University of BRISTOL

Semiclassical theory of spectral statistics: n-point correlation functions

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non-oscillatory and oscillatory contributions

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Weyl term $\equiv 1$ — per. orbits γ — stability amplitude — action

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get only non-oscillatory term $R_2(\epsilon_1, \epsilon_2) = 1 - \frac{1}{2\pi(\epsilon_1 - \epsilon_2)}$

Improved semiclassics
(Riemann-Siegel lookalike formula)

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- use that $det(E - H) \in \mathbb{R}$ for $E \in \mathbb{R}$

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- expand exponential

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Weyl factor

sum over sets of classical orbits shorter than $T_H/2 = \pi\hbar\overline{\rho}$

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- relation between long and short orbits

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here $F_A^{(+1)} = F_A$, $F_A^{(-1)} = F_A^*$

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- n-point correlations:
  Shukla 1997;
  Nagao, SM 2009

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$$Z \propto \sum_{\sigma_j, \tau_j \, (\sum_j \sigma_j = \sum_j \tau_j)} e^{i\pi(\sum_j \sigma_j \epsilon_j - \sum_j \tau_j \eta_j)} \left(\prod_{j=1}^{n} \sigma_j\right) \frac{\prod_{\sigma_j=1, \tau_k=-1}(\epsilon_j - \eta_k) \prod_{\tau_j=1, \sigma_k=-1}(\eta_j - \epsilon_k)}{\prod_{\sigma_j=1, \sigma_k=-1}(\epsilon_j - \epsilon_k) \prod_{\tau_j=1, \tau_k=-1}(\eta_j - \eta_k)}$$

Result

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Originally derived to compare RMT & number theory!

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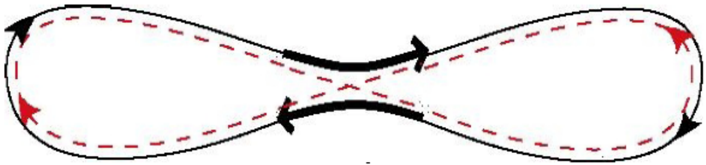

(Sieber & Richter 2001)

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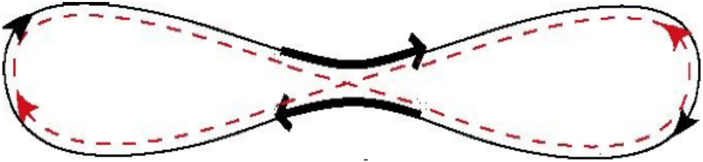

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See Jack Kuipers' talk

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- also get corrections for **finite** $\hbar$, **long wires**