Universal and non-universal features of critical multifractal wave functions

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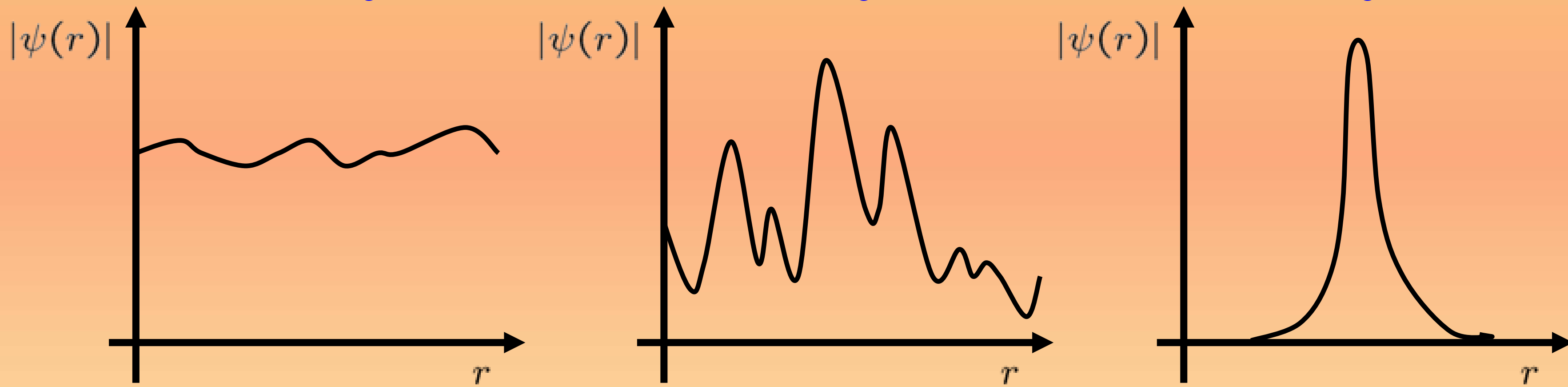
Anderson model

Hamiltonian on a *d*-dimensional lattice:

$$(\hat{H}\psi)_i = v_i\psi_i + \sum_{<ij>} \psi_j, \qquad \overline{v_i} = 0,\ \overline{v_i v_k} = W\delta_{ik}$$

Metal-insulator transition in the three-dimensional case:

$W<W_c$ $\qquad$ $W=W_c$ $\qquad$ $W>W_c$

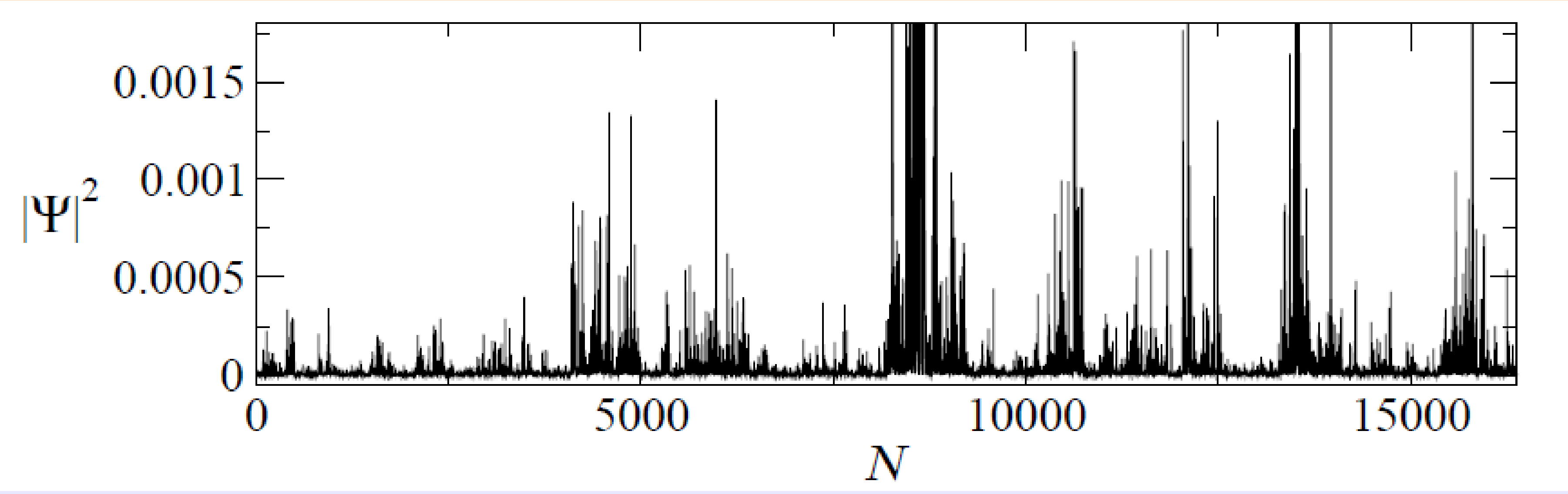


ergodic $\qquad$ (multi)fractal $\qquad$ localized

Wigner-Dyson RM $\qquad$ Power-law Banded RM $\qquad$ Banded RM

P. W. Anderson, Phys. Rev. **109**, 1492 (1958)

Example of a fractal eigenfunction



Outline

1. Power-law banded matrices and ultrametric ensemble: universality of the fractal dimensions

2. Fractal dimensions: beyond the universality

3. Dynamical scaling: validity of Chalker's ansatz

Fractal dimensions

Moments: $$I_q = \sum_{\mathbf{r}} \overline{|\psi_n(\mathbf{r})|^{2q}} \propto L^{-d_q(q-1)}$$

Metal: $d_q = d$

Insulator: $d_q = 0$

Anomalous scaling exponents: $\Delta_q \equiv (d_q - d)(q-1)$

Critical point: $\Delta_q \neq 0$

Spatial correlations : $L^{2d}\overline{|\psi^2(\mathbf{r})\psi^2(\mathbf{r}')|} \propto (|\mathbf{r}-\mathbf{r}'|/L)^{\Delta_2}$

How one can calculate d_q ?

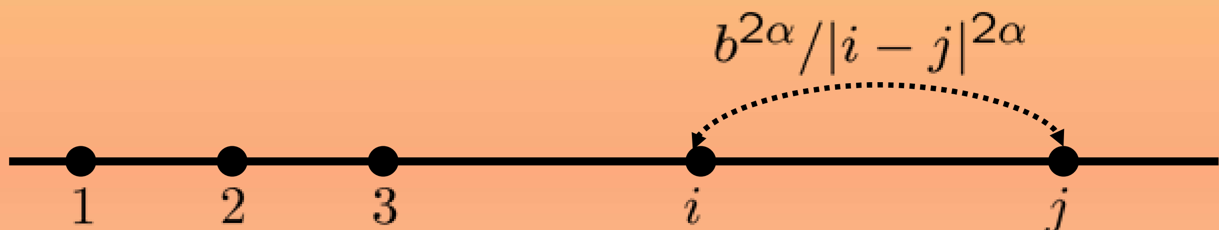
Green's functions: $G^{R/A} = (E \pm i0 - H)^{-1}$

$$I_{m+n} \Longleftrightarrow \overline{(G^R(\mathbf{r},\mathbf{r}))^m (G^A(\mathbf{r},\mathbf{r}))^n}$$

Power-law banded random matrices

$H_{ij} = H_{ji}^*$ Gaussian distributed, independent $\overline{H_{ij}} = 0$

$$\overline{|H_{ij}|^2} = \frac{1}{1 + (|i-j|/b)^{2\alpha}}$$



$\alpha = 1$ critical states at all values of b

$b \gg 1$ mapping onto non-linear σ-model weak multifractality

$b \ll 1$ almost diagonal matrix strong multifractality

A. D. Mirlin et. al., Phys. Rev. E **54**, 3221 (1996)

Almost diagonal matrices

Diagonal approximation: $\overline{I_q^{(0)}} = 1$

2x2 matrix approximation:

$$H(n,m) = \begin{pmatrix} H_{nn} & H_{nm} \\ H_{nm}^* & H_{mm} \end{pmatrix} \qquad \overline{I_q^{(1)}} = \frac{1}{L}\sum_{n\neq m}^{L}\left(\overline{I_q(n,m)} - 1\right)$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\sqrt{\overline{|H_{nm}|^2}}$$

$$S \equiv \frac{1}{L}\sum_{n\neq m}^{L}\sqrt{\overline{|H_{nm}|^2}}$$

determines the nataure of eigenstates in the thermodynamic limit $L \to \infty$

$S \to \text{const}$	localized states
$S \sim L^{\alpha}, \quad \alpha > 0$	extended states
$S \sim \ln L$	critical states

Strong multifractality in the power-law ensemble

$b \ll 1$

General expression:

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}} \frac{\Gamma(q-1/2)}{\Gamma(q-1)} \frac{1}{L} \sum_{n \neq m}^{L} \sqrt{\overline{|H_{nm}|^2}}$$

Power-law banded matrices:

$$\overline{|H_{nm}|^2} = \frac{1}{1+(|n-m|/b)^2}$$

$$\overline{I_q} = 1 - \sqrt{2\pi} \frac{\Gamma(q-1/2)}{\Gamma(q-1)} b \ln L$$

$$\overline{I_q} \propto L^{-d_q(q-1)} = 1 - d_q(q-1)\ln L + \frac{d_q^2(q-1)^2}{2} \ln^2 L + \ldots$$

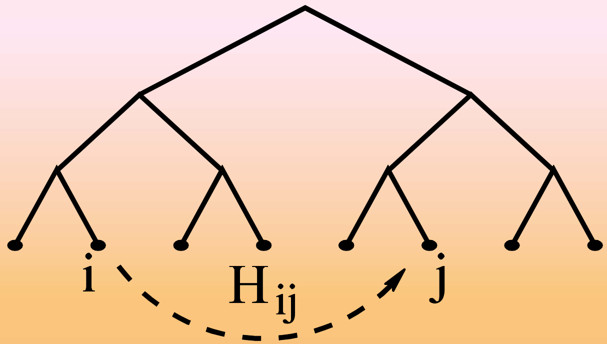
Fractal dimensions:

$$d_q = \sqrt{2\pi} \frac{\Gamma(q-1/2)}{\Gamma(q)} b$$

A. D. Mirlin and F. Evers, Phys. Rev. B **62**, 7920 (2000)

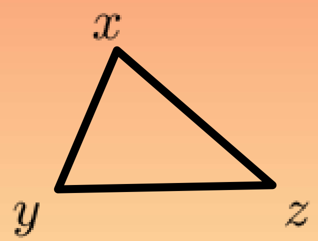
Ultrametric ensemble

Random hopping between boundary nodes of a tree of K generations with coordination number 2

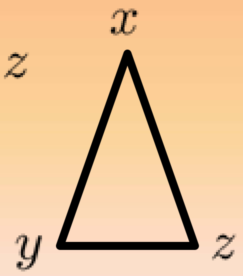


$$\overline{H_{ii}^2} = W^2 \qquad \overline{|H_{ij}|^2} = \frac{J^2}{p^{d(i,j)-2}}$$

Distance $d(i,j) =$ number of edges in the shortest path connecting i and j --- ultrametric



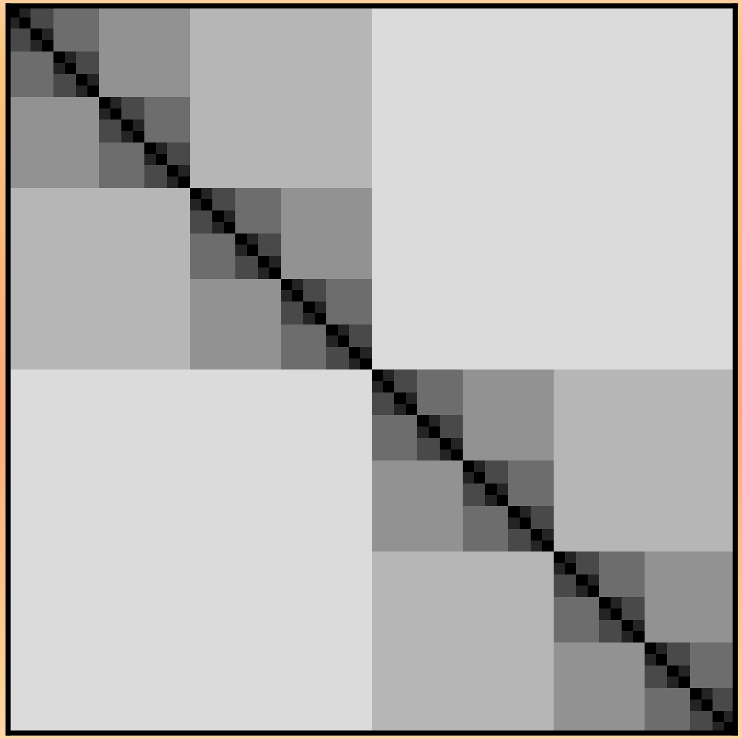
Triangle inequality: $d(x,y) \leq d(x,z) + d(z,y)$



Strong triangle inequality: $d(x,y) \leq \max\{d(x,z), d(z,y)\}$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

Structure of the ultrametric matrix



Metal-insulator transition in the ultrametric ensemble

$J/W \ll 1$

General expression: $$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\frac{\sqrt{\overline{|H_{nm}|^2}}}{W}$$

Ultrametric ensemble: $$\overline{|H_{nm}|^2} = \frac{J^2}{p^{d(n,m)-2}}$$

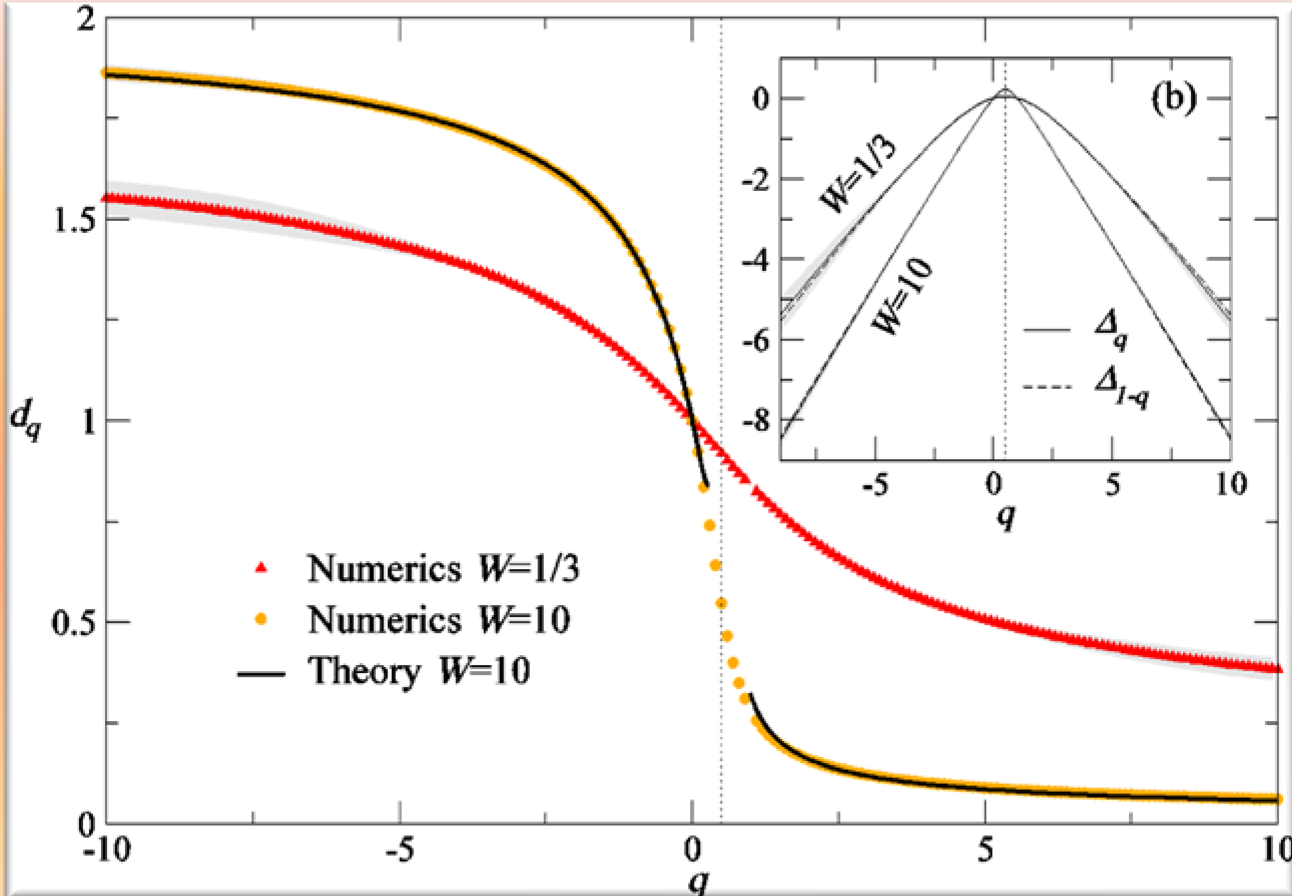
$$S \equiv \frac{1}{L}\sum_{n\neq m}^{L}\sqrt{\overline{|H_{nm}|^2}} = \frac{(2/p)^{\log_2 L}-1}{(2/p)-1}$$

$p > 2$	$S \to \text{const}$	localized states
$p < 2$	$S \sim L^{\alpha}, \quad \alpha > 0$	extended states
$p = 2$	$S \sim \ln L$	critical states

$$d_q = \frac{J}{W}\frac{\sqrt{\pi}}{\sqrt{2}\ln 2}\frac{\Gamma(q-1/2)}{\Gamma(q)}$$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

Fractal dimensions in the ultrametric ensemble



Anomalous exponents:

$$\Delta_q \equiv (d_q - d)(q-1)$$

Symmetry relation:

$$\Delta_q = \Delta_{1-q}$$

Y. V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech., L12001 (2009)

A. D. Mirlin et. al., Phys. Rev. Lett. **97**, 046803 (2007)

Universality of fractal dimensions

$$b = J/W \ll 1$$

Power-law banded matrices:

$$d_q = b\sqrt{2\pi}\frac{\Gamma(q-1/2)}{\Gamma(q)}$$

Ultrametric random matrices:

$$d_q = \frac{J}{W}\frac{\sqrt{\pi}}{\sqrt{2}\ln 2}\frac{\Gamma(q-1/2)}{\Gamma(q)}$$

$$\overline{I_q} = 1 - \sqrt{\frac{\pi}{2}}\frac{\Gamma(q-1/2)}{\Gamma(q-1)}\frac{1}{L}\sum_{n\neq m}^{L}\frac{\sqrt{\overline{|H_{nm}|^2}}}{W}$$

universality

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Fractal dimensions: beyond universality

$$b = J/W \ll 1$$

$$d_q(b) = b f_1(q) + b^2 f_2(q) + \dots$$

$f_1(q)$ can be choosen the same for all models

$f_2(q)$ model specific

Can we calculate $f_2(q)$?

$$I_q(L) \propto L^{-d_q(q-1)} = 1 - d_q(q-1)\ln L + \frac{1}{2} d_q^2 (q-1)^2 \ln^2 L + \dots$$

$$I_q(L) = 1 + I_q^{(1)}(L) + I_q^{(2)}(L) + \dots \qquad I_q^{(1)} \propto b, \quad I_q^{(2)} \propto b^2$$

$$I_q^{(1)}(L) = -f_1(q) b \ln L + \dots \qquad I_q^{(2)}(L) = \frac{f_1(q)^2}{2} b^2 \ln^2 L + f_2(q) b^2 \ln L \,.\, .$$

Fractal dimension d_2 for power –law banded matrices

Supersymmetric virial expansion:

$$d_2 = \frac{\pi b}{\sqrt{2}} + \frac{(\pi b)^2}{4}\left[10 - \frac{56}{3\sqrt{3}} - \ln 4 + \pi\, I\right] + O(b^3)$$

where

$$I = \left(\frac{2}{\pi}\right)^3 \int_0^{\frac{\pi}{2}} \frac{d\varphi_1\, d\varphi_2\, d\varphi_3}{(\cos\varphi_1 + \cos\varphi_2)\,(\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3)} \approx 0.794$$

V. E. Kravtsov, A. Ossipov, O. M. Yevtushenko, E. Cuevas , Phys. Rev. B **82**, 161102(R) (2010)

Weak multifractality

$$b = J/W \gg 1$$

$$d_q(b) = d - \frac{1}{b}F_1(q) + \frac{1}{b^2}F_2(q) + \dots$$

How one can calculate $F_i(q)$?

Mapping onto the non-linear σ-model

Perturbative expansion in the regime $b \gg 1$

$$I_q(L) = \frac{q!}{L^{d(q-1)}}\left(1 + I_q^{(1)}(L) + I_q^{(2)}(L) + \dots\right) \qquad I_q^{(1)} \propto \frac{1}{b}, \quad I_q^{(2)} \propto \frac{1}{b^2}$$

$$I_q^{(1)}(L) = -F_1(q)\frac{1}{b}\ln L + \dots \qquad I_q^{(2)}(L) = \frac{F_1(q)^2}{2}\frac{1}{b^2}\ln^2 L + F_2(q)\frac{1}{b^2}\ln L \dots$$

Non-universal contributions to the fractal dimensions

Anomalous fractal dimensions Δ_q

Orthogonal symmetry class (β=1):

Power-law ensemble $\Delta_q^{(O)} = q(1-q)(t-t^2) + O(t^3) \qquad t \sim \frac{1}{b}$

Ultrametric ensemble $\Delta_q^{(O)} = q(1-q)(\tilde{t} - \tilde{t}^2 \frac{13 \ln 2}{6}) + O(\tilde{t}^3)$

Unitary symmetry class (β=2):

Power-law ensemble $\Delta_q^{(U)} = \frac{1}{2} q(1-q)(t - C_1 t^3) + C_2 q^2 (q-1)^2 t^3 + O(t^4)$

Ultrametric ensemble $\Delta_q^{(U)} = \frac{1}{2} q(1-q)(\tilde{t} - D_1 \tilde{t}^3) + D_2 q^2 (q-1)^2 \tilde{t}^3 + O(\tilde{t}^4)$

$C_2 \neq D_2$ ⟹ models are different

I. Rushkin, A. Ossipov, Y. V. Fyodorov in preparation

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Spectral correlations

$$C(\omega) = \overline{|\psi_n(\mathbf{r})|^2 |\psi_m(\mathbf{r})|^2} \propto \omega^{\nu-1} \qquad \omega = E_n - E_m$$

$$\nu = \frac{d_2}{d}$$

$$I_2 = \sum_{\mathbf{r}} \overline{|\psi_n(\mathbf{r})|^4} \propto L^{-d_2}$$

Strong multifractality: $d_2 \to 0 \quad \Rightarrow \quad \nu - 1 \to -1$

⇓

Strong overlap of two infinitely sparse fractal wave functions!

J.T. Chalker and G.J.Daniell, Phys. Rev. Lett. **61**, 593 (1988)

Return probability

$$P(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} C(\omega)$$

$$C(\omega) \propto \omega^{\nu-1} \quad \Longleftrightarrow \quad P(t) \propto \frac{1}{(bt)^{\nu}}$$

$$\lim_{bt/L\to\infty} P(t) = I_2 \propto \frac{1}{L^{d_2}}$$

Strong multifractality $d_2, \nu \ll 1$:

$$P(t) = \begin{cases} 1 - \nu \ln(bt) + \frac{\nu^2}{2} \ln^2(bt) + \ldots, & bt \ll L \\ 1 - d_2 \ln(L) + \frac{d_2^2}{2} \ln^2(L) + \ldots, & L \ll bt \end{cases}$$

$$\nu = d_2 = \frac{\pi b}{\sqrt{2}} + \frac{(\pi b)^2}{4}\left[10 - \frac{56}{3\sqrt{3}} - \ln 4 + \pi I\right] + O(b^3)$$

V. E. Kravtsov, A. Ossipov, O. M. Yevtushenko, E. Cuevas , Phys. Rev. B **82**, 161102(R) (2010)

Conclusions

- Two critical random matrix ensembles: the power-law random matrix model and the ultrametric model
- Analytical results for the multifractal dimensions in the regimes of the strong and the weak multifractality
- Universal and non-universal contribution to the fractal dimensions
- Equivalence of the spectral and the spatial scaling exponents