Euclidean random matrices for waves in random media

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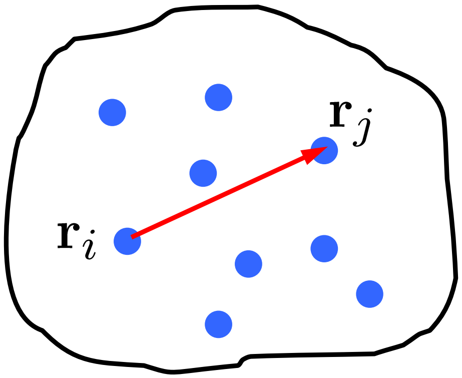
Euclidean matrices



$i, j = 1, \ldots N$

$N \times N$ matrix $F_{ij} = f(\mathbf{r}_i, \mathbf{r}_j)$

$$= f(\mathbf{r}_i - \mathbf{r}_j)$$
$$= f(|\mathbf{r}_i - \mathbf{r}_j|)$$

Eigenvalue problem:

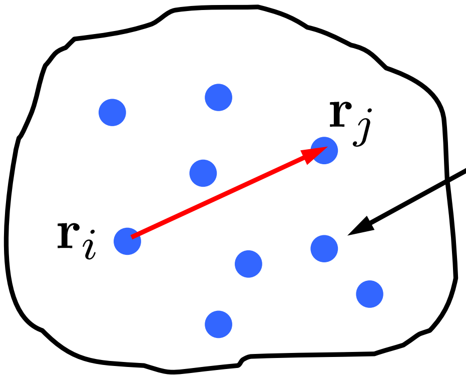
$$\widehat{F}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$

λ_n — eigenvalues

$\vec{\Psi}_n = \{\Psi_n^i\}$ — eigenvectors

$n = 1, \ldots, N$

Our motivation: Why study Euclidean matrices?



$$i, j = 1, \ldots N$$

$$N \gg 1$$

Point-like scatterers or atoms

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

Free-space Green's function:

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k_0^2 G(\mathbf{r}, \mathbf{r}') = -\frac{4\pi}{k_0}\delta(\mathbf{r} - \mathbf{r}')$$

Classical physics:

An ensemble of randomly placed point-like scatterers:

- multiple scattering of waves
- Anderson localization
- random lasing or
- nonlinear-optical phenomena

Quantum physics:

An ensemble of randomly distributed atoms:

- collective spontaneous emission (superradiance)
- random lasing

Previous studies of Euclidean matrices

- Mézard, Parisi and Zee, Nucl. Phys. B 559, 689 (1999)
  Low- and high-density expansions
  A link between $p(\lambda)$ and the Fourier transform of $f(\mathbf{r},\mathbf{r}')$

- Grigera, Martin-Mayor, Parisi and Verrocchio, Phys. Rev. Lett. 87, 085502 (2001)
  J. Phys. Cond. Mat. 14, 2167 (2002)
  Nature 422, 289 (2003)
  Application to phonons in glasses and to the 'boson peak' in supercooled liquids

- Parisi, in *Applications of Random Matrices in Physics* (2006)
  Nice review paper

- Ganter and Schirmacher, arXiv:1003.2514
  Higher-order terms (up to ρ^{-2}) in the high-density expansion
  New terms overlooked in the previous works

- Grigera, Martin-Mayor, Parisi, Urbani and Verrocchio, arXiv:1011.2798
  Advanced analysis of the high-density expansion

Why the previous results are not sufficient

Previous analysis holds for functions $f(\mathbf{r}_i, \mathbf{r}_j)$ that decay fast with $|\mathbf{r}_i - \mathbf{r}_j|$ $\longrightarrow$ the results are independent of the sample size L.

Our $f(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$, as well as $\mathrm{Re} f(\mathbf{r}_i, \mathbf{r}_j)$ and $\mathrm{Im} f(\mathbf{r}_i, \mathbf{r}_j)$ decay slowly with $|\mathbf{r}_i - \mathbf{r}_j| \longrightarrow$ some integrals do not converge.

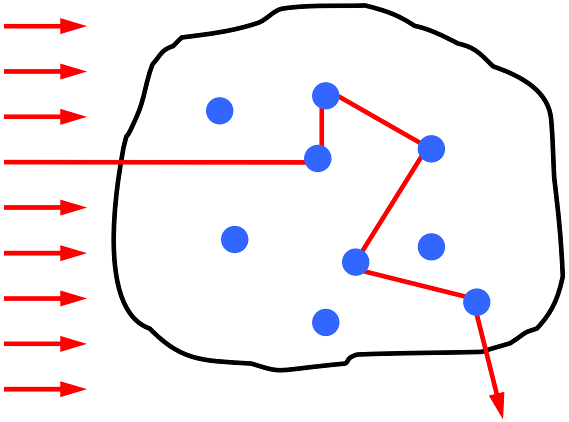
Previous analysis rely on low-ρ or high-ρ expansions.

We want to have results for any ρ

Previous works only treat Hermitian matrices.

Our matrix $\widehat{G}$ is non-Hermitian.

Motivation: Multiple scattering and Anderson localization



Incident wave

$\psi_0(\mathbf{r})$

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\psi(\mathbf{r}_i) = \psi_0(\mathbf{r}_i) - \frac{k_0}{4\pi} t \sum_{j=1}^{N} G_{ij}\, \psi(\mathbf{r}_j)$$

Field on the ith scatterer

Scattering matrix of the scatterer

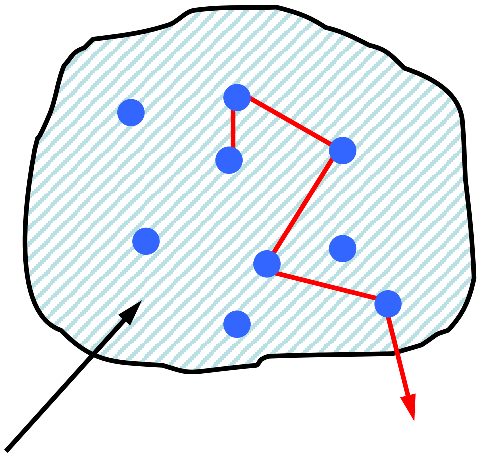
Incident wave

Field on the jth scatterer

Anderson localization can be studied by analyzing the statistics of the imaginary part of eigenvalues of $\widehat{G}$

Rusek, Mostowski and Orlowski, Phys. Rev. A 61, 022704 (2000)
Pinheiro et al., Phys. Rev. E 69, 026605 (2004)
Antezza, Castin and Hutchinson, Phys. Rev. A 82, 043602 (2010)

Motivation: Random lasing



$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

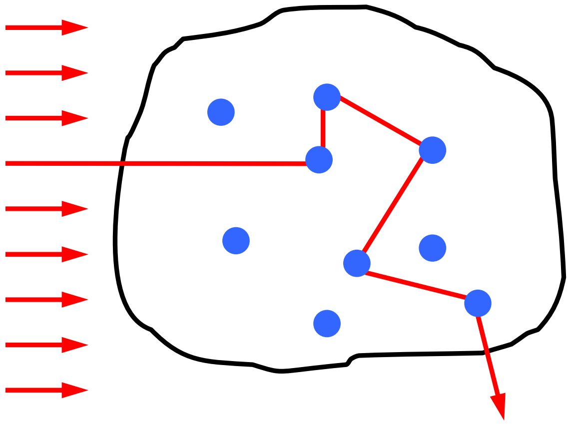
The minimal value of the imaginary part of the eigenvalues of $\widehat{G}$ determines the random laser threshold:

$$\min\,[\mathrm{Im}\lambda] < \text{amplification rate}$$

(physical model behind is questionable...)

Pinheiro and Sampaio, Phys. Rev. A 73, 013826 (2006)

Motivation: Instability in nonlinear random medium



Incident wave

$\psi_0(\mathbf{r})$

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\psi(\mathbf{r}_i) = \psi_0(\mathbf{r}_i) - \frac{k_0}{4\pi}\sum_{j=1}^{N} G_{ij}\, t_j \psi(\mathbf{r}_j)$$

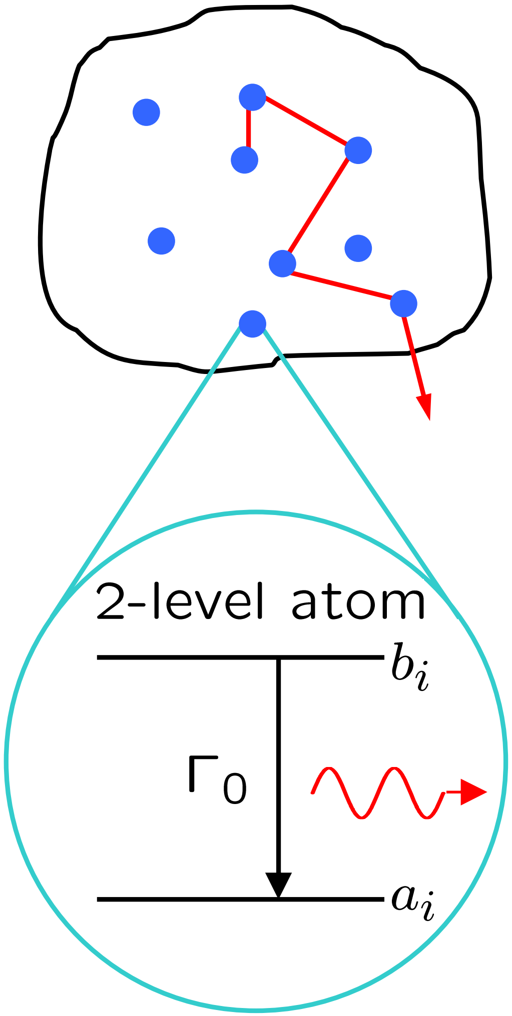
Intensity-dependent

scattering matrix: $t_j = -\frac{2\pi\mathrm{i}}{k_0}\left(e^{\mathrm{i}\alpha|\psi(\mathbf{r}_j)|^2} + 1\right)$

The stationary solution loses its stability at some $\alpha = \alpha_c^{(1)}$.
Multiple solutions appear at $\alpha = \alpha_c^{(2)} > \alpha_c^{(1)}$.
$\alpha_c^{(1)}$ and $\alpha_c^{(2)}$ are determined by the eigenvalues of $\widehat{G}$.

Skipetrov and Maynard, Phys. Rev. Lett. 85, 736 (2000)
Zyuzin and Spivak, Phys. Rev. Lett. 84, 1970 (2000)
Grémaud and Wellens, Phys. Rev. Lett. 104, 133901 (2010)

Motivation: Collective spontaneous emission



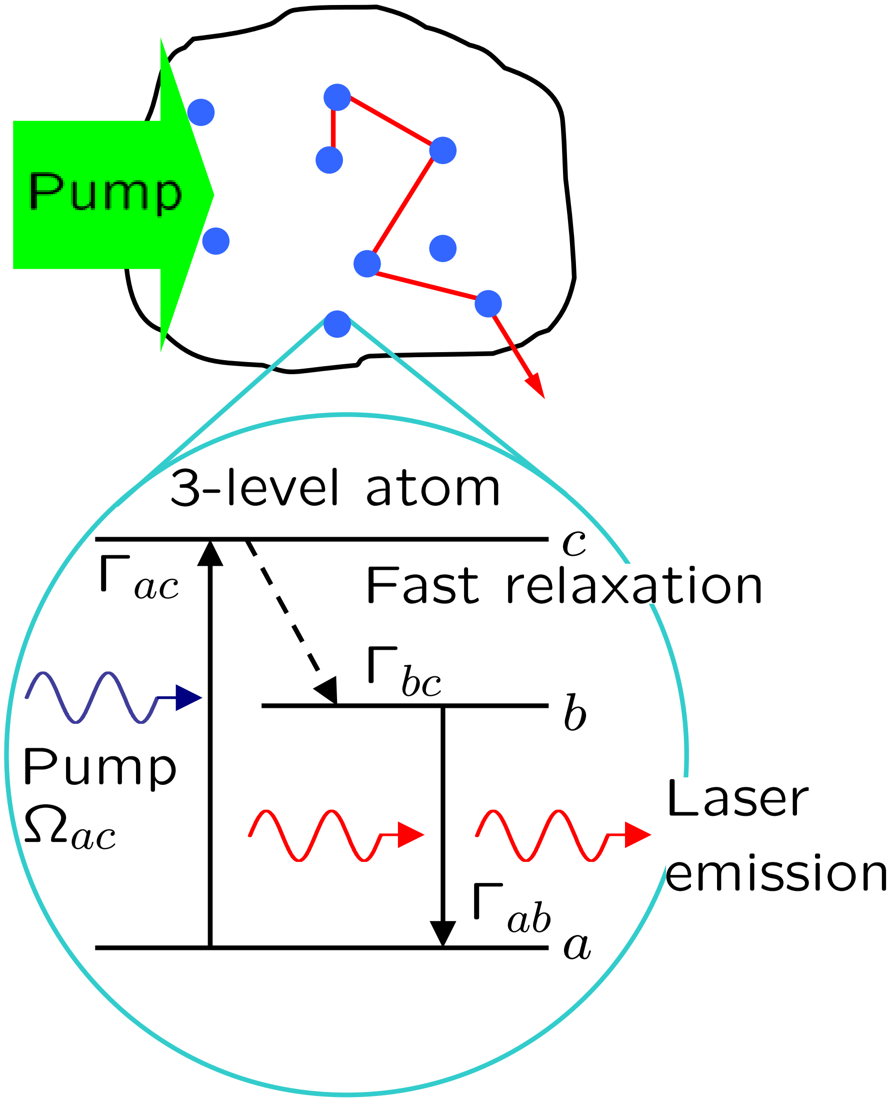
$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\Psi(t) = \sum_{j=1}^{N} \alpha_j(t)|a_1 \cdots b_j \cdots a_N\rangle|0\rangle$$
$$+ \sum_{\mathbf{k}} \gamma_{\mathbf{k}}(t)|a_1 \cdots a_N\rangle|1_{\mathbf{k}}\rangle$$
$$+ \sum_{m<n} \sum_{\mathbf{k}} \beta_{mn,\mathbf{k}}|a_1 \cdots b_m \cdots b_n \cdots a_N\rangle|1_{\mathbf{k}}\rangle$$

$$\frac{d\vec{\alpha}(t)}{dt} = -\Gamma_0\vec{\alpha}(t) + \mathrm{i}\Gamma_0 \widehat{G}\vec{\alpha}(t)$$

Svidzinsky, Chang and Scully, Phys. Rev. A 81, 053821 (2010)

Motivation: Quantum model for the random laser



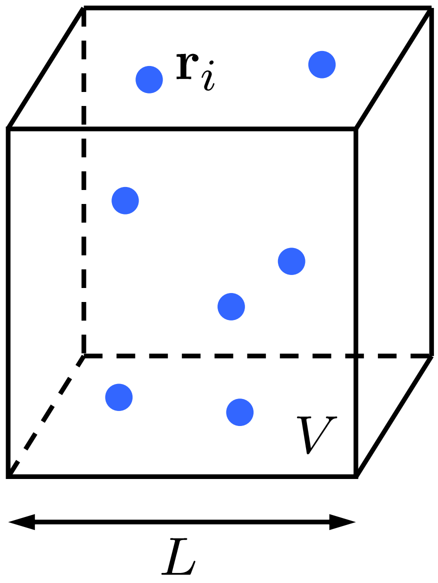
$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\begin{aligned}
\frac{dS_i^+}{dt} &= \left[\mathrm{i}\omega_0 - \frac{\Gamma}{2}(1+W_i)\right] S_i^+ \\
&+ \mathrm{i}\frac{\Gamma}{2}\Pi_i \sum_{j\neq i}^{N} G_{ij}^* S_j^+ \\
&+ F_i^+(\mathbf{r}_i, t)
\end{aligned}$$

$$\begin{aligned}
\frac{d\Pi_i}{dt} &= -\Gamma\left[(1+W_i)\,\Pi_i + (1-W_i)\right] \\
&- 2\Gamma\,\mathrm{Im}\left[S_i^+ \sum_{j\neq i}^{N} G_{ij} S_j^-\right] \\
&+ F_i^{\Pi}(\mathbf{r}_i, t)
\end{aligned}$$

$\Pi_i = |b_i\rangle\langle b_i| - |a_i\rangle\langle a_i|,\ S_i^+ = |b_i\rangle\langle a_i|$:

Representation of Euclidean matrices



$N \times N$ matrix $F_{ij} = f(\mathbf{r}_i, \mathbf{r}_j)$

$$\boxed{\widehat{F} = \widehat{H}\widehat{T}\widehat{H}^{+}}$$

$$T_{mn} = \frac{\rho}{V}\int_V d^3\mathbf{r} \int_V d^3\mathbf{r}' e^{-\mathrm{i}\mathbf{q}_m\mathbf{r}} f(\mathbf{r}, \mathbf{r}') e^{\mathrm{i}\mathbf{q}_n\mathbf{r}'}$$

Eigenmodes of the box:

$$\psi_m(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{\mathrm{i}\mathbf{q}_m\mathbf{r}}$$

$$\mathbf{q}_m = \left\{\frac{2\pi}{L}m_x, \frac{2\pi}{L}m_y, \frac{2\pi}{L}m_z\right\}$$

$$H_{im} = \sqrt{\frac{V}{N}}\psi_m(\mathbf{r}_i) = \frac{1}{\sqrt{N}} e^{\mathrm{i}\mathbf{q}_m \cdot \mathbf{r}_i}$$

$$\langle H_{im} \rangle = 0$$

$$\langle H_{im} H^*_{jn} \rangle = \frac{1}{N}\delta_{ij}\delta_{mn}$$

$$i, j = 1, \ldots, N$$
$$m, n = 1, \ldots, M$$

Hermitian matrices: Generalities

$\hat{F}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$, $\hat{F}^+ = \hat{F}$, λ_n – real eigenvalues

Probability density of eigenvalues λ:

$$p(\lambda) \;=\; \left\langle \frac{1}{N} \sum_{n=1}^{N} \delta(\lambda - \lambda_n) \right\rangle = -\left\langle \frac{1}{N} \mathrm{Im} \sum_{n=1}^{N} \frac{1}{\pi} \frac{1}{\lambda + \mathrm{i}\epsilon - \lambda_n} \right\rangle$$

$$\delta(x) = -\frac{1}{\pi} \mathrm{Im} \frac{1}{x + \mathrm{i}\epsilon}, \quad \epsilon \to 0^+$$

$$\boxed{p(\lambda) \;=\; -\frac{1}{\pi} \mathrm{Im}\mathcal{G}(z = \lambda + \mathrm{i}\epsilon)}$$

Green's function (resolvent):

$$\mathcal{G}(z) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z - \hat{F}} \right\rangle$$

Hermitian matrices: Diagrammatic expansion

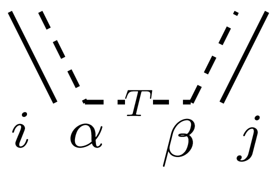
$$\mathcal{G}(z) = \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-\widehat{F}}\right\rangle = \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z}+\frac{1}{z}\widehat{F}\frac{1}{z}+\frac{1}{z}\widehat{F}\frac{1}{z}\widehat{F}\frac{1}{z}+\ldots\right)\right\rangle$$

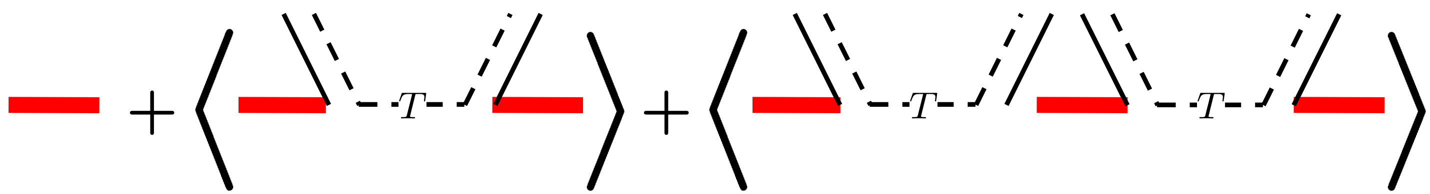
$$= \frac{1}{z-\Sigma(z)}$$

$$\widehat{F} = \widehat{H}\widehat{T}\widehat{H}^{+}$$

Diagrammatic notation:

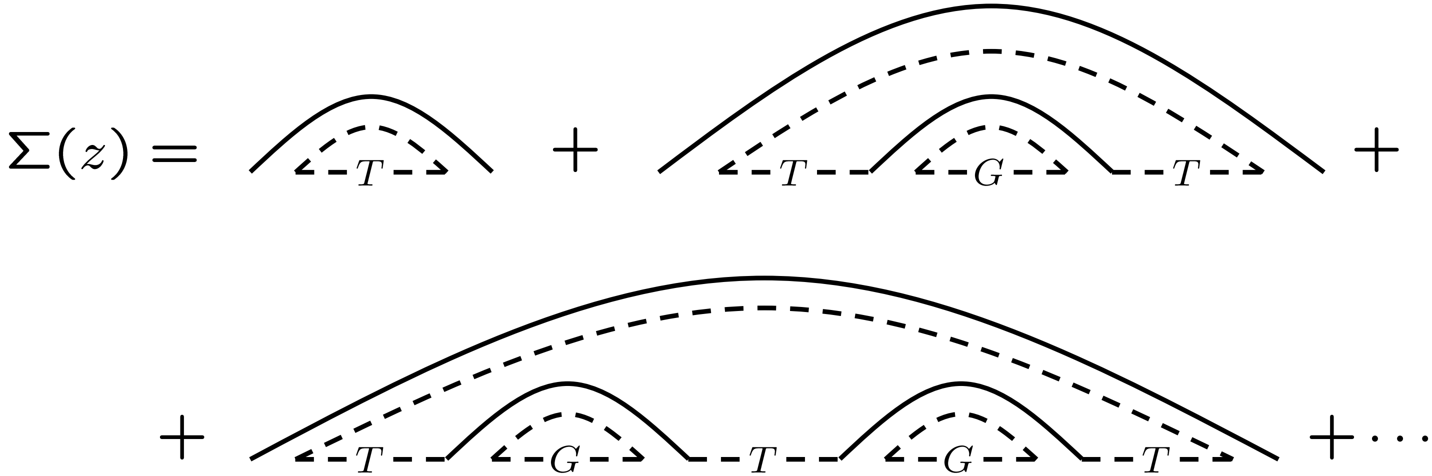
$$\frac{1}{z} =$$ 

$$H_{i\alpha}T_{\alpha\beta}H^{+}_{\beta j} =$$ 

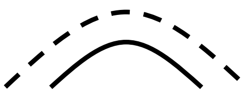
$$\left\langle\frac{1}{z-\widehat{F}}\right\rangle =$$ 

$$+\cdots$$

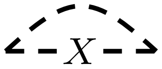
Self-energy



Weight of elements:

 $=$  $= \frac{1}{N}$

 $= N$



$=$ Tr$\hat{X}$

Diagram selection rule:

In the limit $N \to \infty$ only those diagrams survive where all doubles arches  are compensated by  or 

Self-energy

$$\Sigma(z) = \frac{1}{N}\mathrm{Tr}\left[\hat{T}\right] + \frac{1}{N}\mathrm{Tr}\left[\hat{T}\mathcal{G}(z)\hat{T}\right] + \frac{1}{N}\mathrm{Tr}\left[\hat{T}\mathcal{G}(z)\hat{T}\mathcal{G}(z)\hat{T}\right] + \ldots$$

$$\boxed{\Sigma(z) = \frac{1}{N}\mathrm{Tr}\left[\frac{\hat{T}}{1-\mathcal{G}(z)\hat{T}}\right]}$$

This equation together with $\mathcal{G}(z) = [z - \Sigma(z)]^{-1}$

allows to find $\mathcal{G}(z)$ and then $p(\lambda) = -(1/\pi)\mathrm{Im}\mathcal{G}(z = \lambda + \mathrm{i}\epsilon)$

Free random variable theory

Definition:

Consider two Hermitian matrices $\hat{A}$ and $\hat{B}$ and define

$$\phi(\hat{A}) = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr}\hat{A}\rangle.$$

$\hat{A}$ and $\hat{B}$ are asymptotically free if for all ℓ and for all polynomials $p_i(\cdot)$ and $q_i(\cdot)$ with $1 \le i \le \ell$ such that

$$\phi[p_i(\hat{A})] = \phi[q_i(\hat{B})] = 0,$$

we have

$$\phi[p_1(\hat{A})q_1(\hat{B})\dots p_\ell(\hat{A})q_\ell(\hat{B})] = 0.$$

D.V. Voiculescu et al., *Free Random Variables* (1992)

Free random variable theory

Question: Given that everything is known about the random matrices $\hat{A}$ and $\hat{B}$, what can we say about $\hat{A}+\hat{B}$ and $\hat{A}\times\hat{B}$?

Green's function $\mathcal{G}_{\hat{A}}(z) = \frac{1}{N}\mathrm{Tr}\frac{1}{z-\hat{A}}$

Blue function $B_{\hat{A}}\left[\mathcal{G}_{\hat{A}}(z)\right] = z$

R function $R_{\hat{A}}(z) = B_{\hat{A}}(z) - \frac{1}{z}$

S transform $\frac{1+z}{z\, S_{\hat{A}}(z)}\mathcal{G}_{\hat{A}}\left[\frac{1+z}{z\, S_{\hat{A}}(z)}\right] - 1 = z$

Answer: If $\hat{A}$ and $\hat{B}$ are asymptotically free ($\sim$ independent), then

$$R_{\hat{A}+\hat{B}}(z) = R_{\hat{A}}(z) + R_{\hat{B}}(z) \text{ and } S_{\hat{A}\times\hat{B}}(z) = S_{\hat{A}}(z) \times S_{\hat{B}}(z)$$

D.V. Voiculescu et al., *Free Random Variables* (1992)

Free random variable theory

Using the free random variable theory we readily find

$$B_{\hat{F}}(\mathcal{G}_{\hat{F}}) = \frac{1}{\mathcal{G}_{\hat{F}}} + \Sigma(\mathcal{G}_{\hat{F}}) \quad \text{with} \quad \Sigma(\mathcal{G}_{\hat{F}}) \;=\; \frac{1}{N}\mathrm{Tr}\left[\frac{\hat{T}}{1 - \mathcal{G}_{\hat{F}}\hat{T}}\right]$$

or

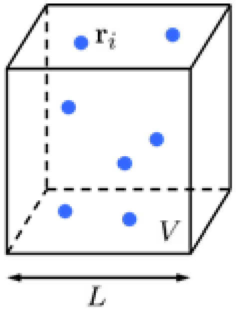
$$\boxed{B_{\hat{F}}(z) = \frac{1}{z}\left\{1 + \frac{M}{N}\left[\frac{1}{z}\mathcal{G}_{\hat{T}}\left(\frac{1}{z}\right) - 1\right]\right\}}$$

Blue function of the matrix $\hat{F} = \hat{H}\hat{T}\hat{H}^{+}$

Green's function of the matrix $\hat{T}$

S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Application: sinc matrix



$$S_{ij} = \mathrm{Im}G(\mathbf{r}_i, \mathbf{r}_j) = \frac{\sin(k_0|\mathbf{r}_i - \mathbf{r}_j|)}{k_0|\mathbf{r}_i - \mathbf{r}_j|} \longrightarrow \hat{S} = \hat{H}\hat{T}\hat{H}^+$$

$k_0 L \gg 1$

$\lambda_0 = \frac{2\pi}{k_0}$

$\rho = \frac{N}{V}$

Diagonal elements: $T_{mm} = \beta = \frac{2.8N}{(k_0 L)^2}$ for $m \le M = \frac{(k_0 L)^2}{2.8}$

Non-diagonal elements: $T_{mn} \sim \frac{T_{mm}}{k_0 L} \sim \rho\lambda_0^3 \ll T_{mm}$

Diagonal approximation: $\hat{T} = \beta \hat{I}_{M \times M}$

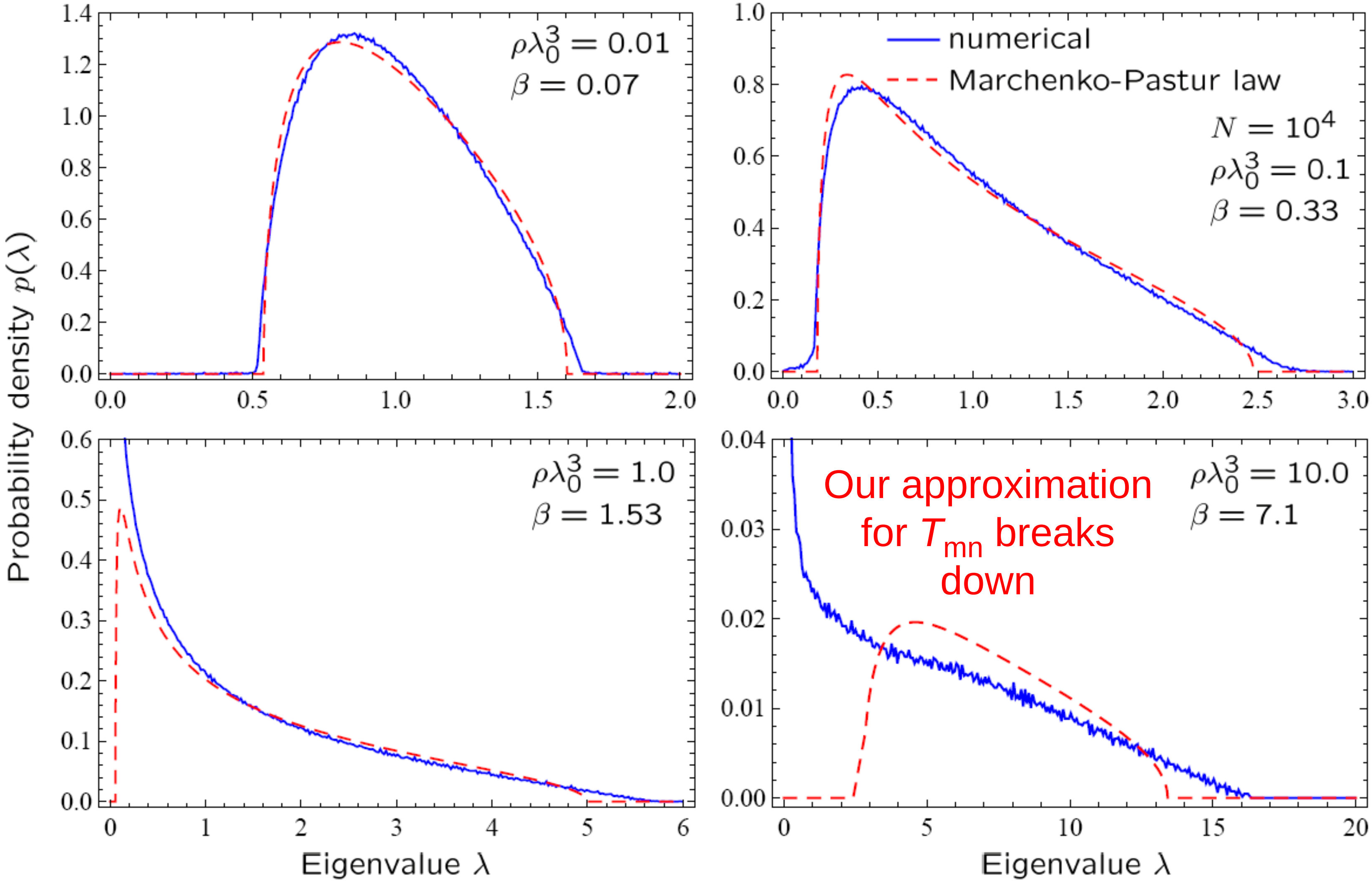
Marchenko-Pastur law:

$$\boxed{p(\lambda) = \left(1 - \frac{1}{\beta}\right)^+ \delta(\lambda) + \frac{\sqrt{(\lambda - \lambda_{\mathrm{min}})^+(\lambda_{\mathrm{max}} - \lambda)^+}}{2\pi\beta\lambda}}$$

$\lambda_{\mathrm{min,max}} = (1 \mp \sqrt{\beta})^2$, $x^+ = \max(x, 0)$

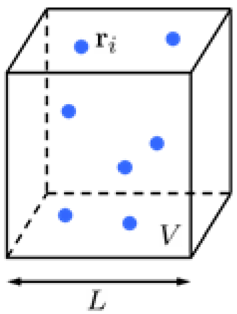
S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Application: sinc matrix



S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Application: cosc matrix



$$C_{ij} = \mathrm{Re}G(\mathbf{r}_i, \mathbf{r}_j) = \frac{\cos(k_0|\mathbf{r}_i - \mathbf{r}_j|)}{k_0|\mathbf{r}_i - \mathbf{r}_j|} \longrightarrow \hat{C} = \hat{H}\hat{T}\hat{H}^+$$

Diagonal approximation: $T_{mn} = \delta_{mn} \frac{\rho\lambda_0^3}{2\pi^2} \frac{1}{(q_m/k_0)^2 - 1}$

$k_0 L \gg 1$

$\lambda_0 = \frac{2\pi}{k_0}$

$\rho = \frac{N}{V}$

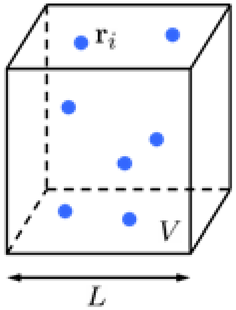
$\beta = \frac{2.8N}{(k_0 L)^2}$

$$\begin{aligned} B_{\hat{C}}(z) &= \frac{1}{z} - \frac{2}{\pi}\operatorname{arccoth}\frac{4\pi^3\beta}{\rho\lambda_0^3} + \frac{2}{\pi}\sqrt{-1 - \frac{\rho\lambda_0^3}{2\pi^2}z} \\ &\times \left[\arctan\frac{1 + \frac{\rho\lambda_0^3}{2\pi^3\beta}}{\sqrt{-1 - \frac{\rho\lambda_0^3}{2\pi^2}z}} - \arctan\frac{1 - \frac{\rho\lambda_0^3}{2\pi^3\beta}}{\sqrt{-1 - \frac{\rho\lambda_0^3}{2\pi^2}z}} - \frac{\pi}{2}\right]. \end{aligned}$$

The Green's function is found from $B_{\hat{C}}\left[\mathcal{G}_{\hat{C}}(z)\right] = z$

S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Application: cosc matrix, low-density limit



$k_0 L \gg 1$

$\lambda_0 = \dfrac{2\pi}{k_0}$

$\rho = \dfrac{N}{V}$

$\beta = \dfrac{2.8N}{(k_0 L)^2}$

$$\rho \lambda_0^3 \ll 1 \longrightarrow B_{\widehat{C}}(z) = \frac{1}{z} - \frac{1}{\pi} \ln \frac{1 - \frac{\pi}{2}\beta z}{1 + \frac{\pi}{2}\beta z}$$

$\underline{\beta \ll 1}$

$$B_{\widehat{C}}(z) = \beta z + \frac{1}{z}$$

Wigner semi-circle law:

$$p(\lambda) = \frac{\sqrt{4\beta - \lambda^2}}{2\pi\beta}$$

$\underline{\beta \gg 1}$

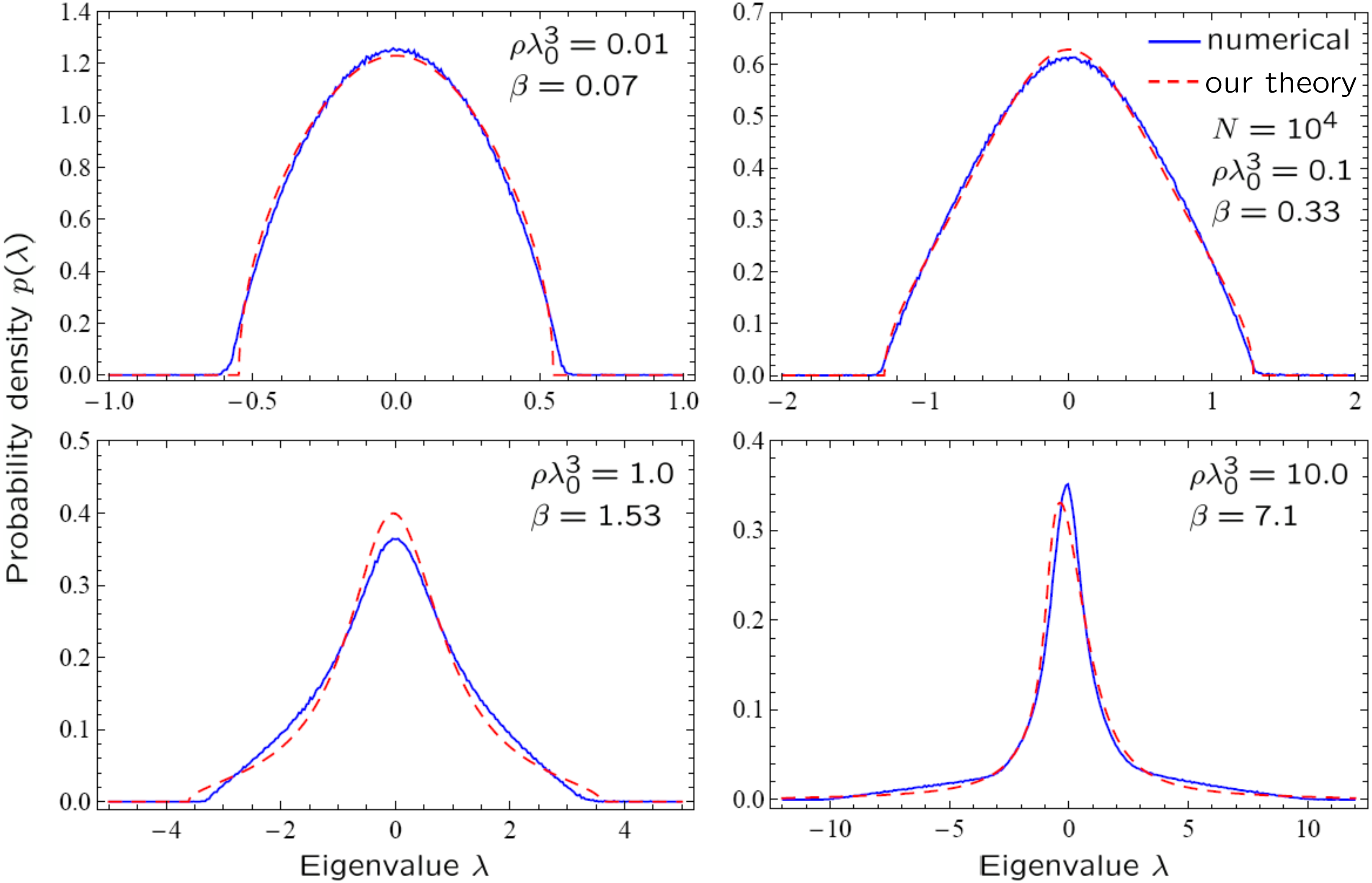
$$B_{\widehat{C}}(z) = -\mathrm{i} + \frac{1}{z}$$

Cauchy distribution:

$$p(\lambda) = \frac{1}{\pi} \frac{1}{1 + \lambda^2}$$

S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Application: cosc matrix



S.E. Skipetrov and A. Goetschy, arXiv:1007.1379

Non-Hermitian matrices: Generalities

$\hat{F}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$, $\hat{F}^+ \neq \hat{F}$, λ_n – complex eigenvalues

Probability density of eigenvalues λ:

$$p(\lambda) = \left\langle \frac{1}{N} \sum_{n=1}^{N} \delta^2(\lambda - \lambda_n) \right\rangle = \frac{1}{\pi} \frac{\partial}{\partial \bar{z}} g(z, \bar{z}) \Big|_{z=\lambda}$$

$$\delta^2(z) = \frac{1}{\pi} \frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2}, \quad \epsilon \to 0^+$$

Green's function (resolvent):

$$g(z, \bar{z}) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{\bar{z} - \hat{F}^+}{(z - \hat{F})(\bar{z} - \hat{F}^+) + \epsilon^2} \right\rangle$$

R.A. Janik et al., Nucl. Phys. B 501, 603 (1997)

Non-Hermitian matrices: Matrix Green's function

$$\hat{F}_{N\times N} \longrightarrow \hat{F}_2 = \begin{pmatrix} \hat{F} & 0 \\ 0 & \hat{F}^+ \end{pmatrix}_{2N\times 2N}$$

$$\hat{\mathcal{G}} = \left\langle \frac{1}{\hat{z}_\epsilon - \hat{F}_2} \right\rangle = \begin{pmatrix} \hat{\mathcal{G}}_{qq} & \hat{\mathcal{G}}_{q\bar{q}} \\ \hat{\mathcal{G}}_{\bar{q}q} & \hat{\mathcal{G}}_{\bar{q}\bar{q}} \end{pmatrix} \quad \text{where} \quad \hat{z}_\epsilon = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix}$$

Green's function (resolvent):

$$g(z,\bar{z}) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{\bar{z} - \hat{F}^+}{(z-\hat{F})(\bar{z}-\hat{F}^+) + \epsilon^2} \right\rangle = \frac{1}{N} \mathrm{Tr} \hat{\mathcal{G}}_{qq}$$

Correlator of right and left eigenvectors:

$$O(z) \;=\; \frac{1}{N} \left\langle \sum_{n=1}^{N} \langle L_n | L_n \rangle \langle R_n | R_n \rangle \delta^2(z - \lambda_n) \right\rangle = -\frac{N}{\pi} (\mathrm{Tr} \hat{\mathcal{G}}_{q\bar{q}})(\mathrm{Tr} \hat{\mathcal{G}}_{\bar{q}q})$$

R.A. Janik et al., Nucl. Phys. B 501, 603 (1997)
Phys. Rev. E 60, 2699 (1999)

Non-Hermitian matrices: Diagrammatic expansion

$$\hat{\mathcal{G}} = \left\langle \frac{1}{\hat{z}_\epsilon - \hat{F}_2} \right\rangle = \left\langle \frac{1}{\hat{z}_\epsilon} + \frac{1}{\hat{z}_\epsilon}\hat{F}_2\frac{1}{\hat{z}_\epsilon} + \frac{1}{\hat{z}_\epsilon}\hat{F}_2\frac{1}{\hat{z}_\epsilon}\hat{F}_2\frac{1}{\hat{z}_\epsilon} + \ldots \right\rangle$$

$$\hat{z}_\epsilon = \begin{pmatrix} z & \mathrm{i}\epsilon \\ \mathrm{i}\epsilon & \bar{z} \end{pmatrix} \qquad \hat{F}_2 = \begin{pmatrix} \hat{F} & 0 \\ 0 & \hat{F}^+ \end{pmatrix}_{2N\times 2N}$$

$$\hat{F} = \hat{H}\hat{T}\hat{H}^+$$

The diagrams can be calculated using the properties of the matrix $\hat{H}$:

$$\langle H_{im} \rangle = 0$$

$$\langle H_{im} H^*_{jn} \rangle = \frac{1}{N}\delta_{ij}\delta_{mn}$$

and keeping only the leading (planar) diagrams in the limit of $N \to \infty$

Non-Hermitian matrices: Results

$$\hat{\mathcal{G}} = \left\langle \frac{1}{\hat{z}_\epsilon - \hat{F}_2} \right\rangle = \begin{pmatrix} \hat{\mathcal{G}}_{qq} & \hat{\mathcal{G}}_{q\bar{q}} \\ \hat{\mathcal{G}}_{\bar{q}q} & \hat{\mathcal{G}}_{\bar{q}\bar{q}} \end{pmatrix}_{2N\times 2N}$$

$$\hat{g} = \frac{1}{N}\mathrm{Tr}_{\mathrm{block}}\hat{\mathcal{G}} = \begin{pmatrix} \frac{1}{N}\mathrm{Tr}\hat{\mathcal{G}}_{qq} & \frac{1}{N}\mathrm{Tr}\hat{\mathcal{G}}_{q\bar{q}} \\ \frac{1}{N}\mathrm{Tr}\hat{\mathcal{G}}_{\bar{q}q} & \frac{1}{N}\mathrm{Tr}\hat{\mathcal{G}}_{\bar{q}\bar{q}} \end{pmatrix}_{2\times 2} = \begin{pmatrix} g & g_2 \\ g_2 & g \end{pmatrix} = \frac{1}{z - \hat{\Sigma}}$$

$$\hat{\Sigma} = \begin{pmatrix} \Sigma & \Sigma_2 \\ \Sigma_2 & \Sigma \end{pmatrix}$$

$$\begin{cases} \Sigma = \dfrac{1}{N}\mathrm{Tr}\left(\dfrac{(1-\bar{g}\hat{T}^{+})\hat{T}}{|1-g\hat{T}|^2 - |g_2\hat{T}|^2}\right) \\[2ex] \Sigma_2 = \dfrac{1}{N}\mathrm{Tr}\left(\dfrac{g_2\hat{T}\hat{T}^{+}}{|1-g\hat{T}|^2 - |g_2\hat{T}|^2}\right) \end{cases}$$

Reminder:

$$p(\lambda) = \frac{1}{\pi}\frac{\partial}{\partial\bar{z}} g(z,\bar{z})\Big|_{z=\lambda}$$

Borderline of the eigenvalue domain

Correlator of right and left eigenvectors:

$$O(z) = \frac{1}{N}\left\langle \sum_{n=1}^{N} \langle L_n|L_n\rangle\langle R_n|R_n\rangle \delta^2(z-\lambda_n)\right\rangle = -\frac{N}{\pi}(\mathrm{Tr}\hat{\mathcal{G}}_{q\bar{q}})(\mathrm{Tr}\hat{\mathcal{G}}_{\bar{q}q})$$

$$= 0$$

$$g_2 = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{1}{|g|^2} = \dfrac{1}{N}\mathrm{Tr}\left(\dfrac{\hat{T}\hat{T}^+}{|1-g\hat{T}|^2}\right) \\[2ex] z = \dfrac{1}{g} + \dfrac{1}{N}\mathrm{Tr}\left(\dfrac{\hat{T}}{1-g\hat{T}}\right) \end{cases}$$

In the $\mathbf{r}$-representation:

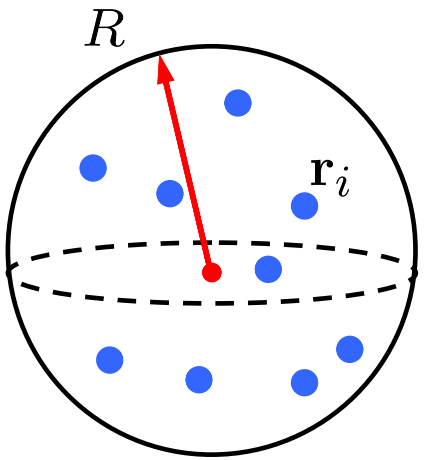
$$\begin{cases} \dfrac{1}{|g|^2} = \dfrac{\rho}{V}\displaystyle\int_V d^3\mathbf{r}\int_V d^3\mathbf{r}'\left|\dfrac{f(\mathbf{r},\mathbf{r}')}{1-\rho g f(\mathbf{r},\mathbf{r}')}\right|^2 \\[2ex] \dfrac{1}{z} = \dfrac{1}{g} + \dfrac{\rho g}{V}\displaystyle\int_V d^3\mathbf{r}\int_V d^3\mathbf{r}'\dfrac{f(\mathbf{r},\mathbf{r}')^2}{1-\rho g f(\mathbf{r},\mathbf{r}')} \end{cases}$$

J.T. Chalker and B. Mehlig, Phys. Rev. Lett. 81, 3367 (1998)
R.A. Janik et al., Phys. Rev. E 60, 2699 (1999)

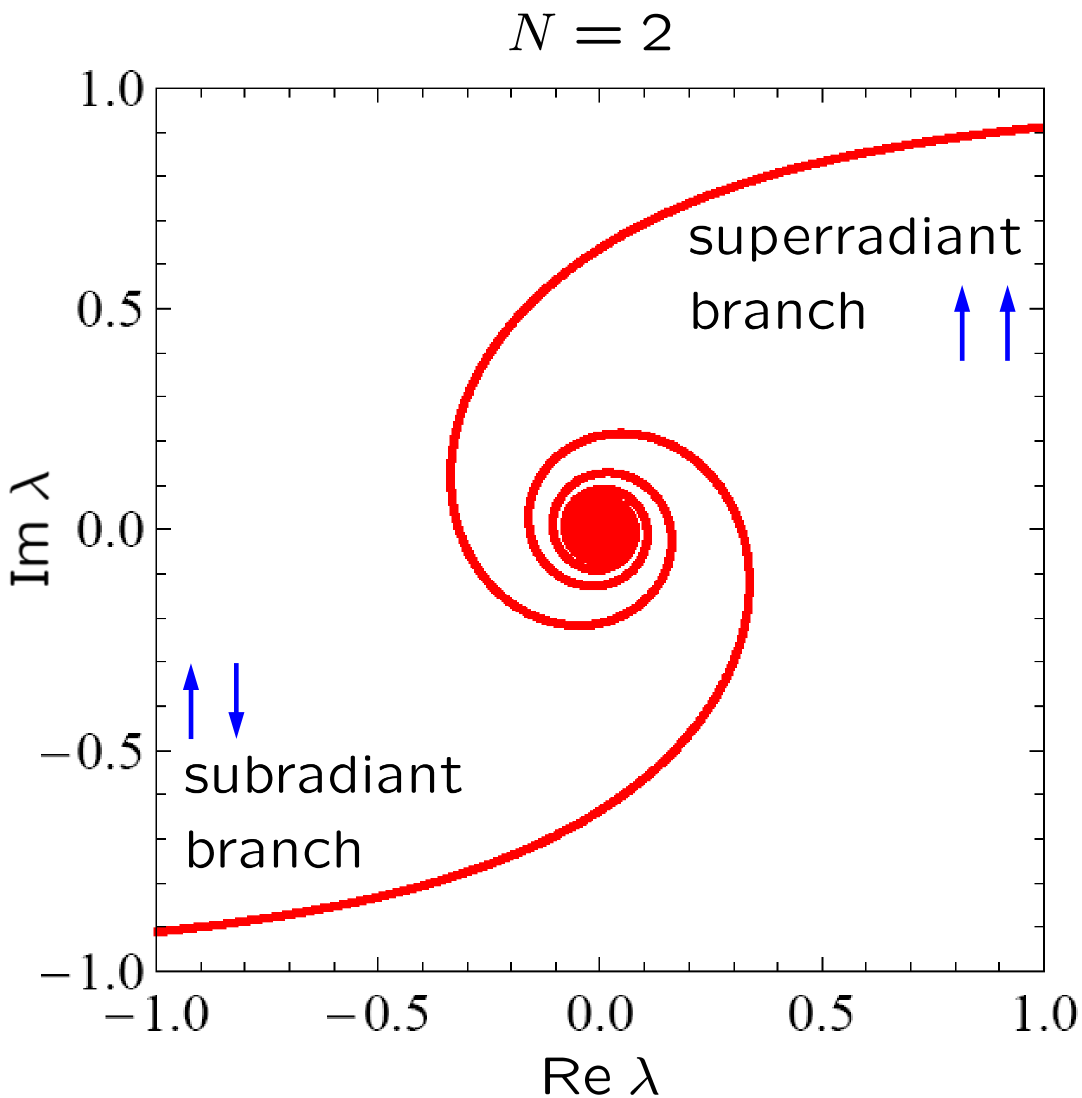
"Green's matrix": $N = 2$

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$



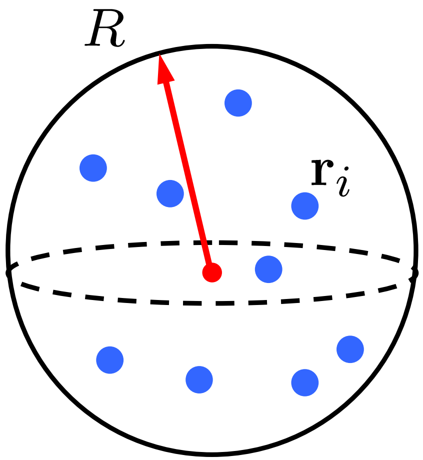
$i = 1, \ldots N$



“Green’s matrix”: N >> 1, low density

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i-\mathbf{r}_j|}}{k_0|\mathbf{r}_i-\mathbf{r}_j|}$$

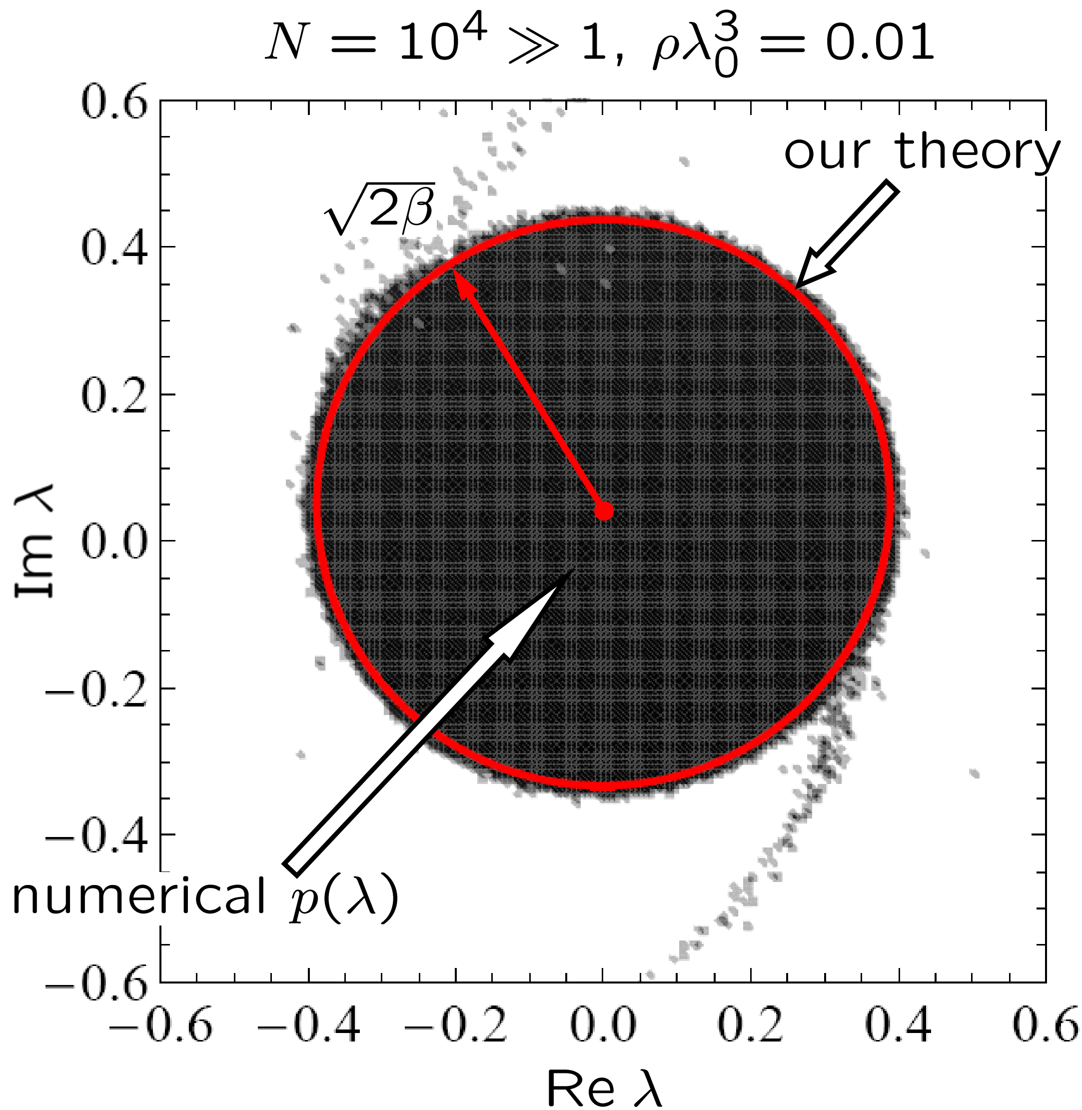
$$\hat{G}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$



$i = 1, \ldots N$

$\lambda_0 = 2\pi/k_0$

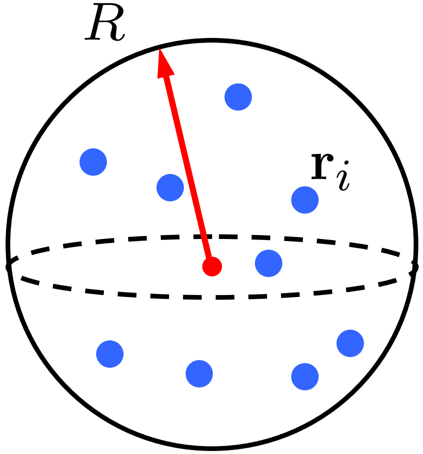
$$\beta = \frac{9}{8}\frac{N}{(k_0 R)^2}$$



"Green's matrix": N >> 1, moderate density

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$

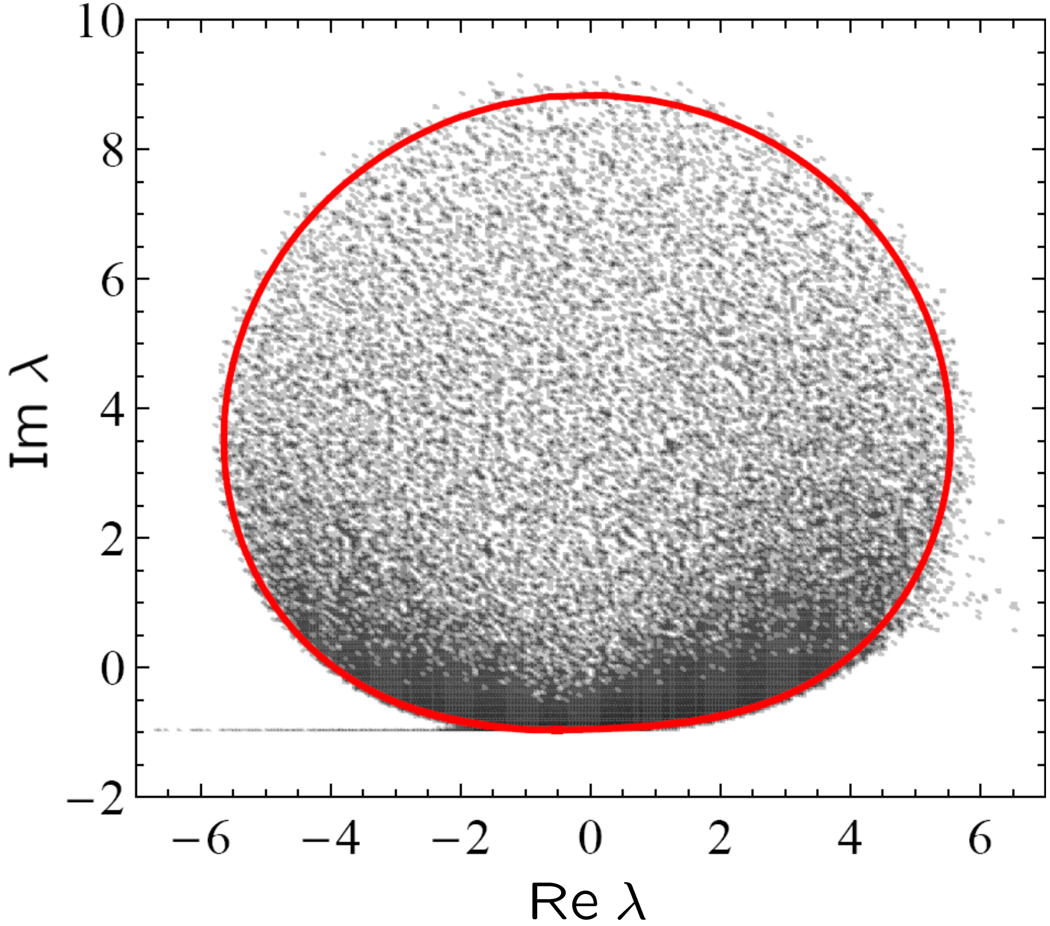


$$i = 1, \ldots N$$

$$\lambda_0 = 2\pi/k_0$$

$$\beta = \frac{9}{8}\frac{N}{(k_0 R)^2}$$

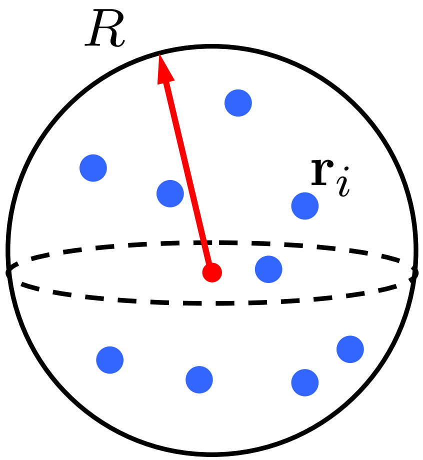
$N = 10^4 \gg 1,\ \rho\lambda_0^3 = 10$



"Green's matrix": *N* >> 1, high density

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$

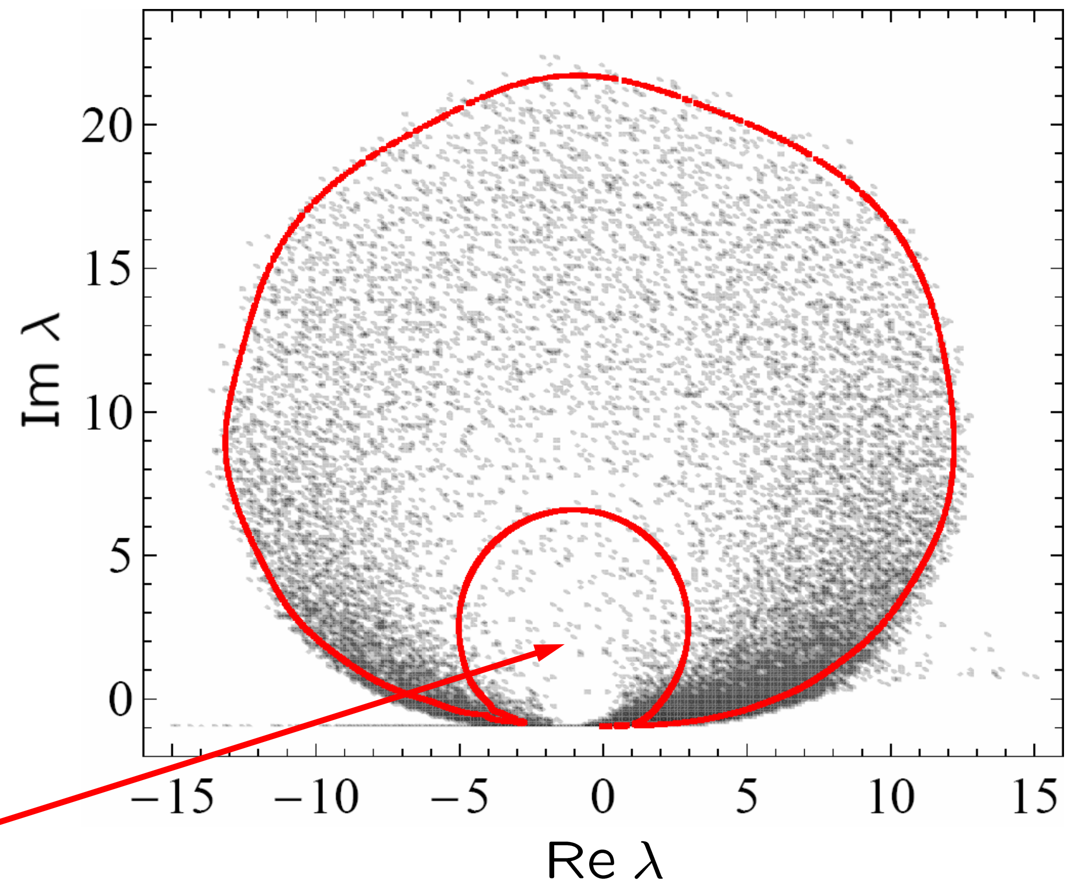


$$i = 1, \ldots N$$

$$\lambda_0 = 2\pi/k_0$$

$$\beta = \frac{9}{8}\frac{N}{(k_0 R)^2}$$

$N = 10^4 \gg 1,\ \rho\lambda_0^3 = 50$

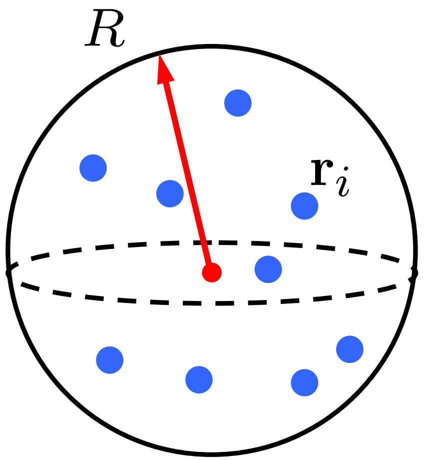


A hole develops in $p(\lambda)$ at $\rho\lambda_0^3 \gtrsim 30$

“Green’s matrix”: *N* >> 1, very high density

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

$$\hat{G}\vec{\Psi}_n = \lambda_n \vec{\Psi}_n$$

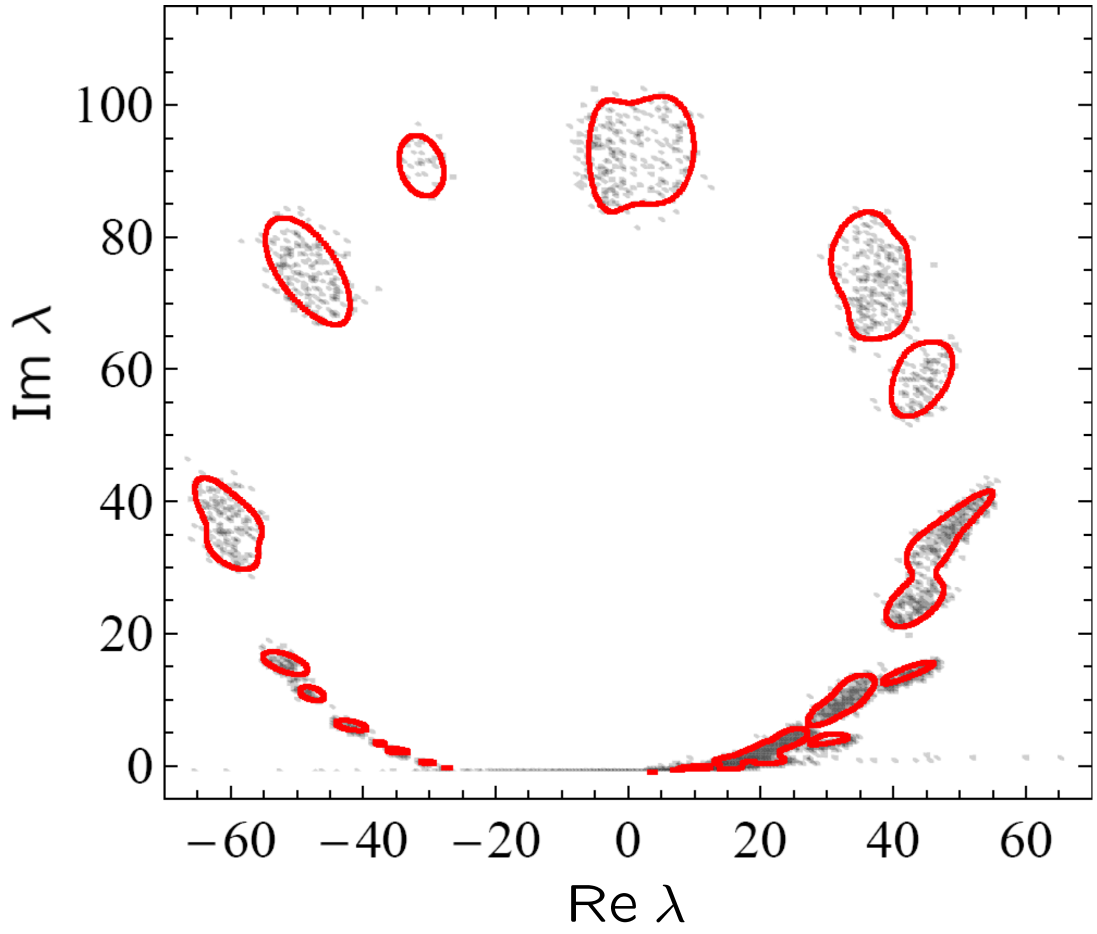


$i = 1, \ldots N$

$\lambda_0 = 2\pi/k_0$

$$\beta = \frac{9}{8}\frac{N}{(k_0 R)^2}$$

$N = 10^4 \gg 1,\ \rho\lambda_0^3 = 500$



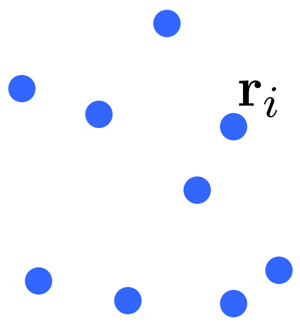
From RMT to scattering theory ... and backwards!

Methods of RMT help to understand the properties
of scattering systems

But can we use our knowledge about wave scattering
to obtain information about the eigenvalue distribution
of the “Green’s matrix” ?

...and the answer is ‘yes’ !

Backward link with the scattering theory



Ensemble of N randomly distributed point-like scatterers with a scattering matrix t

$$G_{ij} = G(\mathbf{r}_i, \mathbf{r}_j) = \frac{e^{\mathrm{i}k_0|\mathbf{r}_i - \mathbf{r}_j|}}{k_0|\mathbf{r}_i - \mathbf{r}_j|}$$

— free-space Green's function

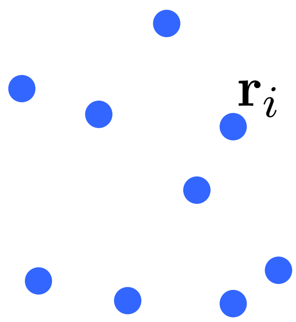
Green's function of Helmholtz equation:

$$\begin{aligned} \mathcal{G}(\mathbf{r}_i, \mathbf{r}_j) &= G(\mathbf{r}_i, \mathbf{r}_j) \quad \textbf{no scattering} \\ &+ \sum_{k=1}^{N} G(\mathbf{r}_i, \mathbf{r}_k)\, t\, G(\mathbf{r}_k, \mathbf{r}_j) \quad \textbf{single scattering} \\ &+ \sum_{k,l=1}^{N} G(\mathbf{r}_i, \mathbf{r}_k)\, t\, G(\mathbf{r}_k, \mathbf{r}_l)\, t\, G(\mathbf{r}_l, \mathbf{r}_j) \quad \textbf{double scattering} \\ &+ \dots \end{aligned}$$

$$\hat{\mathcal{G}} = \frac{\hat{G}}{1 - t\hat{G}} = \frac{1}{\hat{G}^{-1} - t}$$

We set $-k_0/4\pi = 1$ to simplify notation

Backward link with the scattering theory



Ensemble of N randomly distributed point-like scatterers with a scattering matrix t

Green's function of Helmholtz equation:

$$\hat{\mathcal{G}} = \frac{\hat{G}}{1 - t\hat{G}} = \frac{1}{\hat{G}^{-1} - t}$$

Average intensity: $I_{ij} = \langle \mathcal{G}_{ij} \bar{\mathcal{G}}_{ij} \rangle$

Sum over i and average over j:

$$I = \frac{1}{N} \sum_{i,j=1}^{N} I_{ij} = \mathrm{Tr} \left\langle \frac{1}{[t - \hat{G}^{-1}][\bar{t} - (\hat{G}^{-1})^{+}]} \right\rangle$$

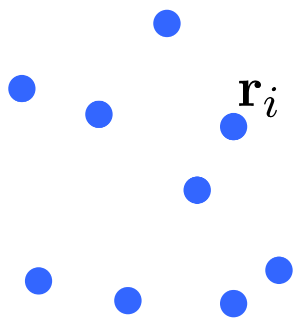
$$= \frac{1}{N} \mathrm{Tr} \left\langle \frac{1}{[t - \hat{G}^{-1}][\bar{t} - (\hat{G}^{-1})^{+}] + \epsilon^2} \right\rangle$$

add $\epsilon^2 \to 0^+$ in the denominator

Outside the domain of existence of eigenvelues of $\hat{G}^{-1}$:

$$g_2 = \frac{-\mathrm{i}\epsilon}{N} \mathrm{Tr} \left\langle \frac{1}{[t - \hat{G}^{-1}][\bar{t} - (\hat{G}^{-1})^{+}] + \epsilon^2} \right\rangle = 0 \quad \Longleftrightarrow \quad -\mathrm{i}\epsilon I = 0$$

Backward link with the scattering theory



Ensemble of N randomly distributed point-like scatterers with a scattering matrix t

$$I = \frac{1}{N} \sum_{i,j=1}^{N} I_{ij}$$

Outside the domain of existence of eigenvelues of $\widehat{G}^{-1}$:

$\epsilon I = 0$

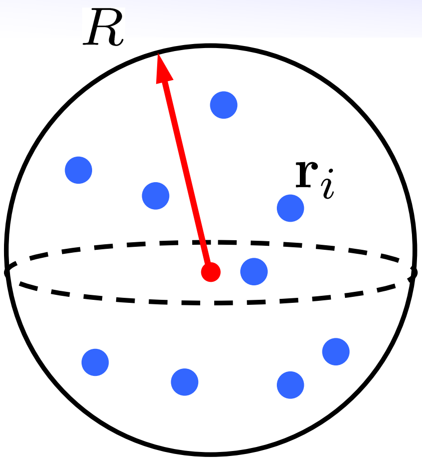
Inside the domain of existence of eigenvelues of $\widehat{G}^{-1}$:

$\epsilon I > 0$

I diverges at the domain boundary

Condition $I \to \infty$ defines the boundary of the domain of existence of eigenvalues of $\widehat{G}^{-1}$ on the complex plane $\lambda = t$ and of $\widehat{G}$ on the complex plane $\lambda = z = 1/t$

Diffusion theory for the eigenvalue domain



Ensemble of N randomly distributed point-like scatterers with a scattering matrix t

$$I = \frac{1}{N} \sum_{i,j=1}^{N} I_{ij}$$

$$I_{ij} = I(\mathbf{r}_i, \mathbf{r}_j)$$

Scattering mean free path: $\ell = \frac{4\pi}{\rho |t|^2}$

Extinction length: $\ell_{\text{ex}} = \frac{k_0}{\rho\, \text{Im} t}$

Absorption/amplification length: $\frac{1}{\ell_a} = \frac{1}{\ell_{\text{ex}}} - \frac{1}{\ell}$

Macrosocpic absorption/amplification length: $L_a^2 = \frac{\ell \ell_a}{3}$

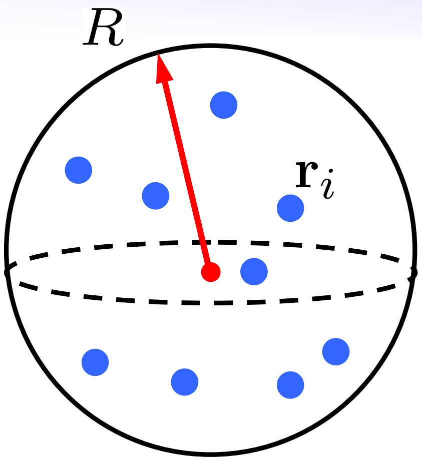
Diffusion equation for the average intensity:

$$\nabla^2 I(\mathbf{r}, \mathbf{r}_j) - \frac{1}{L_a^2} I(\mathbf{r}, \mathbf{r}_j) = \delta(\mathbf{r} - \mathbf{r}_j)$$

Boundary conditions in a sphere:

$$I(\mathbf{r}, \mathbf{r}_j) = 0 \text{ for } r = \frac{2}{3}\ell \frac{1}{1 + \frac{2}{3}\frac{\ell}{R}}$$

Diffusion theory for the eigenvalue domain



Ensemble of N randomly distributed point-like scatterers with a scattering matrix t

$$I = \frac{1}{N}\sum_{i,j=1}^{N} I_{ij}$$

$$I_{ij} = I(\mathbf{r}_i, \mathbf{r}_j)$$

Diffusion equation for the average intensity:

$$\nabla^2 I(\mathbf{r}, \mathbf{r}_j) - \frac{1}{L_a^2} I(\mathbf{r}, \mathbf{r}_j) = \delta(\mathbf{r} - \mathbf{r}_j)$$

Boundary conditions in a sphere:

$$I(\mathbf{r}, \mathbf{r}_j) = 0 \text{ for } r = \frac{2}{3}\ell \frac{1}{1 + \frac{2}{3}\frac{\ell}{R}}$$

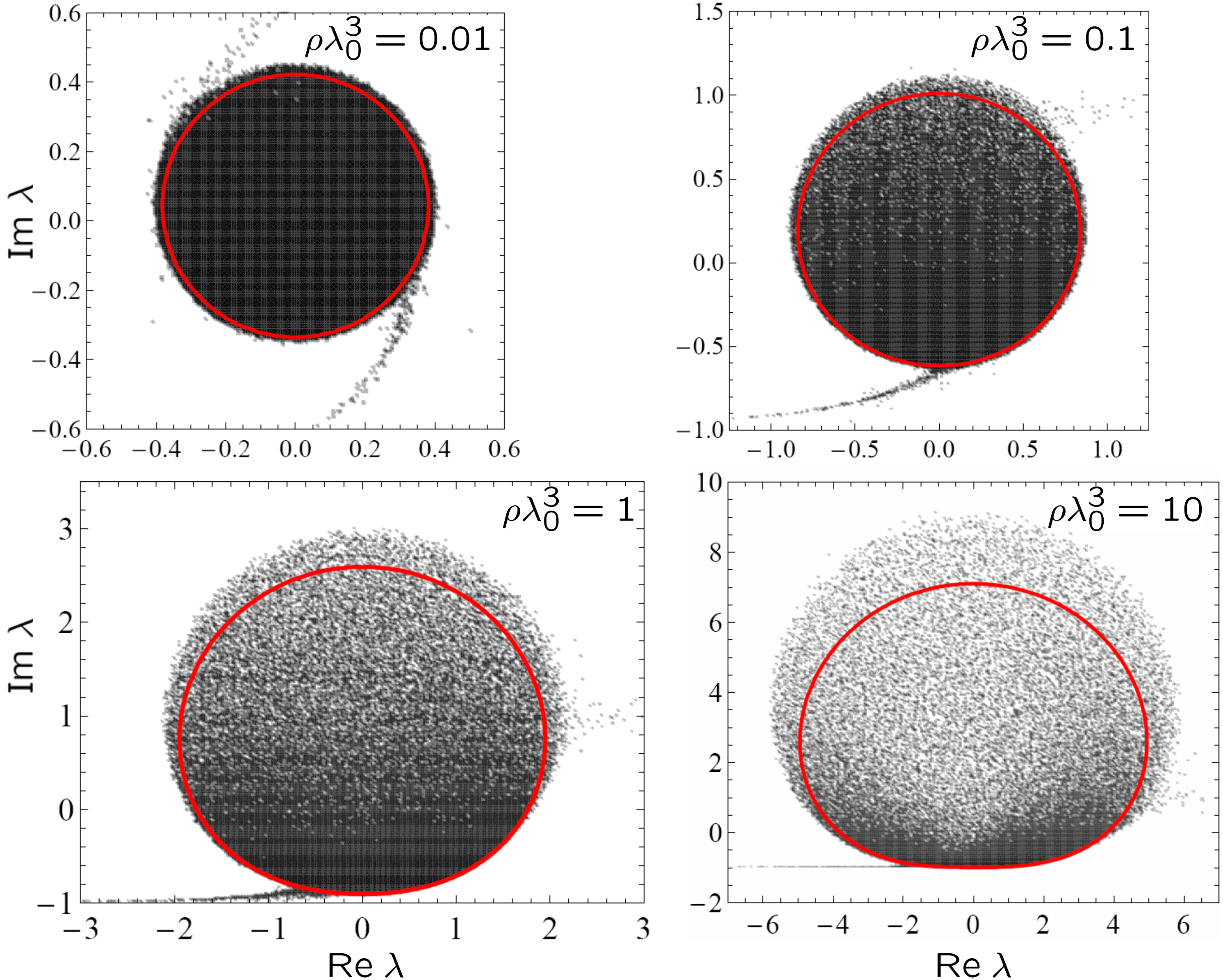
Solution diverges for

$$|z|^4 = \frac{3b_0^2}{(2\pi)^2}(1 + \mathrm{Im}z)\left(1 + \frac{|z|^2}{|z|^2 + \frac{3}{4}b_0}\right)^2$$

optical thickness $b_0 = \frac{16}{3}\beta$

$$\beta = \frac{9}{8}\frac{N}{(k_0 R)^2}$$

Diffusion theory for the eigenvalue domain



Conclusions

An approach is developed to deal with large Hermitian and non-Hermitian Euclidean random matrices

The approach is applied to matrices that appear in the context of wave propagation and scattering in random media

A link between the Euclidean RMT and the “standard” scattering theory (diffusion approximation) is established

The results should be particularly useful to study random lasers