Random matrices with equi-spaced external source

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Random matrices with external source

- space of $n \times n$ Hermitian matrices with probability measure

$$\frac{1}{Z_n} \exp(-n\text{Tr}\,(V(M) - AM))dM,$$

where

- V is a polynomial of even degree with positive leading coefficient,
- $dM = \prod_{i<j} d\text{Re}\, M_{ij} d\text{Im}\, M_{ij} \ \prod_{j=1}^{n} dM_{jj}$
- A is a Hermitian $n \times n$ matrix (can be assumed diagonal $A = \text{diag}(a_1, \ldots, a_n)$)

(Zinn-Justin '97, Brézin-Hikami '98)

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- if $A = 0$, unitary ensemble

$$\frac{1}{Z_n} \exp(-n \operatorname{Tr} V(M)) dM.$$

- we will study the case

$$A = \frac{1}{n} \operatorname{diag}(0, 1, \ldots, n-1)$$

  - for $V(x) = cx^2$, eigenvalues behave like n non-intersecting Brownian motions starting at 0 and ending at $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$ *(Johansson)*

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- Joint probability distribution of eigenvalues in the ensemble

$$\frac{1}{Z_n} \exp(-n \text{Tr} (V(M) - AM)) dM$$

is given by

$$\frac{1}{\tilde{Z}_n} |\det(e^{n a_i \lambda_j})_{i,j=1,\dots,n}| \prod_{i<j} |\lambda_i - \lambda_j| \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j$$

  - if $A = \frac{1}{n}\text{diag}(0, 1, \dots, n-1)$,

$$\frac{1}{\tilde{Z}_n} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j.$$

- Behavior of eigenvalues for large n?

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- $A = \text{diag}(a, \ldots, a, -a, \ldots, -a)$ *(Bleher-Kuijlaars, Bleher-Delvaux-Kuijlaars, Adler-van Moerbeke)*
  - vector equilibrium problem
  - critical point: Pearcey kernel
- $A = \text{diag}(a_1, a_2, \ldots, a_k, 0, \ldots, 0)$ with k fixed *(Baik-Wang, Bertola-Buckingham-Lee-Pierce, Adler-Délépine-van Moerbeke)*
  - every non-zero eigenvalue of A is responsable for at most one outlier-eigenvalue of M
- External source matrix with n different eigenvalues *(Eynard-Orantin)*

External source

- $A = \frac{1}{n}\text{diag}(0, 1, \ldots, n-2, n-1)$, jpdf

$$\frac{1}{\hat{Z}_n} \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i<j} (e^{\lambda_i} - e^{\lambda_j}) \prod_{j=1}^{n} e^{-nV(\lambda_j)} d\lambda_j.$$

- eigenvalue configurations for which

$$\frac{1}{2n^2} \sum \sum_{i \neq j} \log(\lambda_i - \lambda_j)^{-1} + \frac{1}{2n^2} \sum \sum_{i \neq j} \log(e^{\lambda_i} - e^{\lambda_j})^{-1} + \frac{1}{n} \sum_{j=1}^{n} V(\lambda_j)$$

  is small are most likely
- natural to expect that limiting mean eigenvalue distribution is the equilibrium measure which minimizes

$$\frac{1}{2} \iint \log|t-s|^{-1} d\mu(t) d\mu(s) + \frac{1}{2} \iint \log|e^t - e^s|^{-1} d\mu(t) d\mu(s) + \int V(s) d\mu(s)$$

Limiting mean eigenvalue density

- limiting mean distribution minimizes

$$\frac{1}{2}\iint \log|t-s|^{-1} d\mu(t)d\mu(s) + \frac{1}{2}\iint \log|e^t - e^s|^{-1} d\mu(t)d\mu(s) + \int V(s)d\mu(s)$$

among all probability measures on $\mathbb{R}$

- equilibrium measure exists and is unique
- supported on a finite number of intervals
- other properties
  - $\rightarrow$ smooth density in the interior of the support
  - $\rightarrow$ typically square root vanishing near the endpoints
  - $\rightarrow$ if V is convex, the support consists of one interval

Eigenvalue correlation kernel

Random matrices without external source:

- eigenvalues in a unitary ensemble have a correlation kernel

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) p_k(y)$$

  - p_k's are normalized orthogonal polynomials on $\mathbb{R}$ w.r.t. weight $e^{-nV(x)}$
  - information about eigenvalues is contained in this kernel

- Christoffel-Darboux formula

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y}.$$

Eigenvalue correlation kernel

Random matrices with external source
$A = \frac{1}{n}\text{diag}(0, 1, \ldots, n-2, n-1)$:

- correlation kernel for eigenvalues is given by

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) q_k(e^y)$$

  - polynomials p_k of degree k and q_j of degree j are determined by the orthogonality conditions

$$\int_{\mathbb{R}} p_k(x) q_j(e^x) e^{-nV(x)} dx = \delta_{kj}$$

  - p_k's are type II multiple OPs with n orthogonality weights $1, e^x, e^{2x}, \ldots, e^{(n-1)x}$

Eigenvalue correlation kernel

Random matrices with external source
$A = \frac{1}{n}\text{diag}(0, 1, \ldots, n-2, n-1)$:

- correlation kernel for eigenvalues is given by

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) q_k(e^y)$$

  - polynomials p_k of degree k and q_j of degree j are determined by the orthogonality conditions

$$\int_{\mathbb{R}} p_k(x) q_j(e^x) e^{-nV(x)} dx = \delta_{kj}$$

  - q_j's are related to type I multiple orthogonal polynomials

Eigenvalue correlation kernel

Interpretation of the polynomials in terms of the random matrix ensemble

$$\frac{1}{Z_n}\exp(-n\mathrm{Tr}\,(V(M)-AM))dM$$

or the determinantal point process

$$\frac{1}{\tilde{Z}_n}\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i<j}(e^{\lambda_i}-e^{\lambda_j})\prod_{j=1}^{n}e^{-nV(\lambda_j)}d\lambda_j.$$

- $p_n(\lambda)=\mathbb{E}(\det(\lambda I-M))=\mathbb{E}'(\prod_j(\lambda-\lambda_j))$ is the average characteristic polynomial *(Bleher-Kuijlaars)*
- $q_n(e^\lambda)=\mathbb{E}(\det(e^{\lambda I}-e^M))=\mathbb{E}'(\prod_j(e^\lambda-e_j^\lambda))$ *(C-Wang)*
  - asymptotics for p_n and q_n?

Eigenvalue correlation kernel

- Assume V is such that μ_V is one-cut supported and regular (e.g. convex), with support $[a, b]$
- Result:
  - for z away from $[a, b]$,

$$p_n(z) = F(z)e^{n\int \log(z-x)d\mu_V(x)}(1 + \mathcal{O}(n^{-1})), \qquad n \to \infty,$$

$$q_n(z) = \hat{F}(z)e^{n\int \log(e^z-e^x)d\mu_V(x)}(1 + \mathcal{O}(n^{-1})), \qquad n \to \infty,$$

  - for $x \in (a, b)$

$$p_n(x) = G(x)\cos(n\int_x^b d\mu_V(x) + \theta)(1 + \mathcal{O}(n^{-1})), \qquad n \to \infty,$$

$$q_n(e^x) = \hat{G}(x)\cos(n\int_x^b d\mu_V(x) + \hat{\theta})(1 + \mathcal{O}(n^{-1})), \qquad n \to \infty.$$

  - Airy asymptotics near a and b

Eigenvalue correlation kernel

- Similar to asymptotics for orthogonal polynomials
  - asymptotics determined by the equilibrium measure away from $[a, b]$
  - oscillatory on $[a, b]$
  - Airy asymptotics near the edges
- Proofs are more complicated
  - RH problem
    $\rightarrow$ of dimension 2×1
    $\rightarrow$ functions living on different domains
  - asymptotic analysis of the RH problem
    $\rightarrow$ transformation to shifted scalar RH problem

Eigenvalue correlation kernel

- RH problem for usual OPs *(Fokas-Its-Kitaev '92)*

(a) Y is analytic in $\mathbb{C} \setminus \mathbb{R}$,

(b) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$,

(c) $Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$,

- Unique solution given by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \kappa_n^{-1} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{p_n(s) w(s)}{s - z} ds \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(s) w(s)}{s - z} ds \end{pmatrix},$$

Eigenvalue correlation kernel

- polynomials defined by

$$\int_{\mathbb{R}} p_k(x) q_j(e^x) e^{-nV(x)} dx = \delta_{kj}$$

- standard RH problem for MOPs is of size $n+1$ - inconvenient for n large
- let

$$Y_1(z) = \kappa_n^{-1} p_n(z)$$

  and

$$Y_2(z) = \frac{-\kappa_n^{-1}}{2\pi i} \int_{\mathbb{R}} \frac{p_n(s)}{e^z - e^s} e^{-nV(s)} ds.$$

- RH problem for $Y = (Y_1, Y_2)$

RH problem for polynomials

1. $Y = (Y_1, Y_2)$, where Y_1 is analytic in $\mathbb{C} \setminus \mathbb{R}$, and Y_2 is analytic in $\mathcal{S} \setminus \mathbb{R}$, where $\mathcal{S} = \{z \in \mathbb{C} : |\text{Im } z| < \pi\}$
2. Y has continuous boundary values $Y_\pm$ on $\mathbb{R}$, and

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-x} e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in \mathbb{R},$$

3. as $z \to \infty$, Y_1 behaves as $Y_1(z) = z^n + \mathcal{O}(z^{n-1})$,
4. as $e^z \to \infty$ (i.e., $z \to +\infty$), Y_2 behaves as $Y_2(z) = \mathcal{O}(e^{-(n+1)z})$,
5. Y_2 has the periodic boundary condition

$$Y_2(x + \pi i) = Y_2(x - \pi i), \qquad \text{for } x \in \mathbb{R}.$$

RH problem for polynomials

- there is also a 2×2 matrix RH problem
  - unlike for usual orthogonal polynomials, $\det Y(z) \neq 1$
  - taking inverses is not possible
  - no advantage
- there is a dual RH problem for $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)$, where

$$\tilde{Y}_1 = -\kappa_n q_n(e^z), \qquad \tilde{Y}_2(z) = \frac{-\kappa_n}{2\pi i} \int_{\mathbb{R}} \frac{q_n(e^s)}{z-s} e^{-nV(s)} ds.$$

RH problem for polynomials

1. $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)$, where $\tilde{Y}_2$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and $\tilde{Y}_1$ is analytic in $\mathcal{S} \setminus \mathbb{R}$, where $\mathcal{S} = \{z \in \mathbb{C} : |\text{Im } z| < \pi\}$
2. $\tilde{Y}$ has continuous boundary values $\tilde{Y}_\pm$ on $\mathbb{R}$, and
$$\tilde{Y}_+(z) = \tilde{Y}_-(z) \begin{pmatrix} 1 & e^{-nV(z)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \mathbb{R},$$
3. as $z \to \infty$, $\tilde{Y}_2$ behaves as $\tilde{Y}_2(z) = z^{-n-1} + \mathcal{O}(z^{-n-2})$,
4. as $e^z \to \infty$ (i.e., $z \to +\infty$), $\tilde{Y}_1$ behaves as $\tilde{Y}_1(z) = \mathcal{O}(e^{nz})$,
5. $\tilde{Y}_1$ has the periodic boundary condition
$$\tilde{Y}_1(x + \pi i) = \tilde{Y}_1(x - \pi i), \qquad \text{for } x \in \mathbb{R}.$$

RH problem for polynomials

- Asymptotic analysis of the RH problem if the support of μ is one interval: Deift/Zhou steepest descent analysis
- Modifications compared to analysis for OPs
  - construction of two g-functions

$$g(z) := \int \log(z-y)d\mu(y)$$

$$\tilde{g}(z) := \int \log(e^z - e^y)d\mu(y).$$

  - Crucial step: transformation of the RH problem to a non-local scalar RH problem in the complex plane

RH problem for polynomials

- Transformation to shifted RH problem of the form
  1. $F : \mathbb{C} \setminus \Sigma \to \mathbb{C}$ is analytic
  2. for $z \in \Sigma$, we have

$$F_+(z) = F_-(z) J_n(z) + F_\pm(f(z)) \tilde{J}_n(z),$$

     with $f : \Sigma \to \Sigma$,
  3. $\lim_{z \to \infty} F(z) = 1$
- Large n asymptotics for this RH problem can be found if
  - $J_n(z) = 1 + o(1)$
  - $\tilde{J}_n(z) = o(1)$

Outlook

- Universality
  - sine kernel
  - Airy kernel
- multi-cut case
- large n behavior in more general point processes of the form

$$\frac{1}{\tilde{Z}_n}\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i<j}(f(\lambda_i)-f(\lambda_j))\prod_{j=1}^{n}e^{-nV(\lambda_j)}d\lambda_j.$$