Quantum chaotic thermalization

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to be exemplified for

Dicke model:
spin and oscillator coupled

$$\hat{H} = \hbar\left\{\omega_0 \hat{J}_z + \omega a^\dagger a + g\sqrt{\frac{2}{j}}(a + a^\dagger)\hat{J}_x\right\}$$

$\vec{J}^2 = j(j+1)$ Bloch sphere

quantum phase transition at $g = g_c = \frac{\sqrt{\omega\omega_0}}{2}$

in superradiant phase: $*$ $\langle a \rangle \neq 0$ Hepp (1973) ...

$*$ chaos Graham (1984) ...

experiment with double condensate in optical resonator:

T. Esslinger 2010

phase transition observed

we propose extension for

chaos,

equilibration,

giant fluctuations

oscillator coherent states

$$|\alpha\rangle$$

complex α fixes mean displacement and momentum

$$\langle x\rangle \propto \alpha + \alpha^*, \qquad \langle p\rangle \propto \alpha - \alpha^*$$

with minimal uncertainty: one Planck cell

spin coherent states

$$|z\rangle$$

complex amplitude $z = \mathrm{e}^{\mathrm{i}\phi} \tan\frac{\theta}{2}$ fixes spin orientation as

$$\langle z|\hat{J}_x|z\rangle = j\sin\theta\cos\phi$$

$$\langle z|\hat{J}_y|z\rangle = j\sin\theta\sin\phi$$

$$\langle z|\hat{J}_z|z\rangle = j\cos\theta$$

with minimum uncertainty: single Planck cell

Glauber Q-function (Husimi)

$$Q(\alpha, z) \propto \langle \alpha, z|\rho|\alpha, z\rangle$$

exists for any ρ, real, non-negative

$$\langle a^m a^{\dagger n}\rangle = \int d^2\alpha d^2 z\, \alpha^m \alpha^{*\,n}\, Q$$

converges to classical phase-space density as $\hbar \to 0$

evolution equation is Fokker-Planck

$$\dot{Q} \quad = (\mathcal{L}_{\mathrm{drift}} + \mathcal{L}_{\mathrm{diff}})Q$$

$$\begin{aligned}\mathcal{L}_{\mathrm{drift}} = {} & \mathrm{i}\partial_\alpha \left(\omega\alpha + g\sqrt{2j}\,\frac{z+z^*}{1+|z|^2}\right) \\ & +\mathrm{i}\partial_z \left(-\omega_0 z + \frac{g}{\sqrt{2j}}(1-z^2)(\alpha+\alpha^*)\right) + \mathrm{c.c.}\end{aligned}$$

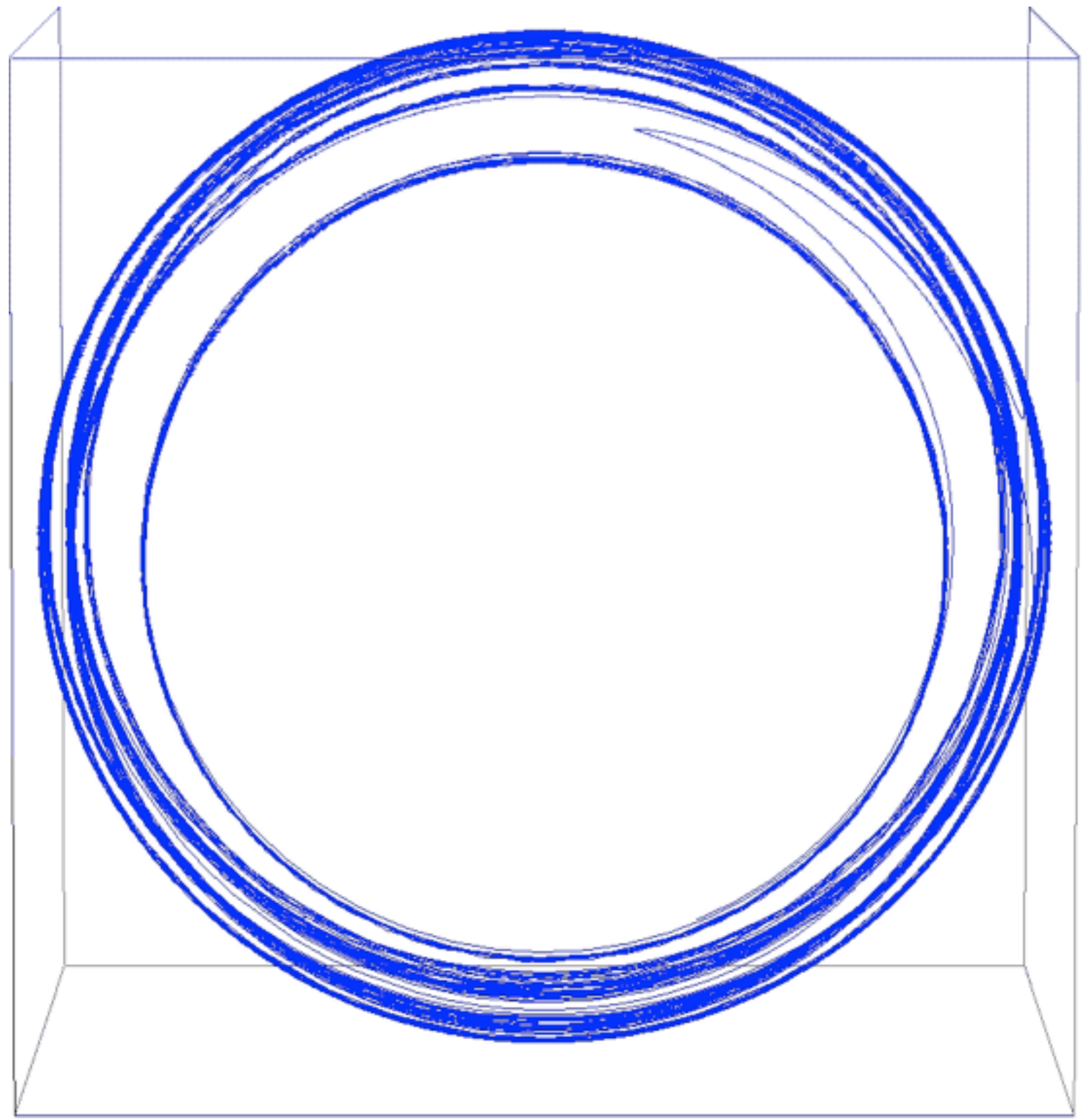
$$\mathcal{L}_{\mathrm{diff}} = \frac{\mathrm{i}g}{\sqrt{2j}}\partial_\alpha\partial_z(1-z^2) + \mathrm{c.c.}$$

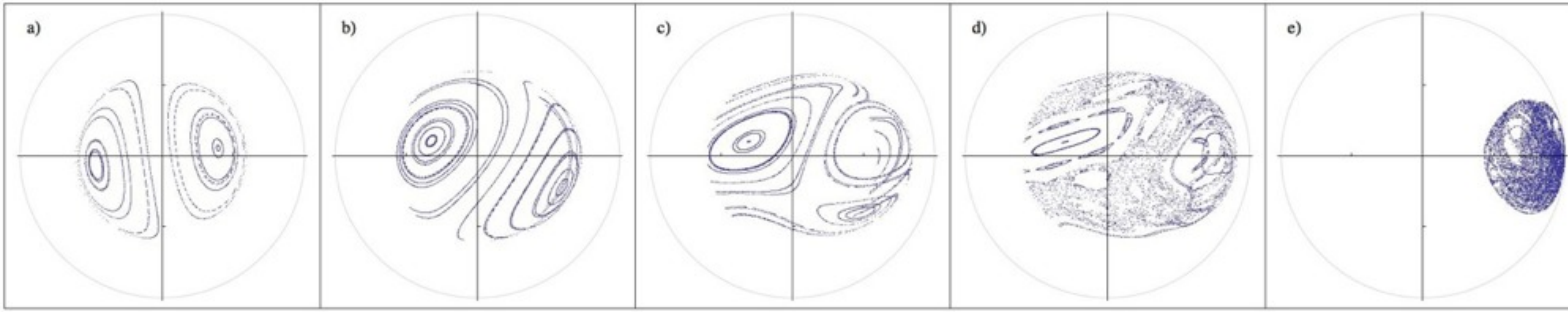
relative order $\frac{1}{j}$

interlude:

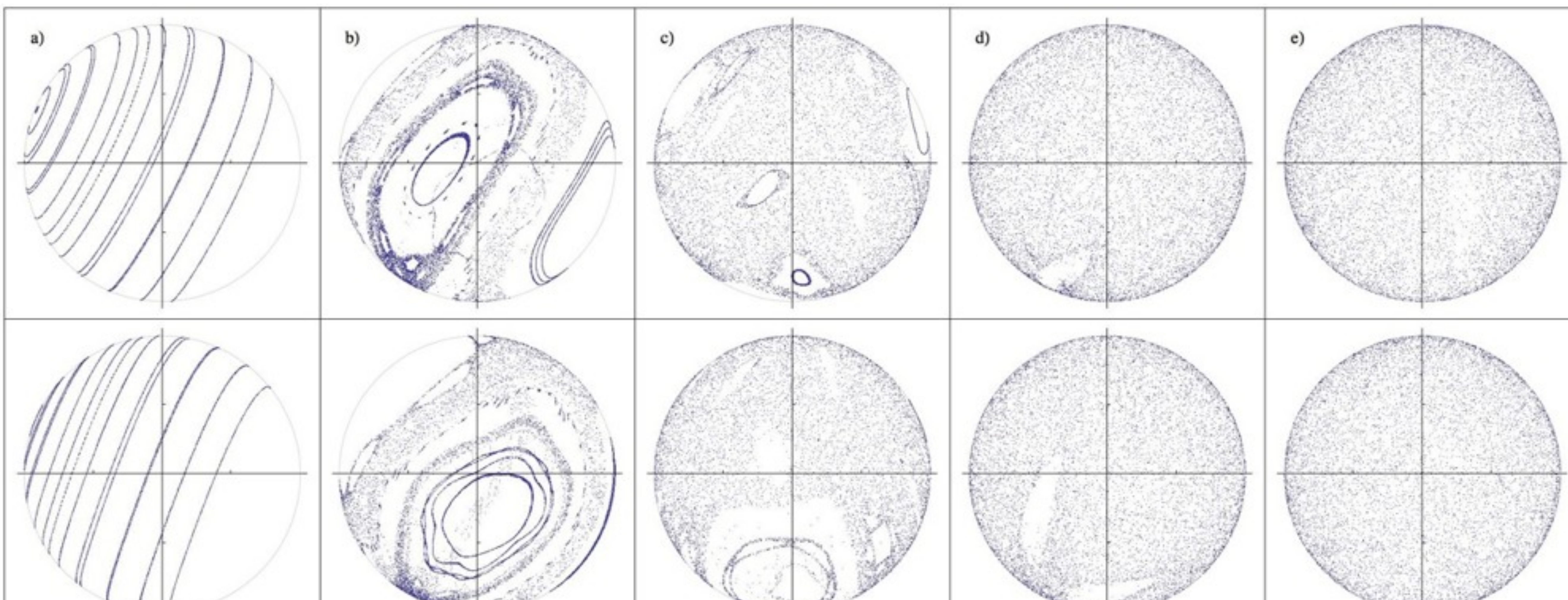
classical dynamics according to drift

single classical trajectory on Bloch sphere





$$\frac{g}{g_c} = 0.2 \qquad 0.7 \qquad 0.9 \qquad 1.01 \qquad 1.5$$



mimick initial coherent state
(with `tiny circular support´ in energy shell)
as swarm of initial points

gives rise to bundle of trajectories

initially circular support deforms while preserving `area´:

squeezes, stretches, bends, folds, without end,

visits everywhere in the energy shell

until

until

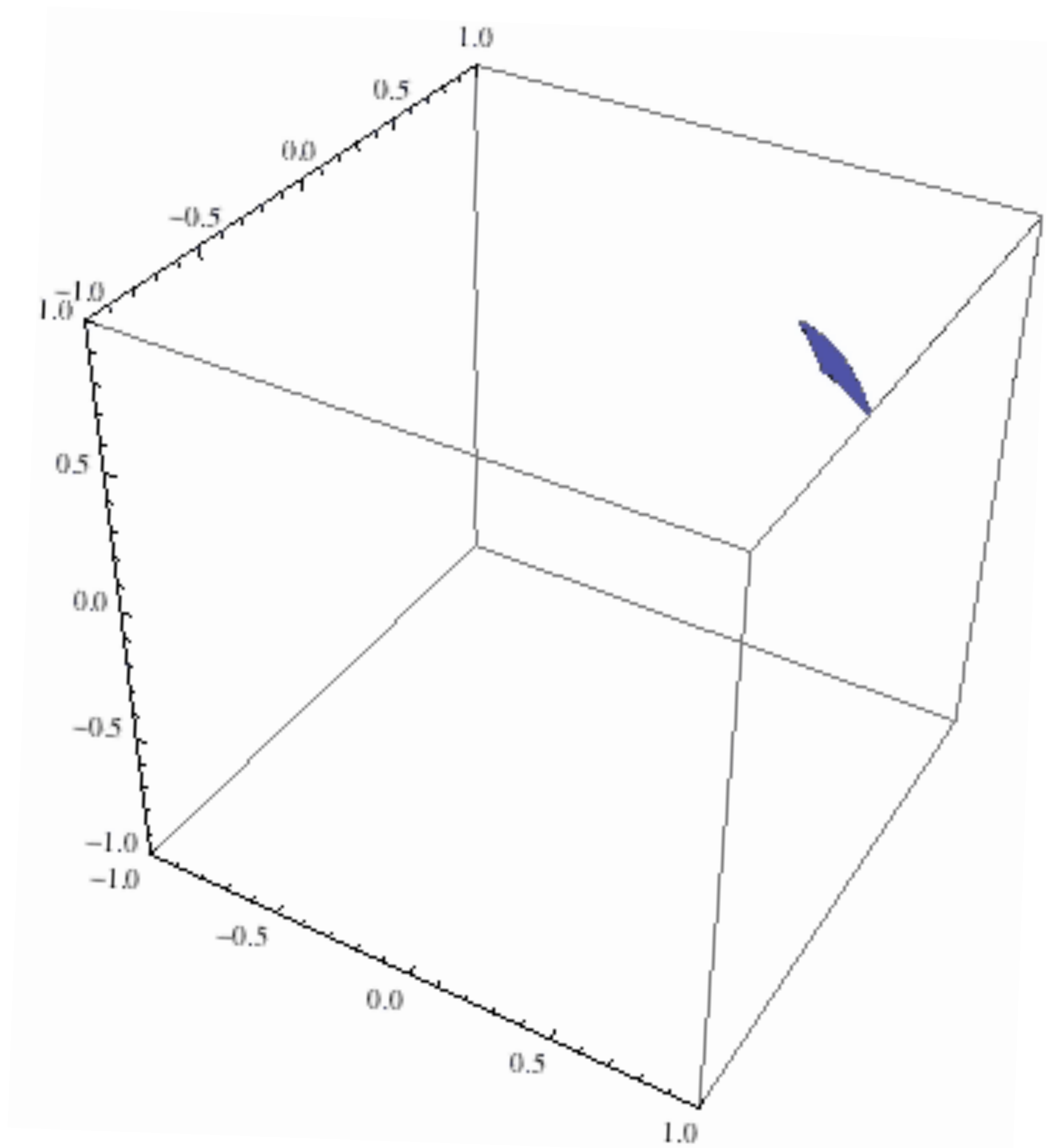
Q forms an `infinitely´ fissured landscape,

over its infinitely wriggling support,

finite resolution, however, suggests

constant Q over the energy shell,

microcanonical distribution


1.0
0.5
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−0.5
0.0
0.5
1.0

quantum mechanics

forbids

classical nonsense like infinitely fine structures

quantum diffusion washes out singular fissures in Q

so as to corroborate microcanonical distribution

but how?

quantum diffusion

$$\mathcal{L}_{\text{diff}} = \frac{\mathrm{i}g}{\sqrt{2j}}\partial_\alpha \partial_z (1 - z^2) + \text{c.c.}$$

$$D = \begin{pmatrix} 0 & d \\ d^\dagger & 0 \end{pmatrix}$$

real symmetric, chiral

$$\lambda^4 - \lambda^2 \ \mathrm{tr}\, dd^\dagger + \det dd^\dagger = 0$$

eigenvalues come as two real $\pm$ pairs

2 eig‘vec‘s of D `expansive´ $\longleftrightarrow$ $\lambda > 0$ genuine diffusion

2 `contractive´ $\longleftrightarrow$ $\lambda < 0$ antidiffusion

classical chaotic drift also has 4 distinguished directions:

2 neutral (along flow and transverse to energy shell),

1 expansive (unstable), 1 contractive (stable)

classical vs quantum expansion/contraction: projection onto co-moving Poincaré section

fix phase space point X_0 and choose deflection δX

write Fokker Planck eqn with δX as independent variables

$$\dot{Q} = \Big(\partial_{\delta X_i} d_i(X_0 + \delta X) + \partial_{\delta X_i}\partial_{\delta X_j} D_{ij}(X_0 + \delta X)\Big)Q$$

truncate wrt deflection, leading order

express deflection in terms of local coordinates along

classically stable, unstable, and neutral directions: s, u, ϵ, τ

integrate out neutrals to get reduced Fokker-Planck eqn:

$$\dot{Q} = \left(\lambda \partial_s s - \lambda \partial_u u + \partial_s^2 D_{ss} + \partial_u^2 D_{uu} + 2\partial_s \partial_u D_{su}\right) Q$$

Lyapounov exponent λ independent of X_0

diffusion matrix with known dependence on X_0

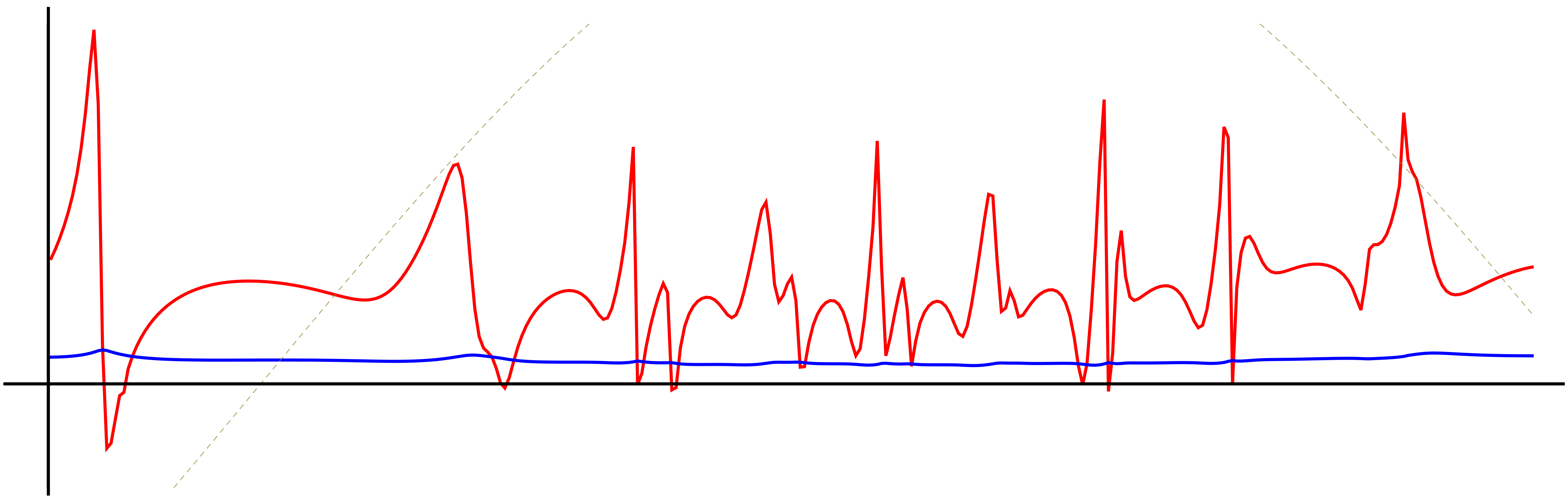
same Fokker-Planck eqn for $X_0 \longrightarrow X_t$

along classical trajectory emanating from X_0

time dependent diffusion matrix

$$\mathrm{v}ar_t(s) = \mathrm{e}^{-\lambda t}\mathrm{var}_0(s) + \int_0^t dt' \mathrm{e}^{-\lambda s} D_{ss}(t-t')$$

must be positive at all times since Q is



minimal scale for stable direction $\propto \dfrac{1}{\sqrt{j}}$

equilibration mechanism:

chaos provides endless stretching
and folding in unstable direction

equipartition
in energy shell

quantum diffusion smoothes
in classically stable direction,
to minimal scale $\propto 1/\sqrt{j} \propto \sqrt{\hbar}$

how general?

Fokker-Planck equations often Q describe unitary dynamics

(kicked top, SU(3)-dynamics, Bose-Hubbard model)

given chaos, equilibration as above

in general, given chaos,

$$\dot{Q} = \mathcal{L}Q$$

$$\mathcal{L} = \sum_{n=1,2,\ldots} \hbar^{n-1} \partial_X^n f_n(X)$$

for classically stable coordinates, along classical trajectory

$$\partial_{\tilde{s}} \lambda \tilde{s} + \sum_{n=2,,\ldots} \partial_{\tilde{s}}^n \hbar^{(n-2)/2} f_n(X_t)$$

suggests asymptotic validity of Fokker-Planck eqn

summary

Q obeys Fokker-Planck equation

equilibration to microcanonical distribution,

due to classically chaotic drift and quantum diffusion

(stretching, folding, quantum smoothing)

The end