The cavity approach to the localization transition in random matrices

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Outline

Part I

Introduction

Examples of localization

- Anderson transition:

  ⇒ particle can hop between sites with random energies $E_1, \dots, E_N$;

  ⇒ strong disorder $\rightarrow$ localization of $\psi_i(t)$.

- Gene coexpression networks:

  ⇒ nodes $\rightarrow$ genes; connectivity matrix $\rightarrow$ correlation between genes;

  ⇒ localized eigenvectors $\rightarrow$ important or influential genes of the network.

  S Jalan *et al*, PRE **81**, 2010.

- Diffusion on random graphs

  J-Y Fortin, JPA **38**, 2005.

- Dynamics of glasses $\rightarrow$ spherical model

  G Semerjian and LF Cugliandolo, EL **61**, 2003.

The inverse participation ratio (IPR)

- Ensemble of $N \times N$ real symmetric matrices:

$$\boldsymbol{J}\boldsymbol{\psi}_\mu = \lambda_\mu \boldsymbol{\psi}_\mu \ , \ \ \mu = 1, \ldots, N$$

$$\lambda_\mu \in \mathbb{R} \, , \quad \sum_{i=1}^{N} |\psi_\mu^i|^2 = 1 \, .$$

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- The IPR corresponding to a given state μ: $Y_{\mu}^{N} = \sum_{i=1}^{N} |\psi_{\mu}^{i}|^{4} \, .$

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- If $|\psi_\mu^i|^2 = |\psi_\mu|^2 \neq 0$ for d sites and zero otherwise:

$$|\psi_\mu|^2 = O(1/d) \Rightarrow Y_\mu^N = O(1/d) \, .$$

  - Extended states: $d = O(N) \Rightarrow \lim_{N \to \infty} Y_\mu^N = 0$.
  - Localized states: d is finite $\Rightarrow \lim_{N \to \infty} Y_\mu^N = O(1/d)$.

- IPR $\Rightarrow$ distinction between localized and extended states.

Random matrix parameters

- The average density of states (DOS):

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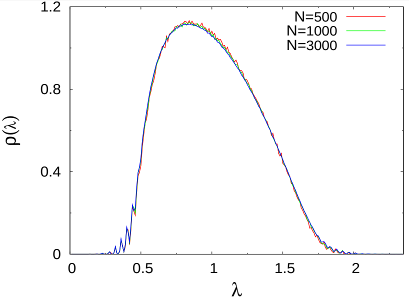
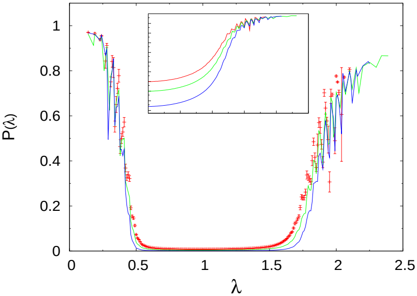
- The average inverse participation ratio:

$$P(\lambda) = \frac{1}{N\rho(\lambda)}\left\langle \sum_{\mu=1}^{N} Y_\mu^N \delta(\lambda - \lambda_\mu) \right\rangle_{\boldsymbol{J}} .$$

Diagonalization results





- Laplacian matrix of a sparse graph:

  $J_{ij} = c_{ij} K_{ij} - \delta_{ij} \sum_{k=1}^{N} c_{ik} K_{ik}$,

  $p_c(c_{ij}) = (1 - \frac{c}{N})\delta_{c_{ij},0} + \frac{c}{N}\delta_{c_{ij},1}$,

  $K_{ij} = -1/c$.

  $\Rightarrow 500$ matrices with $c = 20$.

- Eigenvectors undergo a localization transition.
- Tails of the spectrum $\rightarrow$ fluctuations.
- Localization threshold $\rightarrow$ one needs large values of N.
- Diagonalization is computationally expensive.

Aim of this work

Study the localization transition by calculating $P(\lambda)$ for $N \to \infty$.

- This work:

$\Rightarrow$ Cavity method: P Cizeau and JP Bouchaud, PRE **50**, 1994,
T Rogers and IP Castillo, PRE **79**, 2009.

$\Rightarrow$ Obtain results for the localization threshold λ_c.

$\Rightarrow$ Ensembles: Laplacian matrices of sparse random graphs and fully-connected Lévy matrices.

Part II

The cavity method

The Green function

- The Green function of a single instance $\boldsymbol{J}$:

$$G_{ij}(z) = (z\boldsymbol{I} - \boldsymbol{J})^{-1}_{ij} = \sum_{\mu=1}^{N} \frac{\psi^i_\mu (\psi^j_\mu)^*}{z-\lambda_\mu}\ , \ \ z = \lambda - i\epsilon.$$

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- The DOS and the IPR in terms of $G_{ii}(z)$:

$$\rho(\lambda) = \lim_{\epsilon \to 0^+} \frac{1}{\pi N} \left\langle \sum_{i=1}^{N} \mathrm{Im} G_{ii}(\lambda - i\epsilon) \right\rangle_{\boldsymbol{J}}$$

$$P(\lambda) = \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi N \rho(\lambda)} \left\langle \sum_{i=1}^{N} |G_{ii}(\lambda - i\epsilon)|^2 \right\rangle_{\boldsymbol{J}}$$

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where

$$\langle f(\omega) \rangle_{\epsilon,\lambda} = \int d\omega W_{\epsilon,\lambda}(\omega) f(\omega) \,, \quad W_{\epsilon,\lambda}(\omega) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta \left[\omega - G_{ii}(\lambda - i\epsilon) \right] \right\rangle_{\boldsymbol{J}} .$$

- $W_{\epsilon,\lambda}(\omega) \Rightarrow$ distribution of the real and imaginary parts of $G_{ii}(\lambda - i\epsilon)$.

Models defined on random graphs

- We can represent $G_{ii}(z)$ as a Gaussian integral:

$$G_{ii}(z) = i \int d\boldsymbol{x} \, x_i^2 \mathcal{P}_z(\boldsymbol{x}) \ , \ x_i \in \mathbb{R} \, ,$$

where

$$\mathcal{P}_z(\boldsymbol{x}) = \frac{\exp\left[-H_z(\boldsymbol{x})\right]}{\int d\boldsymbol{x} \exp\left[-H_z(\boldsymbol{x})\right]} \ , \ \ H_z(\boldsymbol{x}) = \frac{i}{2} \sum_{ij=1}^{N} x_i (z - \boldsymbol{J})_{ij} x_j \, .$$

- Analogy with models defined on random graphs:

  $\Rightarrow x_1, \ldots, x_N \rightarrow$ state variables of N interacting nodes;

  $\Rightarrow J_{ij} \rightarrow$ interaction strength between nodes i and j;

  $\Rightarrow H_z(\boldsymbol{x}) \rightarrow$ Hamiltonian;

  $\Rightarrow \mathcal{P}_z(x_i) \rightarrow$ single-site marginals.

- Calculation of $\mathcal{P}_z(x_i) \Rightarrow$ cavity method.

Equations for the single-site functions

- Cavity graph $\Rightarrow$ graph without a node and its connections;

  $\Rightarrow \mathcal{P}_z^{(l)}(x_k) \rightarrow$ local marginal in a graph without node l.

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- Local marginals in the original graph ($J_{ii} = 0$):

$$\mathcal{P}_z(x_k) \sim \int \left[\prod_{j\in\partial_k} dx_j\right] \mathcal{P}_z^{(k)}(\boldsymbol{x}_{\partial_k}) \exp\left(-\frac{i}{2} z x_k^2 + i x_k \sum_{j\in\partial_k}^{N} J_{kj} x_j\right),$$

$\partial_k \to$ set of nodes connected to node k

$\boldsymbol{x}_{\partial_k} \to$ state variables of the nodes belonging to ∂_k

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For $l \in \partial_k$:

$$\mathcal{P}_z^{(l)}(x_k) \sim \int \left[\prod_{j \in \partial_k \setminus l} dx_j\right] \mathcal{P}_z^{(k)}(\boldsymbol{x}_{\partial_k \setminus l}) \exp\left(-\frac{i}{2} z x_k^2 + i x_k \sum_{j \in \partial_k \setminus l}^N J_{kj} x_j\right).$$

Main assumption

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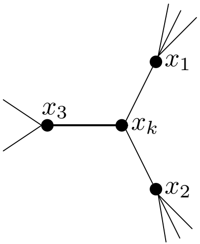
⇒ Sparse random matrices:



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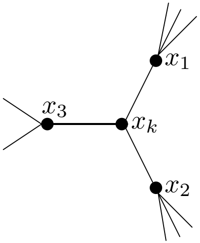

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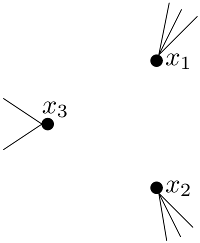

∂_k is uncorrelated

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$\Rightarrow$ Sparse random matrices:

∂_k is correlated

∂_k is uncorrelated

$\Rightarrow$ Fully-connected random matrices:

$\rightarrow$ each node interacts with all the other $N-1$ nodes;

$\rightarrow$ fully-connected spin systems $\rightarrow$ factorization holds on the cavity graph.

Equations for the Green function

- Gaussian *ansatz*: $\mathcal{P}_z(x_i) \sim \exp\left[-\frac{i x_i^2}{2G_{ii}(z)}\right], \quad \mathcal{P}_z^{(k)}(x_i) \sim \exp\left[-\frac{i x_i^2}{2G_{ii}^{(k)}(z)}\right].$

  $G_{ii}^{(k)}(z) \Rightarrow$ Green function of $\boldsymbol{J}$ where node $k \in \partial_i$ has been removed.

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- The diagonal elements of $\boldsymbol{G}(z)$ are obtained from:

$$G_{ii}(z) = \frac{1}{z - \sum_{j \in \partial_i} H\left[J_{ij}, G_{jj}^{(i)}(z)\right]}, \quad G_{ii}^{(k)}(z) = \frac{1}{z - \sum_{j \in \partial_i \setminus k} H\left[J_{ij}, G_{jj}^{(i)}(z)\right]}.$$

$$H[J, \omega] = J^2 \omega$$

- These equations determine $\{G_{ii}(z)\}$ for a single instance of $\boldsymbol{J}$.

Part III

Laplacian random matrices

Laplacian matrix of a random graph

$$J_{ij} = c_{ij}K_{ij} - \delta_{ij} \sum_{k=1}^{N} c_{ik}K_{ik} \ , \qquad \sum_{j=1}^{N} J_{ij} = 0 \, .$$

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- Elements of the connectivity matrix ($c_{ii} = 0$):

$$p_c(c_{ij}) = (1 - \tfrac{c}{N}) \delta_{c_{ij},0} + \tfrac{c}{N} \delta_{c_{ij},1} \longrightarrow p(k) = \frac{c^k \exp(-c)}{k!}$$

$\Rightarrow c$ is the average number of connections per node.

$\Rightarrow \boldsymbol{J}$ has a sparse structure for $N \to \infty$.

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- Nonzero edges of the graph:

Fixed values $\Rightarrow p_K(K_{ij}) = \delta(K_{ij} + 1/c) \, .$

Gaussian disorder $\Rightarrow p_K(K_{ij}) \sim \exp\left[-\frac{c}{2} K_{ij}^2\right] .$

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- Cavity method: $H(K,\omega) = \frac{K^2\omega}{1+K\omega} - K$.

Distribution of the Green function

$$W_{\epsilon,\lambda}(\omega) = \sum_{k=0}^{\infty} p(k) \int \left[\prod_{l=1}^{k} d\omega_l \, dK_l \, W_{\epsilon,\lambda}(\omega_l) \, p_K(K_l)\right] \delta\left(\omega - \frac{1}{z - \sum_{l=1}^{k} H(K_l,\omega_l)}\right)$$

- Population dynamics:

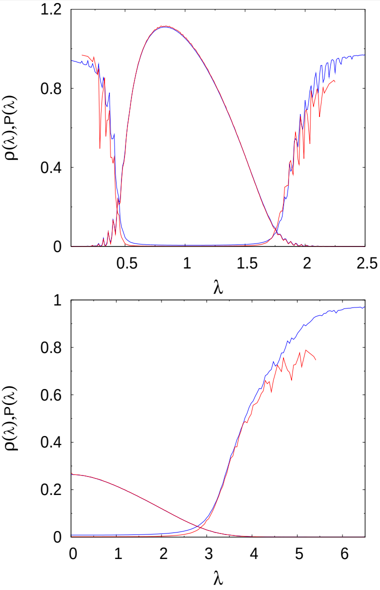
  1) Numerical solution for $W_z(\omega) \rightarrow$ very small ϵ (R Kühn, JPA **41**, 2008).

  2) Calculate $\rho(\lambda)$ and $P(\lambda)$ by using:

$$P(\lambda) = \frac{1}{\pi\rho(\lambda)} \lim_{\epsilon\rightarrow 0^+} \epsilon \langle |\omega|^2 \rangle_{\epsilon,\lambda} \, ,$$

$$\rho(\lambda) = \frac{1}{\pi} \lim_{\epsilon\rightarrow 0^+} \langle \mathrm{Im}\omega \rangle_{\epsilon,\lambda} \, .$$

Results from population dynamics - $c = 20$

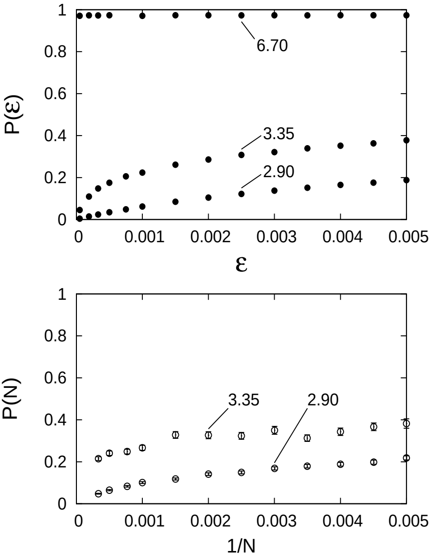


- Upper graph $\Rightarrow$ fixed edges
  Lower graph $\Rightarrow$ Gaussian disorder
- Pop. dynamics (blue lines):
  $N_s = 5 \times 10^6$, $\epsilon = 0.001$.
- Diagonalization (red lines):
  500 samples of $N = 3000$.

$\Rightarrow$ Fixed edges $\rightarrow$ regular peaks.

$\Rightarrow P(\lambda) \rightarrow 1$ for $\lambda \gg 1$.

The ϵ dependence

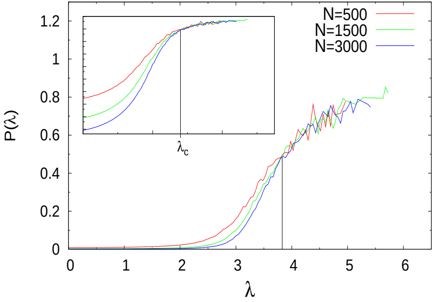


- Effect of ϵ and N in the case of Gaussian disorder.
- Localization transition becomes sharper for $N \to \infty$.
- We use the ϵ independence as a criterion to estimate λ_c.

The mobility edge λ_c

Gaussian disorder: $c = 20$

Diagonalization:



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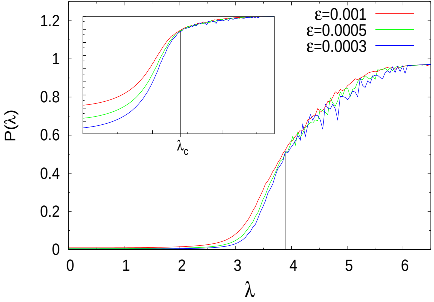

Part IV

Lévy random matrices

Lévy random matrices

- The elements J_{ij} ($J_{ii} = 0$) are drawn from:

$$P_\alpha(J) = \int \frac{dq}{2\pi} \exp\left(-iqJ\right) L_\alpha(q) \ , \qquad \ln L_\alpha(q) = -\left|\frac{q}{\sqrt{2}N^{1/\alpha}}\right|^\alpha \ .$$

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- $\alpha < 2 \Rightarrow$ $\boxed{\lim_{|J|\to\infty} P_\alpha(J) = \frac{C_\alpha}{N|J|^{\alpha+1}}}$

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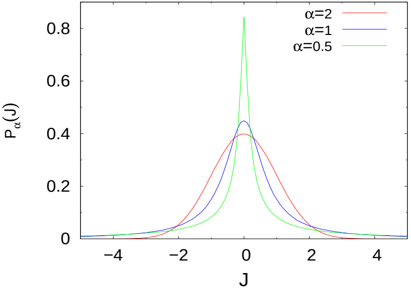

$N \to \infty$:

$\Rightarrow$ extensive number of elements of order $O(1/N^{1/\alpha})$

$\Rightarrow$ finite number of elements of order $O(1)$.

Introduction of a cutoff

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$$\sum_{j\neq i} J_{ij}^2 G_{jj}^{(i)}(z) = \sum_{\{j|J_{ij}>\gamma\}} J_{ij}^2 G_{jj}^{(i)}(z) + \underbrace{\sum_{\{j|J_{ij}<\gamma\}} J_{ij}^2 G_{jj}^{(i)}(z)}_{\sigma_\gamma^2\langle\omega\rangle} .$$

I. Neri, FLM and D. Bollé, JSM, 2010.

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- Variance of the weak matrix elements in a given row:

$$\sigma_\gamma^2 = N \int_{-\gamma}^{\gamma} dJ P_\alpha(J) J^2 = \frac{2\gamma^{2-\alpha} C_\alpha}{2-\alpha}.$$

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$$c_\gamma = N \int_{|J|>\gamma} dJ P_\alpha(J) = \frac{2C_\alpha}{\alpha\gamma^\alpha}.$$

- Distribution of the strong elements: $p_J(J;\gamma) = \begin{cases} \frac{\alpha\gamma^\alpha}{2|J|^{\alpha+1}} & |J| > \gamma\,, \\ 0 & |J| < \gamma\,. \end{cases}$

The distribution of the Green function

$$W_{\epsilon,\lambda,\gamma}(\omega) = \sum_{k=0}^{\infty} \frac{c_\gamma^k \exp(-c_\gamma)}{k!} \left[\int \prod_{l=1}^{k} \, d\omega_l W_{\epsilon,\lambda,\gamma}(\omega_l)\right]$$

$$\times \left[\int \prod_{l=1}^{k} \, dJ_l \;\, p_J(J_l;\gamma)\right] \delta\left(\omega - \frac{1}{z - \sum_{l=1}^{k} J_l^2 \omega_l - \sigma_\gamma^2 \langle\omega\rangle}\right)$$

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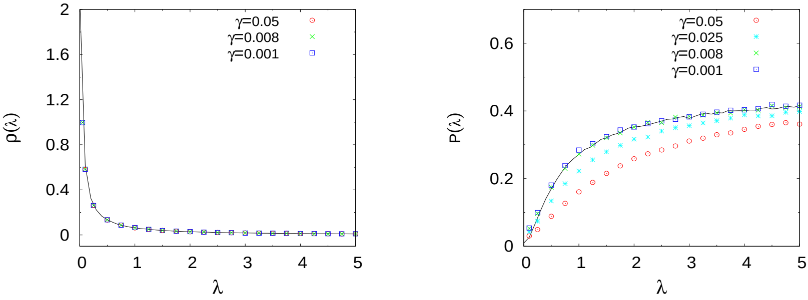
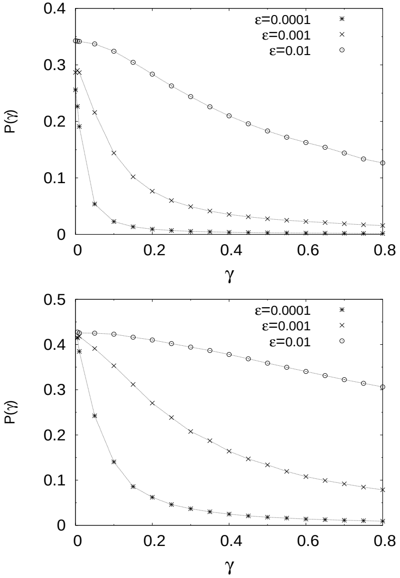
- $\alpha \to 2$: $W_{\epsilon,\lambda,\gamma}(\omega) = \delta\left(\omega - \frac{1}{z-\langle\omega\rangle}\right) \Rightarrow$ Wigner semi-circle law.
- $\alpha = 0.5$, $N = 3000$ and $\epsilon = 0.0005$.



Localized and extended states



- Pop. dynamics results:

  $N_s = 5 \times 10^6$ and $\alpha = 0.5$.

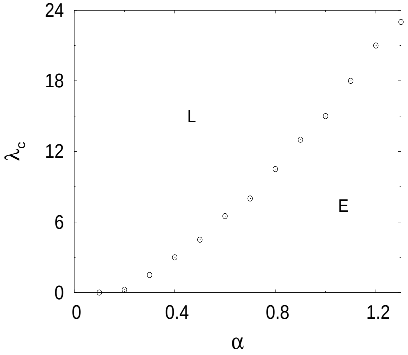
- Upper graph ($\lambda = 1$):

  $\Rightarrow$ Extended states: the IPR depends on ϵ for $\gamma \to 0$.

- Lower graph ($\lambda = 5$):

  $\Rightarrow$ Localized states: the IPR is independent of ϵ for $\gamma \to 0$.

The phase diagram



$N_s = 10^6$ and $\gamma = 0.008$.

FLM, I Neri and D Bollé, PRE **82**, 2010.

- Small α:

  $\Rightarrow$ the sparse character is highlighted $\rightarrow$ localized states.

- Large α:

  $\Rightarrow$ the fully-connected character is highlighted $\rightarrow$ extended states

  $\Rightarrow$ large λ_c $\rightarrow$ inaccurate results

- $P(\lambda) \rightarrow 1/2$ for $\lambda \gg 1$.

Final remarks

- We have shown how an equation for the distribution of Green functions $W(\omega)$ can be obtained in the limit $N \to \infty$ with the cavity method.

  $\Rightarrow$ Numerical solution for $W(\omega)$: average IPR and mobility edge.

- Laplacian matrices:

  $\Rightarrow \lambda \gg 1$: states localized on a single node with very high connectivity.

- Lévy matrices:

  $\Rightarrow \lambda \gg 1$: states localized on pairs of very strongly interacting nodes.

- Localization in asymmetric matrices $\Rightarrow$ fluctuations in the symmetry of the related graph?

- Calculation of the distribution of $|\psi_{\mu}^{i}|^2$ over the sites $\Rightarrow$ cavity method?

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Group of Disordered Systems.