s) = \sum_{n=1} \tilde{m}_n s^n, \qquad F_1(s) = -\frac{\xi s}{(1-s)(1-s+4\xi s)}$$

with $\xi = \frac{M_1 M_2}{M^2}$, cf Brouwer and Beenakker, J Math Phys 1996

- Delay times:

$$F(s) = \sum_{n=1} m_n \mu^n s^n, \qquad MF_1(s) = \frac{1-3s-\sqrt{1-6s+s^2}}{2(1-6s+s^2)}$$

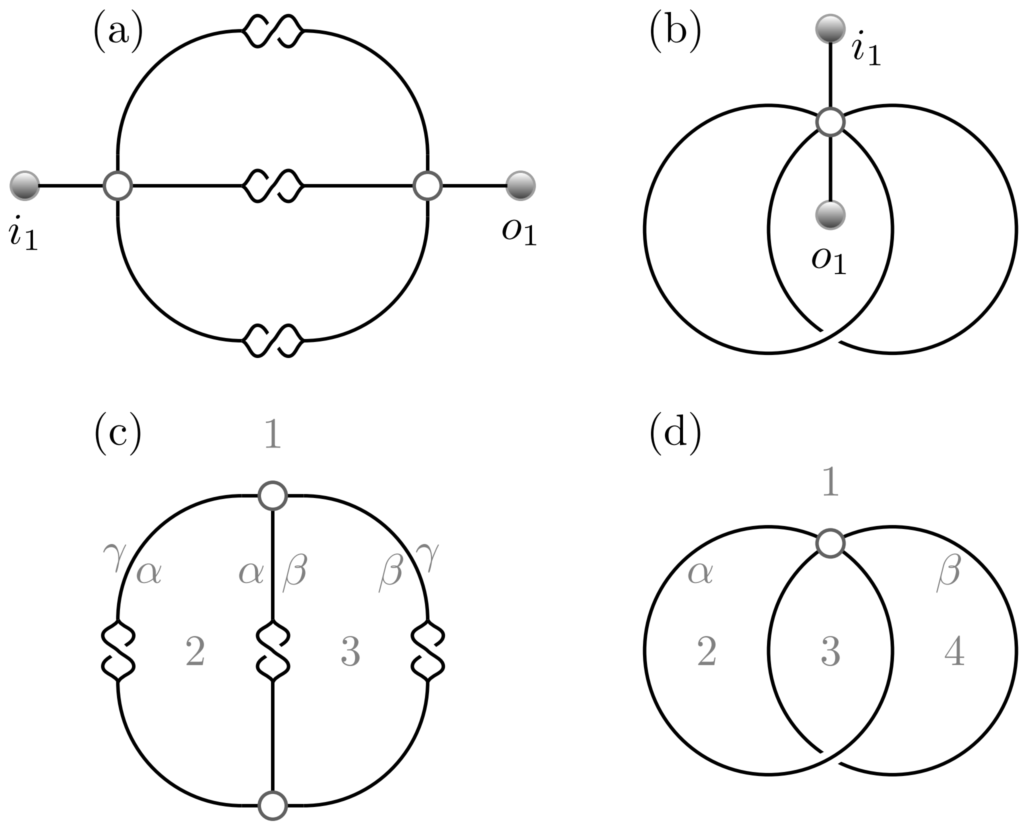
Next order
Unitary case

- Without time reversal symmetry
- Transmission eigenvalues:

$$-\frac{\xi^2 s^2}{(1-s)^{\frac{3}{2}}(1-s+4\xi s)^{\frac{5}{2}}}$$

- Delay times:

$$\frac{2s^2}{(1-6s+s^2)^{\frac{5}{2}}}$$

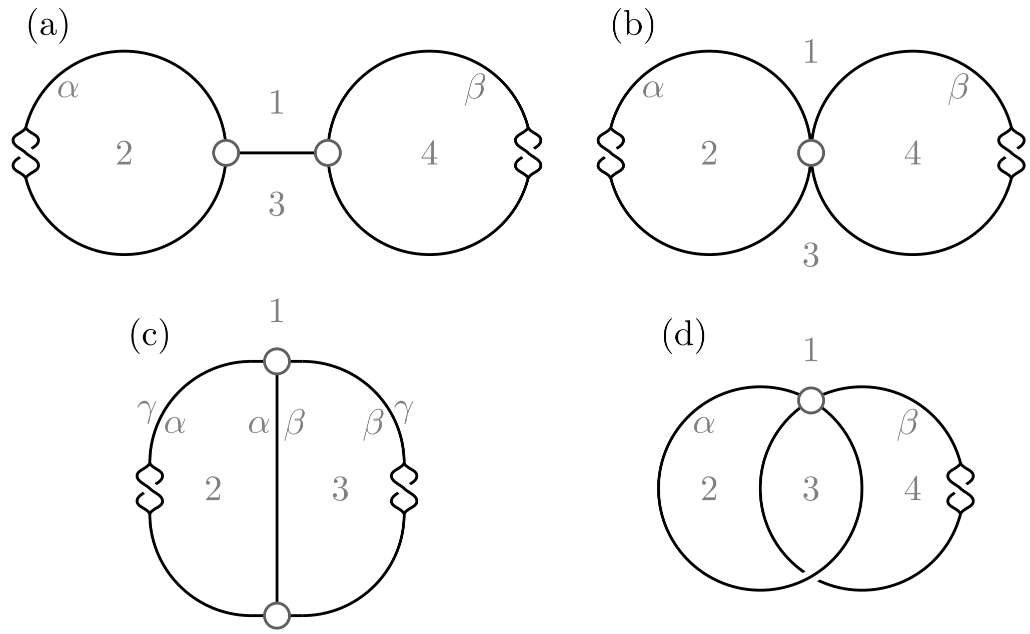


Next order

Orthogonal case

- With time reversal symmetry
- Transmission eigenvalues:

$$\frac{\xi s\left[\xi s(4s-3)+1-s^2\right]}{(1-s)^{\frac{3}{2}}(1-s+4\xi s)^{\frac{5}{2}}}$$

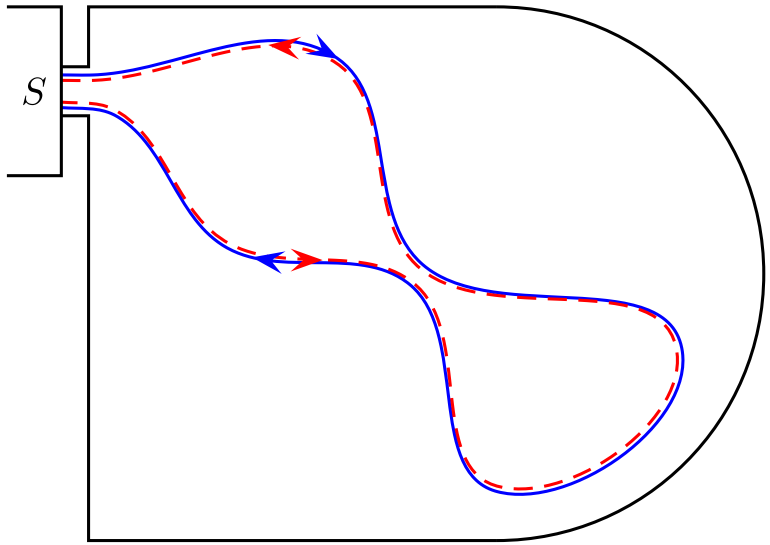


- Delay times:

$$\frac{s(s-3)}{(1-6s+s^2)^2}+\frac{3s(s-1)^2+2s^2}{(1-6s+s^2)^{\frac{5}{2}}}$$

ANDREEV BILLIARDS

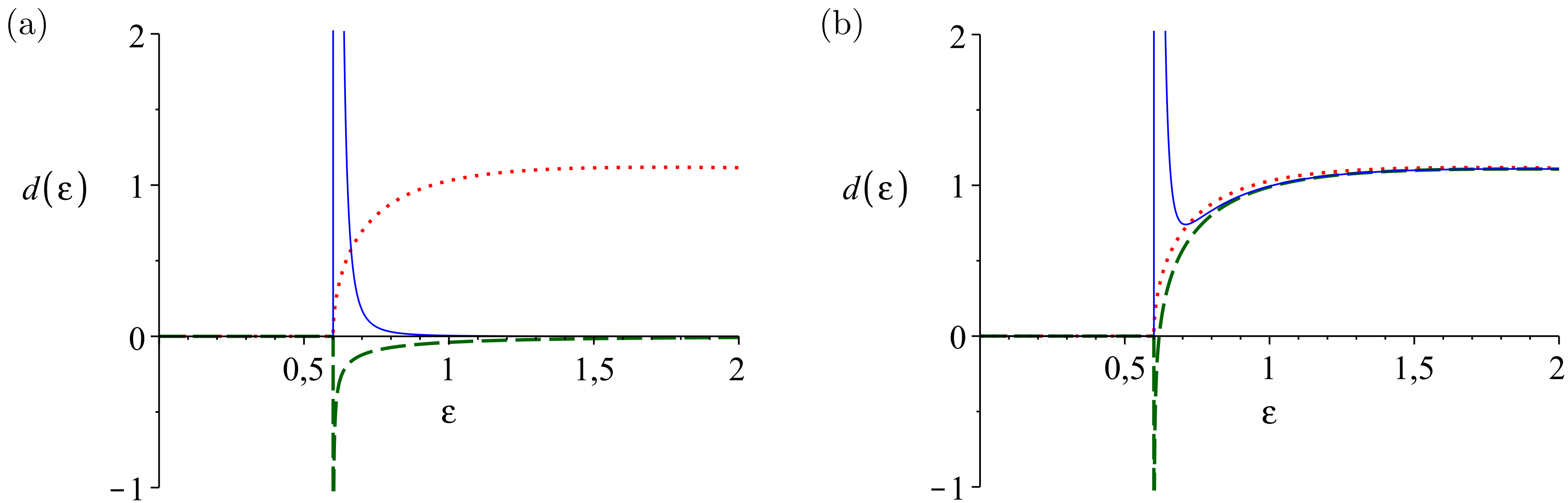
- Replace lead by superconducting block
  - electrons reflected back as holes
- Density of states Ihra et al, EPJB 2001
  - renormalised by average density



$$d(\epsilon) = 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Im}\frac{\partial C(\epsilon,n)}{\partial\epsilon}$$

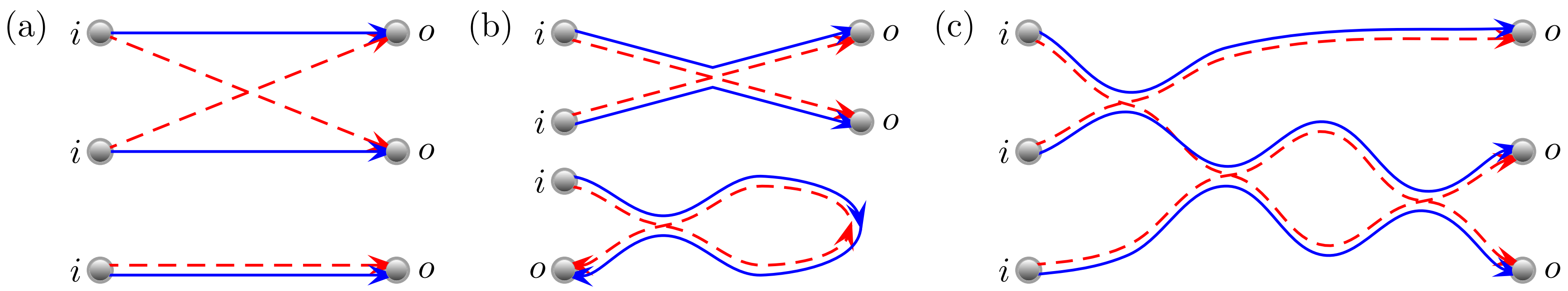
expanded in terms of the correlation functions

ANDREEV BILLIARDS
DENSITY OF STATES

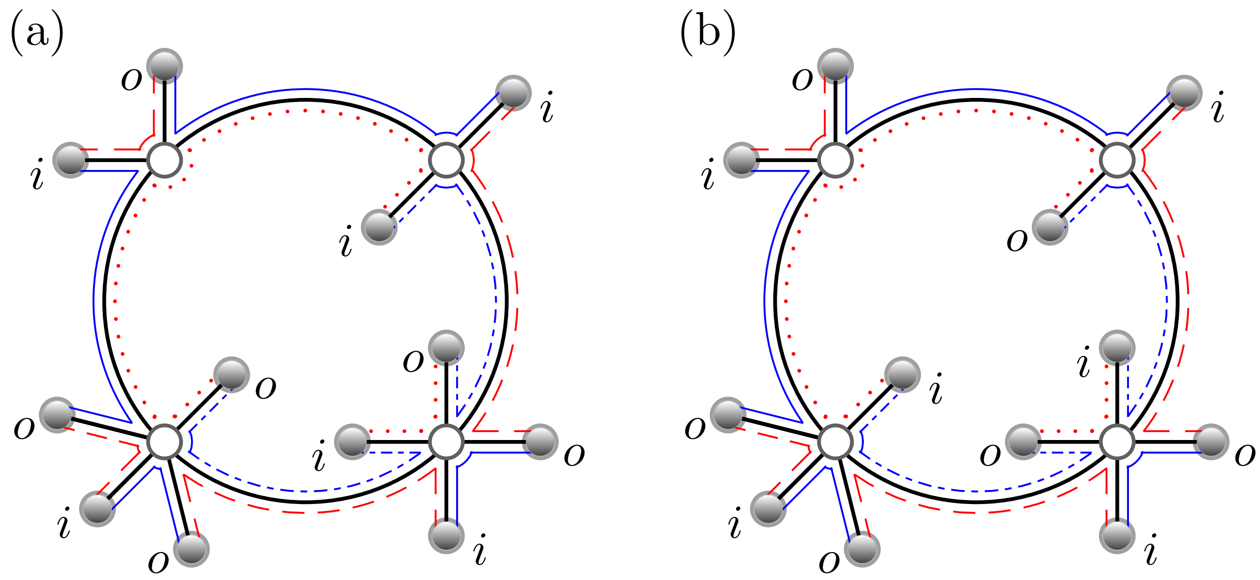


- Gap in the density of states persists
- Expansion is poorly convergent
- Leading order is RMT result Melsen et al, EPL 1996

NONLINEAR STATISTICS
TWO TRACES



- Leading order base diagram is a ring



CONCLUSIONS

- Leading order, we obtained semiclassically
  - moments of the delay times JPA (2010) **43** 035101
  - density of states of Andreev billiards PRL (2010) **104** 027001; arXiv:1004.1327
- Next two corrections, we can obtain arXiv:1012.3526
  - transmission eigenvalues
  - delay times
  - Andreev billiards
  - nonlinear statistics
- Can we go to all orders?