# Higher Order Analogues of the Tracy Widom Distribution and Integrable Hierarchies.

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Based on work with Gernot Akemann in math-ph/1208.3645

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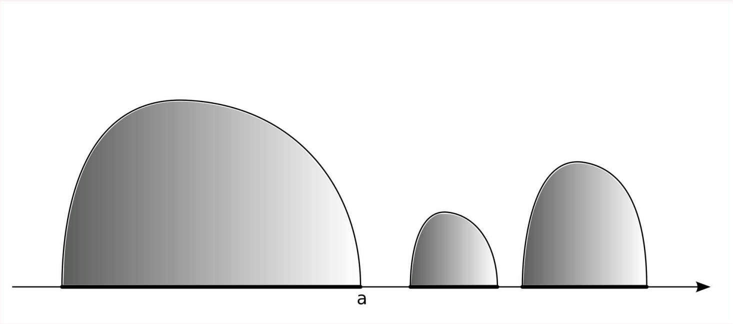
# Outline

# Motivation

Consider the partition function for a $N \times N$ random hermitian matrix,

$$Z = \int [dM] e^{-N\mathrm{Tr}V(M)} = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \Delta(\lambda)^2 e^{-NV(\lambda_i)}$$

The eigenvalues lie on disjoint intervals of the real line.



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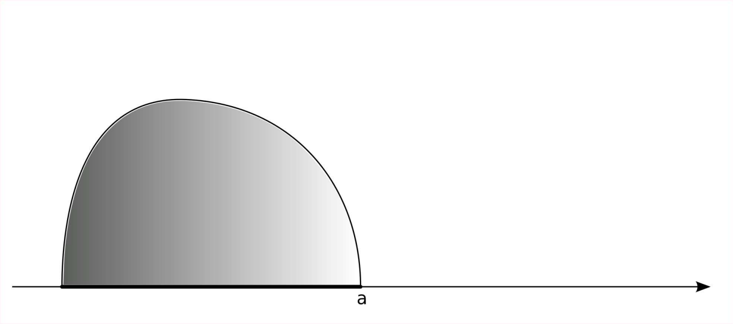
Consider the case of a connected support.



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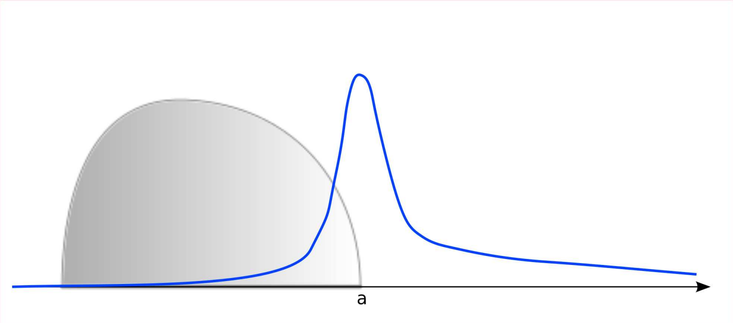
The value of the maximum eigenvalue will be described by some distribution.



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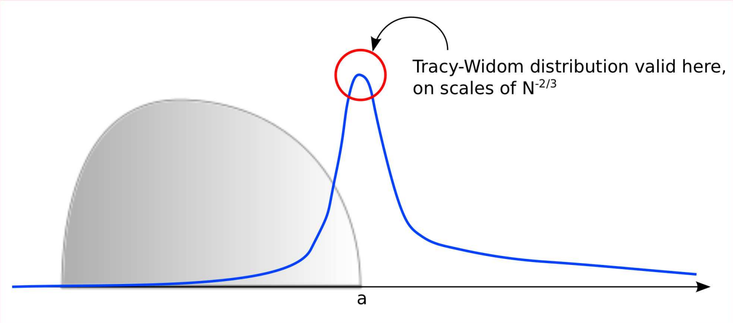
For small fluctuations about the mean, the distribution is given by the Tracy-Widom distribution.

$$\lim_{N\to\infty} \mathbb{P}\left(cN^{2/3}(\lambda_{\max} - a) < s\right) = \exp\left(-\int_s^{+\infty}(y-s)q_0^2(y)dy\right)$$

in terms of the Hastings-McLeod solution $q_0$ of the Painlevé II equation.



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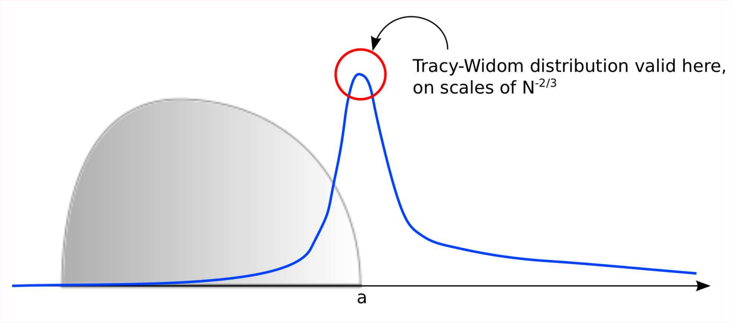
This is a solution to the Painlevé II equation,

$$q_{xx} = xq + 2q^3$$

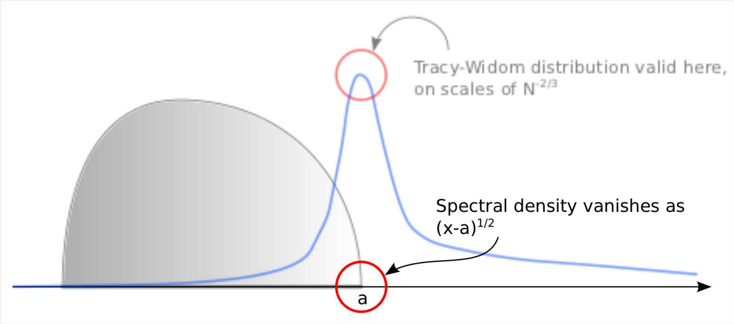
which is defined by the asymptotic behaviour,

$$q_0(x) \sim \mathrm{Ai}(x), \text{ as } x \to +\infty,$$



# Motivation

The appearance of Tracy-Widom is a universal property because the spectral density generically ends with square root behaviour.



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The Tracy-Widom distribution will not hold in these cases.

$$\text{TW} \Rightarrow k\text{-TW}$$

# Painlevé II Hierarchy

## Painlevé II hierarchy

$$\left(\frac{d}{dx}+2q\right)\mathcal{L}_n[q_x-q^2]+\sum_{l=1}^{n-1}\tau_l\left(\frac{d}{dx}+2q\right)\mathcal{L}_l[q_x-q^2]=xq-\alpha,$$

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## Lenard recursion relation

$$\frac{d}{dx}\mathcal{L}_{j+1}f=\left(\frac{d^3}{dx^3}+4f\frac{d}{dx}+2f_x\right)\mathcal{L}_j f,\quad \mathcal{L}_0 f=\frac{1}{2},\quad \mathcal{L}_j 0=0,$$

# Higher Order TW distributions

The $k$-TW distribution has been computed by RH methods.

**Theorem [Claeys, Its, Krasovsky]**

The $2k+1$-TW takes the form,

$$\frac{d}{ds}\log \mathbb{P}^{(2k+1)}(\lambda_{\max} < s) = \int_{-\infty}^{x(s)} u(\xi;\tau_1(s),\ldots,\tau_{2k}(s))^2 d\xi,$$

with $u(x) = u(x;\tau_1,\ldots,\tau_{2k})$ satisfying

$$u''(x) = [q_x(x) + q(x)^2]u(x),$$

## Theorem [Claeys, Its, Krasovsky] (cont.)

$q(x;\tau_1,\ldots,\tau_{2k})$ is a solution of the Painlevé II Hierarchy with

$$x(s) = -2^{\frac{4k+1}{4k+3}} b_0(s), \qquad \tau_j(s) = (2j+1)2^{\frac{4(k-j)+1}{4k+3}} b_j(s)$$

where,

$$b_j(s) = \frac{\Gamma(2k+\frac{3}{2})}{\Gamma(j+\frac{3}{2})\Gamma(2k+2-j)} s^{2k+1-j}$$

## Asymptotics [Claeys, Its, Krasovsky]

For $x \to \infty$ and $x \to -\infty$ we have respectively,

$$q(x) = \frac{1}{2x} + O(x^{-\frac{4n+1}{2n}}), \quad q(x) = \left(\frac{n!^2}{(2n)!}|x|\right)^{\frac{1}{2n}} + O(|x|^{-1}),$$

$$u(x) = 2^{-\frac{4k+1}{4k+3}}\sqrt{\frac{x}{2}}(1+o(1)), \quad u(x) = o(1),$$

# TW Using Orthogonal Polynomials

Recently a simpler derivation of TW was given by [Nadal, Majumdar] using,

$$\mathbb{P}_N(\lambda_{\max} < y; \alpha, \{g_l\}) = \frac{Z_N(y; \alpha, \{g_l\})}{Z_N(\infty; \alpha, \{g_l\})},$$

where,

$$Z_N(y; \alpha, \{g_l\}) \equiv \frac{1}{N!} \int_{-\infty}^{y} \prod_{i=1}^{N} d\lambda_i e^{-N\alpha V(\lambda_i)} \prod_{k>j}^{N} (\lambda_k - \lambda_j)^2$$

$$V(\lambda) \equiv \sum_{l=1}^{\infty} \frac{1}{l} g_l \lambda^l$$

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Introduce polynomials $\{\pi_n\}$ with $\pi_n(\lambda) = \frac{1}{h_n^{1/2}}\lambda^n + \ldots$ and orthonormal with respect to the inner product,

$$\langle \pi_n | \pi_m \rangle \equiv \int_{-\infty}^{y} d\lambda \; \mathrm{e}^{-NV(\lambda)} \pi_n(\lambda)\pi_m(\lambda) = \delta_{nm}.$$

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It is then easy to rewrite,

$$Z_N(y;\alpha,\{g_l\}) = \prod_{i=0}^{N-1} h_i = h_0^N \prod_{i=1}^{N-1} r_i^{N-i}.$$

Here we have defined the ratios

$$r_n = \frac{h_n}{h_{n-1}}.$$

# TW Using Orthogonal Polynomials

The inner product has the simple property,

$$\langle \lambda\pi_n | \pi_m \rangle = \langle \pi_n | \lambda\pi_m \rangle,$$

which leads to the standard 3-term recurrence,

$$\lambda\pi_n = \sqrt{r_{n+1}}\pi_{n+1} + s_n\pi_n + \sqrt{r_n}\pi_{n-1}$$

The derivation of [Nadal, Majumdar] proceeds from here and derives recursion relations between $r_n$ and $s_n$.

# Main Result

In recent work we extended the orthogonal polynomial analysis to the $k$-TWs and found more compact expressions,

**Theorem [Akemann, MA]**

$$\log \mathbb{P}^{(k)}(s) = -\int_{-\infty}^{0} dx \frac{x}{2} \left(u(x,s) - u(x,\infty)\right)$$

$u(x,s)$ satisfies,

$$\mathcal{L}'_{k+1}[u] - 4s\mathcal{L}'_k[u] = xu'(x) + 2u(x) - 2s \ ,$$

This is Painlevé 34.

# Proof of the Main result: The Plan

- Introduce operators $B$, $A$ and $C$, acting on the vector $\pi = (\pi_1, \ldots, \pi_n)$, which have the effects $B\pi = \lambda\pi$, $A\pi = \partial_\lambda \pi$ and $C\pi = \partial_y \pi$.
- Construct local operators $P$ and $H$, from $A$,$B$ and $C$ which are anti-symmetric and find their commutation relations. This gives recursion relations for computing $Z_N(y; \alpha, \{g_l\})$ at finite $N$.
- Using the technique of pseudo-differential operators take the scaling limit of these commutation relations.
- Compute the scaling limit of $Z_N(y; \alpha, \{g_l\})$.

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# Introduce operators $B$, $A$ and $C$

In the following a “local” operator will mean a matrix with a finite number of non-zero diagonals,

$$\begin{pmatrix} \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots \end{pmatrix} \to \partial$$

- Due to the three term recurrence relation $B$ is local.
- $A$ and $C$ are not local.
- However, they have the commutations relations with $B$ of,

$$[B, A] = 1 \qquad [B, C] = -\partial_y B$$

# Flow Equation - finite $N$

We want to redefine the operators $A$ and $C$ so that they are local.

We mimic the derivation of the usual string equations. Introduce

$$P_{nm} \equiv A_{nm} + C_{nm} - \frac{N}{2} V'(B)_{nm}$$

It can be shown by integration by parts that $P$ is local and anti-symmetric. Hence,

$$P = -\frac{N}{2}(V'(B)_+ - V'(B)_-)$$

It also satisfies,

**Flow Equation**

$$[P, y - B] = \partial_y(y - B) = 1 - \partial_y B$$

# String Equation - finite $N$

A second local operator can be constructed by noting,

$$\int_{-\infty}^{y} d\lambda \partial_\lambda \left[ (\lambda - y) \ \mathrm{e}^{-NV(\lambda)} \pi_n(\lambda) \pi_m(\lambda) \right] = 0$$

and expressing the action of the derivative in terms of $A$, $B$ and $C$.

This leads to the local anti-symmetric operator,

$$H = -\frac{N}{2} \Big( (V'(B)(B-y))_+ - (V'(B)(B-y))_- \Big)$$

which satisfies,

**String Equation**

$$[B - y, H] = B - y \tag{1}$$

# Large $N$ limit

As $N \to \infty$, $n/N \to \xi$ and in the **one-cut** case,

$$\pi_n \to \pi(\xi, y), \qquad r_n \to r(\xi, y), \qquad s_n \to s(\xi, y)$$

which are smooth functions.

The operators, $B$, $P$ and $H$ can be expanded in powers of $N$, with each order a differential operator in $\xi$.

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The operators, $B$, $P$ and $H$ can be expanded in powers of $N$, with each order a differential operator in $\xi$.

The partition function can then be written as,

$$\log[Z_N(y; \alpha, \{g_l\})] = N^2 \int_0^1 d\xi\, (1-\xi) \log[r(\xi, y)] + O(1/N)$$

# Double Scaling limit

We want to double scale about the point when $y$ collides with the spectral density, set;

$$y = y_c + \epsilon c_1 s \qquad N = \epsilon^{-(2k+1)/2} \tag{2}$$

and let $\epsilon \to 0$.

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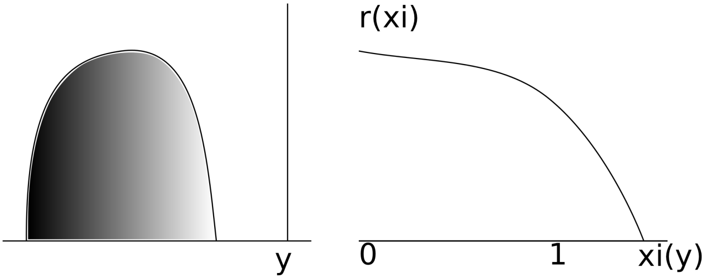
One can think of $y$ colliding with the eigenvalues as a phase transition.



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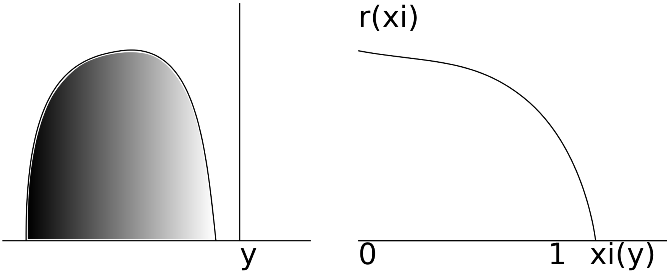

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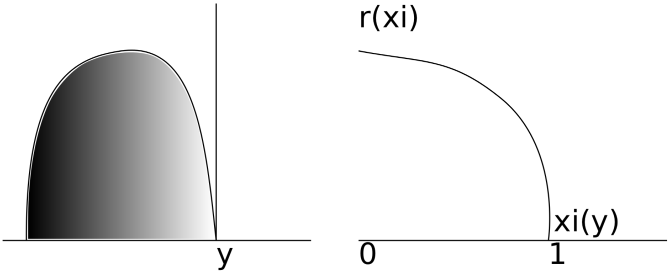

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The functions $r$ and $s$ will also scale with $\epsilon$.

$$\begin{aligned} r(\xi, y) &= r_c(1 + \epsilon\rho(x, s) + O(\epsilon^{3/2})) \ , \\ s(\xi, y) &= \sqrt{r_c}(s_c + \epsilon\sigma(x, s) + O(\epsilon^{3/2})) \end{aligned}$$

It will also be useful to introduce the functions,

$$u \equiv \rho + \sigma \ , \quad v \equiv \rho - \sigma \ .$$

# Double Scaling Limit of $[B, P]$

As $\epsilon \to 0$,

**d.s.l for B**

$$B \to \left(B_c + \epsilon\sqrt{r_c}\mathcal{B} + O(\epsilon^2)\right)$$

independent of $k$ with $\mathcal{B} \equiv d_x^2 + u(x)$. This is because $N^{-1}\partial_\xi \to -\epsilon^{(2k+1)/2}\epsilon^{-k}d_x = -\epsilon^{1/2}d_x$.

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## d.s.l for P

From,

$$P = -\frac{N}{2}(V'(B)_+ - V'(B)_-)$$

we see $P \to \epsilon^{m/2-(2k+1)/2} \times (d^m + \ldots)$.

How do we fix $m$?

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Use the flow equations,

$$[P, y - B] = \partial_y (y - B) = 1 - \partial_y B$$

In order for scaling dimension of both sides of the flow equation to match, $P$ must scale scale as $P \to \epsilon^{-1}\mathcal{P} + \ldots$. Hence,

$$m = 2k - 1 \tag{6}$$

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**Reduction of Commutators**

The equation, $[\mathcal{B}, \mathcal{P}] = f(x)$ can be reduced to an ODE by using pseudo-differential operators.

# Pseudo-differential Operators

A pseudo-differential operator is a Laurent series in $\partial$, where $\partial^{-1}$ acts as an anti-derivative.

## Useful Properties

- It can be shown that $[\mathcal{B}, \mathcal{P}] = f(x)$ is solved by $\mathcal{P} = (\mathcal{B}^{l/2})_+$ for any $l$.
- $[\mathcal{B}, (\mathcal{B}^{(2m-1)/2})_+] = -4^{1-m}\mathcal{L}'_m[u]$

We now can study the double scaling limit of both the flow and string equations.

# Back to the Double Scaling Limit

Lets see how this works for the flow equation,

**d.s.l of flow equation**

$$[P, y - B] = 1 - \partial_y B \to \partial_s u(x,s) = \partial_x \left(x - 2\mathcal{L}_k[u]\right)$$

and crucially $v$ satisfies an equation independent of $s$.

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and crucially $v$ satisfies an equation independent of $s$.

For the string equation,

$$\epsilon(\epsilon^{(m-2k-1)/2}[\mathcal{B},\mathcal{H}] + O(\epsilon^{(m-2k-1)/2+1})) = \epsilon(d^2 + u - s) + O(\epsilon^2),$$

Having both sides of the string equation scale with the same dimension fixes the scaling dimension of $H$.

# Back to the Double Scaling Limit

This fixes $\mathcal{H}$ to be a differential operator of order $2k+1$
We have,

$$[\mathcal{B}, \mathcal{H}] = (d^2 + u - s)$$

To reduce the scaled string equation we must modify $\mathcal{H}$ slightly since the RHS is a differential operator. Let $\mathcal{H} \equiv \bar{\mathcal{H}} + \frac{1}{4}\{x, d\}$, then,

$$[\mathcal{B}, \bar{\mathcal{H}}] = \frac{1}{2} x u'(x) + u(x) - s$$

which can be reduced by pseudo-differential operators. We have,

$$\bar{\mathcal{H}} = \left( (\mathcal{B}^{(2k+1)/2})_+ - s(\mathcal{B}^{(2k-1)/2})_+ \right).$$

Which after rescalings gives,

$$\mathcal{L}'_{k+1}[u] - 4s\mathcal{L}'_k[u] = xu'(x) + 2u(x) - 2s.$$

# Back to the Double Scaling Limit

Finally we double scale the expression for $Z_N$ when $N$ is large,

$$\log[Z_N(y;\alpha,\{g_l\})] = N^2 \int_0^1 d\xi\,(1-\xi)\log[r(\xi,y)] + O(1/N)\ .$$

it takes the form,

$$\log Z^{(k)}(s) = -\int_{-\infty}^{0} dx \frac{x}{2}\,(u(x,s)+v(x,s))\ ,$$

where we have introduced the scaled partition function $\log Z^{(k)}(s)$. We have for the double scaling limit of the gap probability,

$$\log \mathbb{P}^{(k)}(s) = -\int_{-\infty}^{0} dx \frac{x}{2}\,(u(x,s)-u(x,\infty))$$

# Main Result

**Theorem [Akemann, MA]**

$$\log \mathbb{P}^{(k)}(s) = -\int_{-\infty}^{0} dx \frac{x}{2} \left(u(x,s) - u(x,\infty)\right)$$

$u(x,s)$ satisfies,

$$\mathcal{L}'_{k+1}[u] - 4s\mathcal{L}'_{k}[u] = xu'(x) + 2u(x) - 2s \ ,$$

# Bäcklund Transformation

How does our result relate to the earlier result of Claeys et. al?

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We can relate these two formulation using a BT found in [Clarkson, Joshi, Pickering].

$$2\mathcal{L}_k[u] - x = 2\psi(x)^2 \qquad \text{and} \qquad \psi''(x) + (u-s)\psi(x) = 0$$

and

$$2\mathcal{L}_k[W' - W^2 + s] - x = 2\psi(x)^2 \qquad \text{and} \qquad \psi'(x) + W\psi(x) = 0$$

For,

$$2\mathcal{L}_k[u] - x = 2\psi(x)^2 \qquad \text{and} \qquad \psi''(x) + (u - s)\psi(x) = 0$$

Eliminating $\psi$ gives,

$$\mathcal{L}'_{k+1}[u] - 4s\mathcal{L}'_k[u] = xu'(x) + 2u(x) - 2s \ ,$$

On the other hand eliminating $u$ gives,

$$2\mathcal{L}_k\left[s - \frac{\psi''}{\psi}\right] - x = 2\psi^2(x) \ .$$

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Eliminating $W$ gives again,

$$2\mathcal{L}_k\left[s - \frac{\psi''}{\psi}\right] - x = 2\psi^2(x)$$

On the other hand eliminating $\psi$ gives,

$$(d + 2W)\mathcal{L}_k[W' - W^2 + s] = xW + \frac{1}{2}$$

which looks a lot like the Painlevé II hierarchy.

# Shift Identity

We can make this more explicit by proving a lemma,

**Shift Identity**

For a constant $z$, we have,

$$\mathcal{L}_k[u(x)+z] = \sum_{j=0}^{k} (4z)^{k-j} \frac{\Gamma(k+1/2)}{\Gamma(k-j+1)\Gamma(j+1/2)} \mathcal{L}_j[u(x)]$$

This allows us to rewrite our result in precisely the manner of Claeys et. al!

# Summary

- Reviewed the higher-order analogues of TW that arises when the behaviour of the spectral density changes at the support edge.
- Shown that by extending the method introduced by Nadal and Madjumdar we can significantly simplify the results of Claeys et al.
- Gave a relation between the results of Claeys and ours via a Backlund transformation.
- Future; can we find examples of where such distributions arise? Can a similar analysis be done for other models?