Universal microscopic correlation functions for products of independent Ginibre matrices

Zdzislaw Burda

Institute of Physics
Jagiellonian University

BB RMT Workshop Dec 2012

Based on joint work with Gernot Akemann (University of Bielefelfd)

J. Phys. A: Math. Theor. 45 (2012) 465201

Product of iid Ginibre matrices

- Ginibre matrices
  - $X = \{X_{ij}\}_{1 \le i,j \le N}$
  - X_{ij} i.i.d. normal (standardised) complex variables
  - PDF of X_{ij}:

$$p(z)\, d^2z = \frac{1}{\pi} e^{-|z|^2} d^2z$$

  - Rescaled form: $x = \left\{ X_{ij}/\sqrt{N} \right\}_{1 \le i,j \le N}$
- Product of i.i.d. Ginibre matrices:

$$P = X_1 X_2 \dots X_n$$

- Eigenvalue properties of P in the limit $N \to \infty$?

Product of i.i.d. normal random variables

- PDFs: $p(z_i) = \frac{1}{\pi} \exp\left[-|z_i|^2\right] \quad , \quad i = 1, 2$
- What is the PDF of the product: $z = z_1 z_2 \quad ?$

$$P_2(z) = \int_C \int_C d^2 z_1 d^2 z_2 \, \delta^{(2)} \left(z - z_1 z_2\right) \, p(z_1) p(z_2)$$

- Polar coordinates: $z_i = \sqrt{\rho_i} e^{i\phi_i}; \quad \rho_i = |z_i|^2$

$$P_2(\rho) = \frac{1}{\pi} \int_0^\infty \int_0^\infty d\rho_1 d\rho_2 \, \delta\left(\rho - \rho_1 \rho_2\right) \, e^{-\rho_1} e^{-\rho_2}$$

- Solution:

$$P_2(\rho) = \frac{1}{\pi} \int_0^\infty \frac{d\rho_2}{\rho_2} e^{-\rho/\rho_2} e^{-\rho_2} = \frac{2}{\pi} K_0(2\sqrt{\rho})$$

Mellin transform

- $z = \underbrace{z_1 z_2 \ldots z_n}_{w} \underbrace{z_{n+1}}_{v}$

$$P_{n+1}(z) = \int\int d^2w d^2v\, \delta^{(2)}(z - wv) P_n(w) \frac{1}{\pi} e^{-|v|^2}$$

- Rewritten in the variable $\rho = |z|^2$:

$$P_{n+1}(\rho) = \int_0^\infty \frac{dr}{r} P_n(\rho/r) e^{-r}$$

- Mellin transform $|z|^2 \to s$

$$M_{n+1}(s) = M_n(s)\Gamma(s) \;\to\; M_n(s) = \frac{1}{\pi}\Gamma^n(s)$$

- Inverse Mellin transform

$$P_n(\rho) = \frac{1}{2\pi i} \int_{c-\imath\infty}^{c+\infty} \frac{1}{\pi} \Gamma^n(s) \rho^{-s} ds = \frac{1}{\pi} G^{n\,0}_{0\,n}\left(\left. \begin{matrix} - \\ \vec{0} \end{matrix} \right| \rho \right)$$

- Meijer G-function, $\rho = |z|^2$

Explicit formulae

- PDF for $n = 1$

$$P_1(z) = \frac{1}{\pi} \exp\left[-|z|^2\right] \ = \ \frac{1}{\pi} G^{1\,0}_{0\,1}\left(\left. {- \atop 0} \right| \ |z|^2 \right)$$

- PDF for $n = 2$

$$P_2(z) \ = \frac{2}{\pi} K_0(2|z|) \ = \ \frac{1}{\pi} G^{2\,0}_{0\,2}\left(\left. {- \atop 0,0} \right| \ |z|^2 \right)$$

- PDF for $n = 3$

$$P_3(z) \ = \ \frac{4}{\pi} \int_0^\infty \frac{dr}{r} K_0\left(2|z|/r\right) \exp[-r^2] \ = \ \frac{1}{\pi} G^{3\,0}_{0\,3}\left(\left. {- \atop 0,0,0} \right| \ |z|^2 \right) \ .$$

- etc

Asymptotic behaviour

- For $|z| \to \infty$

$$P_n(z) \sim \frac{1}{\pi\sqrt{n}} (2\pi)^{(n-1)/2} |z|^{(1-n)/n} e^{-n|z|^{2/n}}$$

- For $n = 2, 3$:

$$P_2(z) \sim \pi^{-1/2} z^{-1/2} e^{-2|z|} \ , \ P_3(z) \sim \frac{2}{\sqrt{3}} |z|^{-2/3} e^{-3|z|^{2/3}} \ ,$$

- For $|z| \to 0$

$$P_n(z) \sim \frac{1}{\pi(n-1)!} \left(-\ln|z|^2\right)^{n-1}$$

Eigenvalue density of the product

- Ginibre matrices
- $X = \{X_{ij}\}_{1\le i,j\le N}$
- X_{ij} i.i.d. normal (standardised) complex variables with PDF

$$p(z)\; d^2z = \frac{1}{\pi} e^{-|z|^2} d^2z$$

- Rescaled form: $x = \left\{ X_{ij}/\sqrt{N} \right\}_{1\le i,j\le N}$
- Product $p = x_1 x_2 \ldots x_n$

$$\rho(z) = \lim_{N\to\infty} \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta^{(2)}\left(z - \lambda_i\right) \right\rangle$$

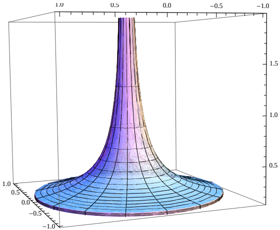
Solution

- Z. Burda, R. A. Janik, B. Waclaw, Phys. Rev. E 81, 041132 (2010)

$$\rho_n(z) = \begin{cases} \frac{1}{n\pi}|z|^{-2+2/n} & |z| \le 1 \\ 0 & |z| > 1 \end{cases}$$

- Example $n = 2$

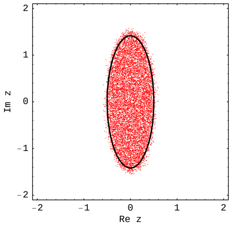
$$\rho_2(z) = \frac{1}{2\pi|z|}$$



Universality

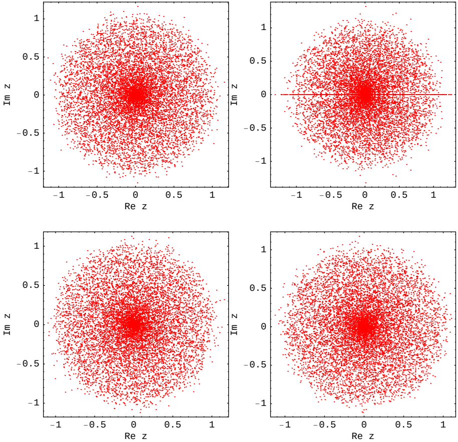
- Non-Gaussian Wigner matrices (Soshnikov, O'Rourke);
- Elliptic ensembles (a, b - iid GUE)

$$x = \cos(\alpha) a + i \sin(\alpha) b$$



- $\alpha = \frac{\pi}{3} \leftrightarrow$ figure;
  $\alpha = 0 \leftrightarrow$ Hermitian; $\alpha = \frac{\pi}{2} \leftrightarrow$ anti-Hermitian; $\alpha = \frac{\pi}{4}$ Ginibre;
- Limiting density of $p = x_1 x_2 \ldots x_n$ is independent of $\alpha_1, \ldots \alpha_n$

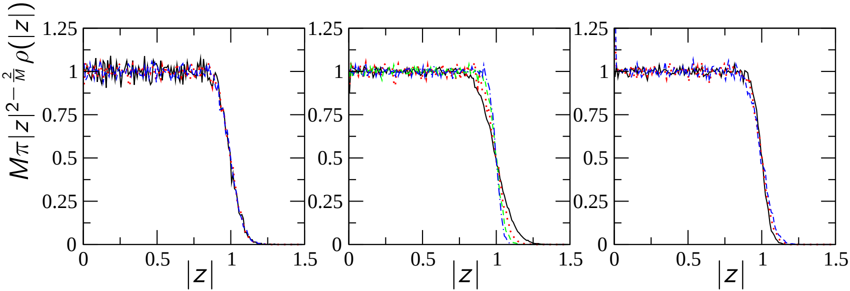
Illustration(MC)



- $X = X_1 X_2$
- MC 100, 100x100

1. G-G · G-G
2. RW · RW (unif. distr.)
3. GUE · G-G
4. GUE · Elliptic ($\alpha = \pi/3$)

Radial profiles



- left: $X = X_1 X_2$ for GUE · GUE; G-G · G-G; Elliptic · GUE;
- middle: $X = X_1 X_2$ for $N = 50, 100, 200, 400$;
- right: $X = X_1 \ldots X_M$ for $M = 2, 3, 4$;

Self averaging and spectral commutativity

- Self-averaging:
- $p = x_1 x_2 \dots x_n$ where x_i iid Ginibre matrices
- $q = x^n$ where x is a single Ginibre matrix
- Self-averaging holds for isotropic matrices $x = hu$; $N \to \infty$
- h semi-positive Hermitian, u - Haar unitary; h, u independent;
- Z.Burda, M.A. Nowak, A. Swiech, PRE in press, arXiv:1205.1625

- Spectral commutativity:
- $xyzyx$ and $x^2 y^2 z$ have the same eigenvalue density in the limit $N \to \infty$
- See poster by Artur Swiech
- U. Haagerup and F. Larsen, J. of Functional Anal., 176, 331 (2000).

Sketch of the solution

- Product matrix $N \times N$:

$$P = X_1 X_2 \dots X_n$$

- Partition function:

$$Z = \int \prod_{j=1}^{n} |DX_j| \; e^{-\mathrm{Tr} X_j^\dagger X_j}$$

- Block matrix $nN \times nN$

$$B = \begin{pmatrix} 0 & X_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & X_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & X_3 & \dots & 0 \\ & & & & \dots & \\ 0 & 0 & 0 & 0 & \dots & X_{n-1} \\ X_n & 0 & 0 & 0 & \dots & 0 \end{pmatrix} .$$

- B^n has n-copies of identical eigenvalues as P

The goal

- Integrate all degrees of freedom except eigenvalues of P (or B)
- Joint probability: $P(z_1, z_2, \ldots, z_N)$
- Correlation functions:

$$R_k(z_1, \ldots, z_k) = \frac{N!}{(N-k)!} \int d^2 z_{k+1} \ldots d^2 z_N P(z_1, z_2, \ldots, z_N)$$

- Unfolding: $\xi = z^{1/n}$

$$\int_{|z| \leq \sqrt{N}^n} \frac{1}{n\pi} z^{-2+2/n} d^2 z \longrightarrow \int_{|\xi| \leq \sqrt{N}} \frac{1}{\pi} d^2 \xi$$

Change of variables

- Generalised Schur decomposition $(n = 2$, Osborn)

$$B = \begin{pmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ X_3 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix} \begin{pmatrix} 0 & \Lambda_1 + T_1 & 0 \\ 0 & 0 & \Lambda_2 + T_2 \\ \Lambda_3 + T_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1^{-1} & 0 & 0 \\ 0 & U_2^{-1} & 0 \\ 0 & 0 & U_3^{-1} \end{pmatrix}$$

- $\Lambda_j = \mathrm{diag}(x_{j1}, \ldots, x_{jN})$; T_j triangular; U_j unitary, coset $U(N)/U(1)^N$;

$$Z = C \int \prod_{j=1}^{3} \prod_{a=1}^{N} dx_{ja} \prod_{b>a}^{N} \left| x_{1b}x_{2b}x_{3b} - x_{1a}x_{2a}x_{3a} \right|^2 \prod_{a=1}^{N} \prod_{i=1}^{3} \frac{1}{\pi} e^{-|x_{ja}|^2}$$

- Eigenvalues of the product: $z_a = x_{1a}x_{2a}x_{3a}$

Solution

- Standard form:

$$Z = C \int \prod_{a=1}^{N} \left(d^2 z_a w(z_a) \right) \prod_{b<a}^{N} \left| z_b - z_a \right|^2$$

- but with a not-standard weight function

$$w(z) = \int \prod_{j=1}^{n} d^2 x_j \ \delta^{(2)} \left(z - \prod_{j=1}^{n} x_j \right) \prod_{j=1}^{n} \frac{1}{\pi} e^{-|x_j|^2} = \frac{1}{\pi} G^{n\,0}_{0\,n} \left(\begin{matrix} - \\ \vec{0} \end{matrix} \middle| \, |z|^2 \right)$$

- $w(z) = P_n(z)$ (toy model) !!!
- Orthogonality

$$\int d^2 z \ z^k \bar{z}^l w(z) = \delta_{kl} (k!)^n$$

Joint probability and correlation functions

- Joint probability

$$P(z_1, z_2, \ldots, z_N) = \frac{1}{N! \prod_{k=0}^{N-1} h_k} \prod_{j=1}^{N} w(z_j) \prod_{i<j}^{N} |z_i - z_j|^2$$

- where $w(z) = \frac{1}{\pi} G^{n\,0}_{0\,n}\left(\begin{array}{c} - \\ \vec{0} \end{array} \middle| \, |z|^2 \right)$ and $h_k = (k!)^n$
- Determinantal structure

$$P(z_1, z_2, \ldots, z_N) = \frac{1}{N!} \det_{1 \le i,j, \le N} [K_N(z_i, z_j)]$$

- k-point correlation functions

$$R_k(z_1, \ldots, z_k) = \det_{1 \le i,j, \le k} [K_N(z_i, z_j)]$$

- Kernel

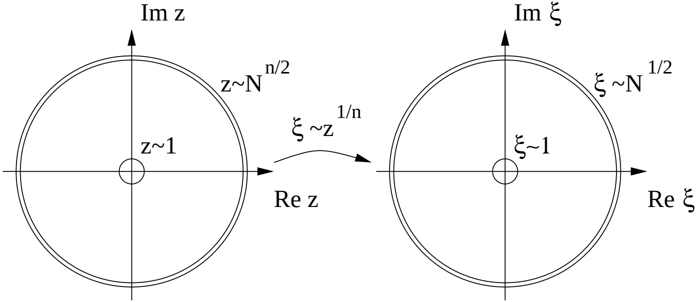
$$K_N(z_i, z_j) = \sqrt{w(z_i) w(z_j)} \sum_{k=0}^{N-1} \frac{1}{(k!)^n} (z_i z_j^*)^k \ .$$

Different scaling limits

- Truncated sum:

$$T_N(x) = \sum_{k=0}^{N} \frac{x^n}{(k!)^n}$$

- Different limits for $N \to \infty$
  - origin: $z \sim 1$
  - bulk: $1 \ll z^{1/n} < \sqrt{N}$
  - edge: $z^{1/n} \sim \sqrt{N}$



Different scaling limits

- Origin

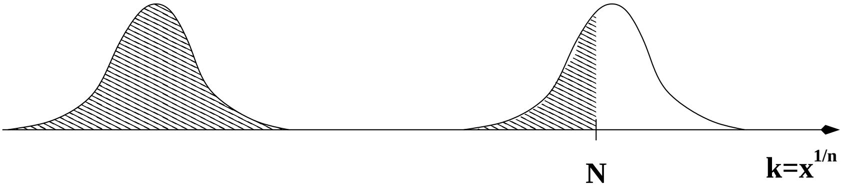
$$T_\infty(x) = \sum_{k=0}^{\infty} \frac{x^n}{(k!)^n} = {}_0F_{n-1}(-, \vec{1}, x)$$

- Bulk

$$T_N(x) \approx (2\pi)^{-n/2} \int_1^N dk k^{-n/2} e^{s(k)}$$

- Saddle point $s(k) = k \ln x - nk \ln k + nk$

$$T_N(x) \approx \sqrt{\frac{2\pi}{n}} (2\pi)^{-n/2} x^{(1-n)/2n} e^{nx^{1/n}} \times \frac{1}{2} \mathrm{erfc}\left(\frac{\sqrt{n}\left(x^{1/n} - N\right)}{\sqrt{2N}} \right)$$



Universality of correlations in the bulk

- Unfolding: $\xi \sim z^{1/n}$
- Kernel in the bulk

$$K_{bulk}(\xi_i, \xi_j) = \frac{1}{\pi} \left(\frac{\xi_i \xi_j^*}{|\xi_i \xi_j|} \right)^{(1-n)/2} \exp\left[-\frac{1}{2} (|\xi_i|^2 + |\xi_j|^2 + \xi_i \xi_j^*) \right].$$

- Consequence: the same universality as Ginibre matrices
- Only at the origin the kernel behaves differently

$$K_{origin}(z_i, z_j) = \sqrt{w(z_i) w(z_j)}\, T_\infty(z_i z_j^*)$$

- where

$$w(z) = G^{n\,0}_{0\,n}\left(\begin{matrix} - \\ \vec{0} \end{matrix} \middle| \rho \right) \quad \text{and} \quad T_\infty(x) = {}_0F_{n-1}(-, \vec{1}, x)$$

Finite size corrections at the edge

- For the product $p = x_1 \dots x_n$

$$\rho_p(w) = \frac{|w|^{\frac{2}{n}-2}}{n\pi} \times \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{2N}{n}}\,(|w|-1)\right) \ .$$

- For the n-th power $q = x^n$

$$\rho_q(w) = \frac{|w|^{\frac{2}{n}-2}}{n\pi} \times \frac{1}{2}\mathrm{erfc}\left(\frac{\sqrt{2N}}{n}\,(|w|-1)\right)$$

Summary

- Products of Gaussian matrices $N \to \infty$
- Self-averaging $p = x_1 \ldots x_n \leftrightarrow q = x^n$
- Joint probability function (standard form, weights as for the product of iid normal variables)
- The same universality as for Ginibre matrices in the bulk
- Kernel - at the origin
- Error function corrections at the edge