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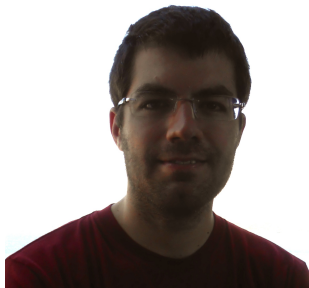
Applications of Random Matrix Theory in Lattice QCD

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Brunel (West London), December 15th, 2012

Our group



our PhD student
Savvas Zafeiropoulos



Jacobus Verbaarschot
& myself

Outline

- ▶ Introduction to Lattice QCD
- ▶ Two-dimensional naive Discretization
- ▶ Wilson RMT
- ▶ What has the future in store for us?

Introduction to Lattice QCD

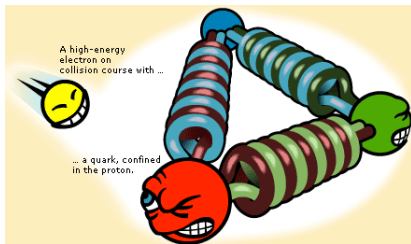


image from www.nobelprize.org (2004)

Action of continuum QCD

The partition function of N_f fermionic flavors

$$Z = \int \exp \left[-S_{\text{YM}}(\mathbf{A}) - \sum_{j=1}^{N_f} \int \bar{\psi}_j (\not{D}(\mathbf{A}) - m_j) \psi_j d^4x \right] D[\mathbf{A}, \psi]$$

The Yang-Mills action of SU(3)

$$S_{\text{YM}}(\mathbf{A}) = \frac{1}{4g^2} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d^4x$$

with the field strength tensor

$$F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g [\mathbf{A}_\mu, \mathbf{A}_\nu]$$

The four components of the vector potential $\mathbf{A}_\mu \in \text{su}(3)$ are 3×3 matrix valued functions.

The continuum Dirac-operator

Fermionic fields ψ_j are Grassmann variables

$$\Rightarrow Z = \int \prod_{j=1}^{N_f} \det(\not{D}(\mathbf{A}) - m_j) \exp[-S_{\text{YM}}(\mathbf{A})] D[\mathbf{A}]$$

The Dirac operator

$$D(\mathbf{A}) = \gamma^\mu (\not{\partial} + g\mathbf{A}_\mu)$$

Index-theorem:

number of zero modes (index)=topological charge

$$\nu = \frac{1}{32\pi^2} \int \varepsilon^{\mu\nu\lambda\kappa} \text{tr} F_{\mu\nu} F_{\lambda\kappa} d^4x$$

QCD in Continuum

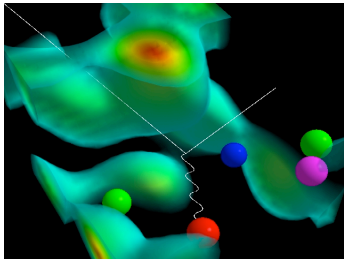


image by Derek Leinweber (CSSM)

Lattice QCD

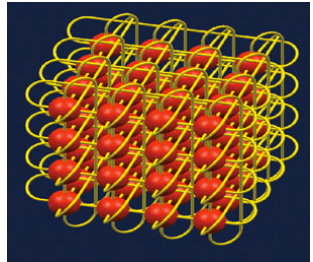
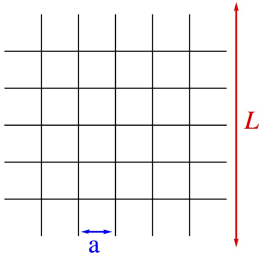


image from www.llnl.gov

Lattice QCD

- ▶ space-time V becomes discrete with lattice spacing a
- ▶ vector field $A_\mu \in \mathfrak{su}(3)$ replaced by $U_\mu \in \text{SU}(3)$



volume of space-time: $V=L^4$

Big question:

How do we perform the limits $a \rightarrow 0$ and $V \rightarrow \infty$ such that we obtain continuum QCD?

Fundamental problem on the lattice

Energy in continuum:

$$E^2 = k_\mu k^\mu + M_0^2$$

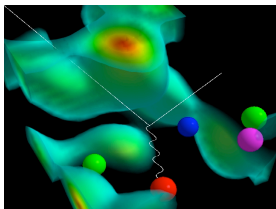


image by Derek Leinweber (CSSM)

Energy on lattice:

$$E^2 = \frac{\sin^2(k_\mu a)}{a^2} + M_0^2$$

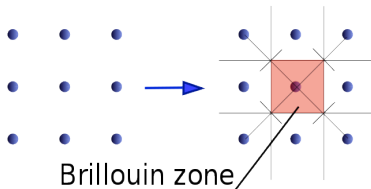


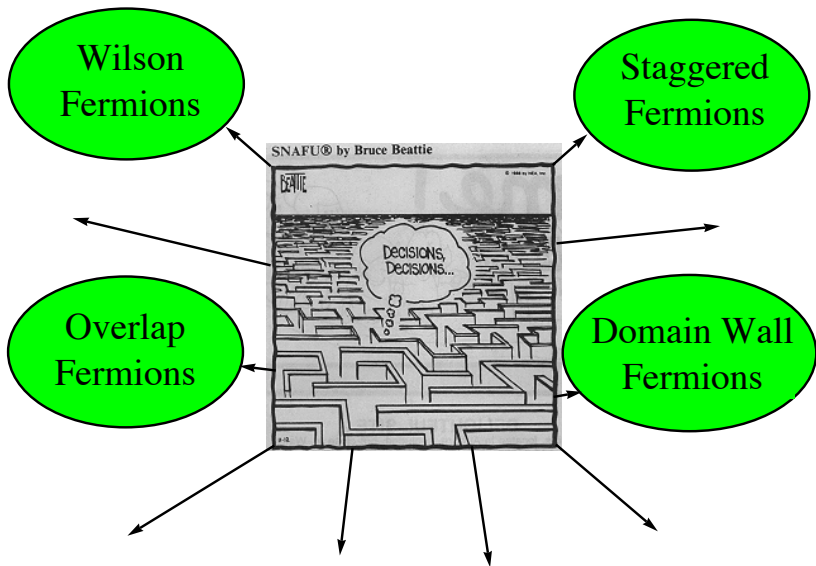
image from Wikipedia.org

Doubler Problem:

$$k_\mu \rightarrow \left\{ \begin{array}{l} k_\mu \\ \frac{\pi}{a} - k_\mu \end{array} \right.$$

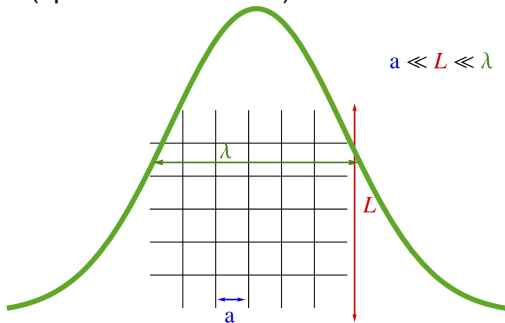
one momentum=($2^4 = 16$) particles

Many ways and no guide



The ϵ -regime of QCD

- ▶ infrared limit of QCD
- ▶ large Compton wavelength of Goldstone bosons \gg box size $V^{1/4} = L$
- ▶ lattice volume (space-time volume) $V \rightarrow \infty$



Saddlepoint approximation:

- ▶ spontaneous breaking of chiral symmetry
- ▶ global Goldstone bosons = Mesons

e.g. $N_f = 2$, SU(2)-integral = zero momentum modes of the three pions

Partition function in the ϵ -regime for N_f flavors

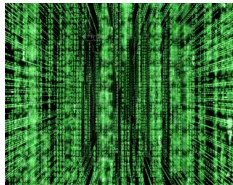
$$\begin{aligned} Z &\propto \int_{\text{SU}(N_f)} \exp[\mathcal{L}(U)] d\mu(U) \\ &\propto \sum_{\nu \in \mathbb{Z}} \int_{\text{U}(N_f)} \exp[\mathcal{L}(U)] \det^\nu U d\mu(U) \end{aligned}$$

Lagrangian of the Goldstone bosons:

$$\mathcal{L}(U) = \frac{\Sigma V}{2} \text{tr } M(U + U^\dagger) + \mathcal{L}_{\text{correction}}(V, a, U)$$

- ▶ index of the Dirac operator: ν
- ▶ masses of the quarks: M
- ▶ low energy constants: Σ, \dots

... and the same for RMT



$$\text{Model: } D_{\text{QCD}} \longrightarrow D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} + D_{\text{correction}}(\mathbf{a})$$

$$Z \propto \sum_{\nu \in \mathbb{Z}} \int_{U(N_f)} \exp[\mathcal{L}(U)] \det^\nu U d\mu(U)$$

Lagrangian of the Goldstone bosons:

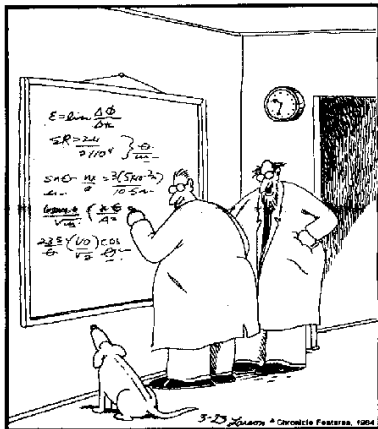
$$\mathcal{L}(U) = \frac{\Sigma V}{2} \text{tr} M(U + U^\dagger) + \mathcal{L}_{\text{correction}}(V, \mathbf{a}, U)$$

Derived by Shuryak and Verbaarschot (90's)!

Two-dimensional naive Discretization

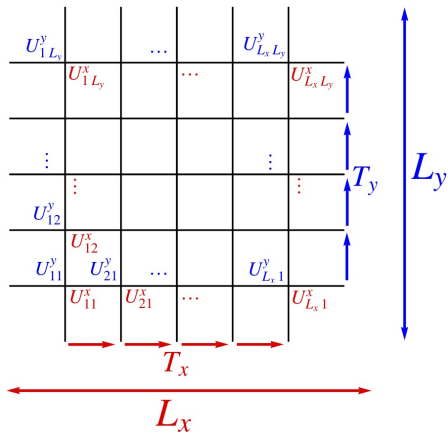
THE FAR SIDE

By GARY LARSON



**"OHHHHHHH . . . Look at that, Schuster . . .
Dogs are so cute when they try to comprehend
quantum mechanics."**

The naive Dirac Operator



Naive Dirac operator:

$$D_{\text{naive}} = \frac{1}{2a} \gamma^\mu (T_\mu - T_\mu^{-1})$$

Translation operator:

$$T_\mu = T_\mu(U_{ij}^\mu)$$

- ▶ has the doubler problem
- ▶ **but** is the starting point for constructing staggered fermions

Why naive fermions?

- ▶ starting point for deriving staggered fermions
- ▶ same universality class as staggered fermions
- ▶ RMT model for staggered fermions by Osborn (2004), immensely complicated

Why naive fermions?

- ▶ starting point for deriving staggered fermions
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Why 2-D?

- ▶ simpler to understand
- ▶ our group has no supercomputer → numerically cheaper



Artificial chiral structure

General RMT model: $D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix}$

Original Classification (Verbaarschot, 90's):

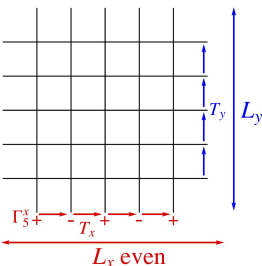
$$W \text{ is } \begin{cases} \text{real,} & \beta = 1 \\ \text{complex,} & \beta = 2 \\ \text{quaternion,} & \beta = 4 \end{cases}$$

Artificial chiral structure

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Reasons:

other dimensions = other universality classes

+

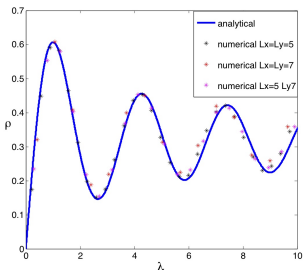
Artificial symmetry: $\Gamma_5^x T_x \Gamma_5^x = -T_x$, $\Gamma_5^x T_y \Gamma_5^x = T_y$
 \Rightarrow change of the universality class

Extensions into Altland-Zirnbauer classification!

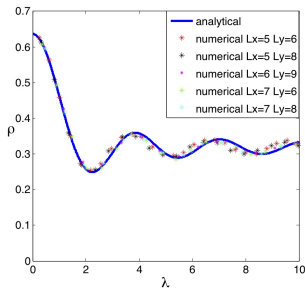
Similar to the classification of topological insulators
 (Schnyder, Ryu, Furusaki, Ludwig (2008))

Comparison: Lattice Data \leftrightarrow RMT

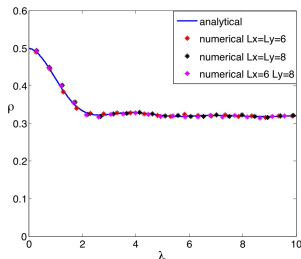
Three colors (SU(3)) & adjoint representation



odd-odd
=
2-d cont. QCD



odd-even
=
level density of
anti-symm. matrices



even-even
=
staggered fermions in 2-d
 $\beta = 1$ in 4-d cont. QCD

Drastic change of the universality class!

Wilson RMT



Kenneth G. Wilson

The Wilson Dirac Operator

Main idea to solve the doubler problem:

- ▶ Make 15 particles infinitely heavy in the continuum limit ($a \rightarrow 0$).
- \Rightarrow too inertial, decouple from the system
- ▶ Wilson-Dirac operator

$$D_W = D_{\text{naive}} + a\Delta$$

$$\propto \gamma^\mu \sin(k_\mu a) + \frac{\sin^2(k_\mu a/2)}{a}$$

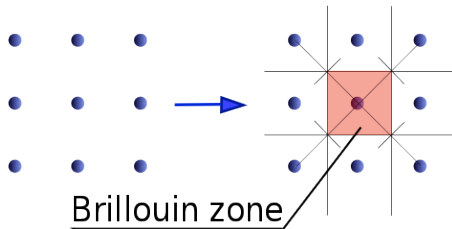


image from Wikipedia.org

- ▶ Laplace operator Δ
 - ▶ additional effective mass
 - ▶ explicitly breaks chiral symmetry
 - ▶ Dirac operator γ_5 -Hermitian: $D_5 = \gamma_5 D_W$ is Hermitian

Wilson RMT

Dirac operator:

$$D_{\text{QCD}} \rightarrow D_{\text{W}} = \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix} + am_6 \mathbf{1} + a\lambda_7 \gamma_5$$

Weight:

$$\exp[-S_{\text{YM}}] \rightarrow P(D_{\text{W}}) : \text{Gaussian}$$

- ▶ Hermitian random matrices A ($n \times n$), B ($(n + \nu) \times (n + \nu)$) and scalar random variables m_6 , λ_7 are the Wilson-terms \Rightarrow breaking of chiral symmetry
- ▶ complex W ($n \times (n + \nu)$) matrix

Damgaard, Splittorff, Verbaarschot (2010)

Partition Function of N_f flavors

$$Z \propto \int_{U(N_f)} \exp[\mathcal{L}(U)] \det^{\nu} U d\mu(U)$$

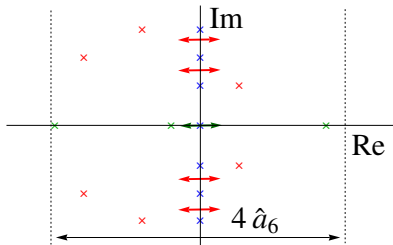
Lagrangian of the Goldstone bosons:

$$\begin{aligned} \mathcal{L}(U) = & \frac{V\Sigma}{2} \text{tr } M(U + U^\dagger) \\ & + a^2 VW_6 \text{tr}^2(U + U^\dagger) + a^2 VW_7 \text{tr}^2(U - U^\dagger) + a^2 VW_8 \text{tr}(U^2 + U^{\dagger 2}) \end{aligned}$$

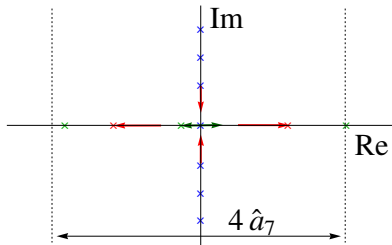
Damgaard, Splittorff, Verbaarschot (2010)

What are the low energy constants Σ and $W_{6/7/8}$?

Effect of the low energy constants



effect of W_6



effect of W_7



Please do not read this now!

$$g_2(x_1, x_2) = g_1(x_1, x_2)k(y_1)k(y_2) + g_1(x_1)k(x_1 - x_2)k(y_1 + y_2). \quad (17)$$

$$n(x_1, x_2) = \exp \left[-\frac{n}{4a^2} (x_1 + x_2 - \frac{a^2(\mu_1 + \mu_2)}{n})^2 + \frac{n}{4} (x_1 - x_2)^2 \right] \times \left[\operatorname{sign}(x_1 - x_2) - \operatorname{erf} \left[\sqrt{\frac{n(1 + a^2)}{4a^2}} (x_1 - x_2) - \sqrt{\frac{a^2}{4n(1 + a^2)}} (\mu_1 - \mu_2) \right] \right]. \quad (18)$$

$$n(z) = -2a \operatorname{sign}(y) \exp \left[-\frac{n}{a^2} \left(x - \frac{a^2(\mu_1 + \mu_2)}{2n} \right)^2 - ny^2 \right], \quad (19)$$

$$n(x) = \exp \left[-\frac{n}{2a^2} \left(x - \frac{a^2\mu_1}{n} \right)^2 \right]. \quad (20)$$

$$K_{(n_1, n_2)}^{(n)}(\vec{z}, \vec{m}) = \frac{1}{\pi^2 [n_1^{n_1+1} n_2^{n_2+1} (m_1, m_2) \Gamma(\frac{n_1}{2}) \Gamma(\frac{n_2}{2})]} \times \exp \left[\frac{K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)})}{K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)})} \frac{K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)})}{K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)})} \frac{K_3^{(n+1)}(m_1, z_1^{(2)})}{K_3^{(n+1)}(m_1, z_1^{(2)})} \right]. \quad (21)$$

with

$$K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) = \left[K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) - K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) \right] \quad (22)$$

$$K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) = \left[K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) - K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) \right] \quad (23)$$

$$K_3^{(n+1)}(m_1, z_1^{(2)}) = \left[K_3^{(n+1)}(m_1, z_1^{(2)}) - K_3^{(n+1)}(m_1, z_1^{(2)}) \right] \quad (24)$$

$$K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) = \left[K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) - K_1^{(n_1+1)}(z_1^{(1)}, z_1^{(2)}) \right] \quad (25)$$

$$K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) = \left[K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) - K_2^{(n_2+1)}(z_2^{(1)}, z_2^{(2)}) \right] \quad (26)$$

$$K_3^{(n+1)}(m_1, z_1^{(2)}) = \left[K_3^{(n+1)}(m_1, z_1^{(2)}) - K_3^{(n+1)}(m_1, z_1^{(2)}) \right] \quad (27)$$

$$\Delta K_2^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) - g_1 \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \times \exp \left[\frac{(-1)^{n-1} \ln^2 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2)}{\sqrt{8a^2}} \int \frac{dx_1 dx_2}{x_1 - x_2} (x_2 + \tilde{z}_1)^n (x_1 - \tilde{z}_1)^n \right. \\ \times \exp \left[-\frac{1}{16a^2} [(x_1 - \tilde{z}_1) + \tilde{z}_2]^2 + (x_2 + \tilde{z}_2 - \tilde{z}_1)^2 \right] \\ \times \left[\frac{g^{(n-1)}(x_2 - \tilde{z}_1)}{(n-1)!(x_1 - \tilde{z}_1)^n} Z_{n-1}^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right. \\ \left. + \operatorname{sign}(\tilde{z}_2) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 + \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right] \\ \times \left[\operatorname{erf} \left[\sqrt{\frac{n}{8a^2}} \right] \right] \\ \left. + 8na^4 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2) \exp \left[-\frac{1}{8a^2} (x_1 - \tilde{z}_1)^2 \right] \Delta K_2^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \right]. \quad (28)$$

$$\rho_1(\vec{z}) = \frac{\exp(-8a^2)}{16a^2} \sum_{j=1}^N (I_j - (8a^2) - I_{j+1} \exp(8a^2)) \quad (29)$$

$$\times \int \exp \left[-\frac{1}{16a^2} [(x_1 - \tilde{z})^2 + (x_2 + \tilde{z})^2] + \frac{\tilde{z}_1^2 + \tilde{z}_2^2}{16a^2} (x_1 - x_2)^2 \right] \\ \times \frac{(-1)^{n-1} \ln^2 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2)}{(j-1)!} Z_{j-1}^{(n)}(x_1, x_2; \tilde{a} = 0) \frac{dx_1 dx_2}{x_1 - x_2}. \quad (30)$$

$$\Delta K_1^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \times \exp \left[\frac{(-1)^{n-1} \ln^2 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2)}{4\pi a^2} \int \frac{dx_1 dx_2}{x_1 - x_2} (x_2 + \tilde{z}_1)^n (x_1 - \tilde{z}_1)^n \right] \quad (31)$$

$$\times \exp \left[-\frac{1}{16a^2} [(x_1 - \tilde{z}_1) + \tilde{z}_2]^2 + (x_2 + \tilde{z}_2 - \tilde{z}_1)^2 \right] \\ \times \left[\frac{g^{(n-1)}(x_1 + \tilde{z}_1)}{(n-1)!(x_1 - \tilde{z}_1)^n} Z_{n-1}^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right. \\ \left. + \operatorname{sign}(\tilde{z}_1) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 + \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right]. \quad (32)$$

$$K_1^{(n_1)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \times \exp \left[\frac{n^2}{4(2\pi)^2 a^2} \int \frac{dx_1 dx_2 \sin^2 \left[\frac{(x_1 - x_2)}{2} \right]}{(x_1 - x_2)^2} \right] \quad (33)$$

$$\times \exp \left[\sum_{j=1}^2 \left(-4a^2 \cos^2 \varphi_j - \frac{\tilde{m}_j + \tilde{\lambda}_j}{2} e^{i\varphi_j} - \frac{\tilde{m}_j - \tilde{\lambda}_j}{2} e^{-i\varphi_j} + i\varphi_j \right) \right] \\ \times \frac{k(\tilde{z}_1, \cos \varphi_1) k(\tilde{z}_2, \cos \varphi_2) - k(\tilde{z}_1, \cos \varphi_2) k(\tilde{z}_2, \cos \varphi_1)}{\cos \varphi_1 - \cos \varphi_2}. \quad (34)$$

$$\Delta K_1^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \times \exp \left[\frac{n^2}{4\pi a^2} \int \frac{dx_1 dx_2 \sin^2 \left[\frac{(x_1 - x_2)}{2} \right]}{(x_1 - x_2)^2} \right] \quad (35)$$

$$\times \exp \left[-\frac{1}{16a^2} [(x_1 + \tilde{z}_1 - \tilde{m}_1)^2 + (x_2 + \tilde{z}_2 - \tilde{m}_2)^2] \right] \quad (36)$$

$$\times \left[\frac{g^{(n-1)}(x_2 - \tilde{z}_1)}{(n-1)!(x_1 + \tilde{z}_1)^n} Z_{n-1}^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right. \\ \left. + \operatorname{sign}(\tilde{z}_2) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 - \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right] \quad (37)$$

$$\times \operatorname{sign}(\tilde{z}_2) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 - \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \quad (38)$$

$$\times \operatorname{sign}(\tilde{z}_1) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 - \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \quad (39)$$

$$\times \operatorname{sign}(\tilde{z}_1) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 - \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \quad (40)$$

$$\times \operatorname{sign}(\tilde{z}_1) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 - \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 - \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \quad (41)$$

$$K_{(n_1, n_2)}^{(n)}(\vec{Z}', m) = \frac{1}{N! \prod_{i=1}^N \det(DW + m_1 \Gamma_{1,1} + m_2 \Gamma_{2,2})} \int \prod_{i=1}^N \det(Z + m_i) \quad (42)$$

$$\times \exp \left[\sum_{j=1}^N \left(-4a^2 \cos^2 \varphi_j - \frac{\tilde{m}_j + \tilde{\lambda}_j}{2} e^{i\varphi_j} - \frac{\tilde{m}_j - \tilde{\lambda}_j}{2} e^{-i\varphi_j} + i\varphi_j \right) \right] \quad (43)$$

$$\times \frac{k(\tilde{z}_1, \cos \varphi_1) k(\tilde{z}_2, \cos \varphi_2) - k(\tilde{z}_1, \cos \varphi_2) k(\tilde{z}_2, \cos \varphi_1)}{\cos \varphi_1 - \cos \varphi_2} \quad (44)$$

$$\Delta K_2^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) - g_1 \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \times \exp \left[\frac{(-1)^{n-1} \ln^2 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2)}{4\pi a^2} \int \frac{dx_1 dx_2}{x_1 - x_2} (x_2 + \tilde{z}_1)^n (x_1 - \tilde{z}_1)^n \right] \quad (45)$$

$$\times \exp \left[-\frac{1}{16a^2} [(x_1 - \tilde{z}_1) + \tilde{z}_2]^2 + (x_2 + \tilde{z}_2 - \tilde{z}_1)^2 \right] \quad (46)$$

$$\times \left[\frac{g^{(n-1)}(x_2 - \tilde{z}_1)}{(n-1)!(x_1 - \tilde{z}_1)^n} Z_{n-1}^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right. \\ \left. + \operatorname{sign}(\tilde{z}_2) \tilde{g}(\tilde{z}_1) [1 - (-1)^n] \left(q^2 + \tilde{\lambda}_1^2 \right) Z_n^{(n)} \left(\sqrt{q^2 - \tilde{\lambda}_1^2}, \sqrt{q^2 + \tilde{\lambda}_2^2}, \tilde{a} = 0 \right) \right] \quad (47)$$

$$\times \left[\operatorname{erf} \left[\sqrt{\frac{n}{8a^2}} \right] \right] \quad (48)$$

$$\left. + 8na^4 \tilde{g}(\tilde{z}_1) \tilde{g}(\tilde{z}_2) \exp \left[-\frac{1}{8a^2} (x_1 - \tilde{z}_1)^2 \right] \Delta K_2^{(n)} \left(\frac{\tilde{z}_1}{2n}, \frac{\tilde{z}_2}{2n} \right) \right]. \quad (49)$$

Will published soon!

Read this instead!

$$\begin{aligned}
 & \times \text{Pf} \left[-K_1^{(n+m)}(z_1^{(n)}, z_2^{(n)}) \mid K_2^{(n+m)}(z_1^{(n)}, z_2^{(n)}) \mid K_3^{(n+m)}(m_j, z_j^{(n)}) \right], \\
 & K_1^{(n)} \left(\frac{z_1}{2n}, \frac{z_2}{2n} \right)^{n+1} \frac{n^2}{4(2\pi)^{5/2}} g_2(z_1, z_2) = g_r(x_1, x_2) \delta(y_1) \delta(y_2) + g_r(z_1) \delta(x_1 - x_2) \delta(y_1 + y_2), \quad (17) \\
 & \times \exp \left[\sum_{j=1}^2 \left(-4\tilde{a}_s^2 \cos \frac{f}{2} \left(\varphi_1 - \varphi_2 \right) \right) \right] g_r(x_1, x_2) = \exp \left[-\frac{n}{4a^2} \left(x_1 + x_2 - \frac{a^2(\mu_r + \mu_l)}{n} \right)^2 + \frac{n}{4} (x_1 - x_2)^2 \right] \quad (18) \\
 & \times \frac{k(z_1, \alpha)}{k(z_1, \alpha)} \times \left[\text{sign}(x_1 - x_2) - \text{erf} \left[\sqrt{\frac{n(1+a^2)}{4a^2}} (x_1 - x_2) - \sqrt{\frac{a^2}{4n(1+a^2)}} (\mu_r - \mu_l) \right] \right], \quad (19) \\
 & g_r(z) = -2\nu \text{sign}(y) \exp \left[-\frac{n}{a^2} \left(x - \frac{a^2(\mu_r + \mu_l)}{2n} \right)^2 - ny^2 \right], \quad (20) \\
 & g_l(x) = \exp \left[-\frac{n}{2a^2} \left(x - \frac{a^2\mu_l}{n} \right)^2 \right], \quad (21) \\
 & \times \left[\text{sign}(x_2) - \text{erf} \left[\frac{z_2}{\sqrt{8a_s^2}} \right] \right] \quad (22) \\
 & + 8m^2 \text{sign} \tilde{a}_l \exp \left[-\frac{1}{8a_s^2} (\tilde{x}_1 - \tilde{a}_l)^2 \right] \Delta K_5^{(n)} \left(\frac{z_1}{2n}, \frac{z_2}{2n} \right). \quad (23)
 \end{aligned}$$

- the average number of the additional real modes for the lowest index:

$$N_{\text{add}}^{\nu=0} \stackrel{\tilde{a} \leq 1}{\approx} 2V\tilde{a}^2(W_8 - 2W_7), \quad (75)$$

- the width of the Gaussian shaped strip of complex eigenvalues:

$$2\sigma \stackrel{\tilde{a} \leq 1}{\approx} 4\tilde{a} \sqrt{\frac{W_8 - 2W_0}{V\Sigma^2}}, \quad (76)$$

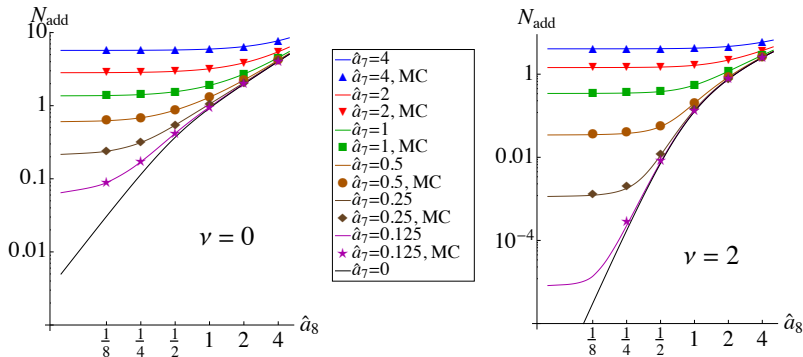
- the variance of the distribution of chirality over the real eigenvalues:

$$\langle (V\Sigma\hat{x})^2 \rangle_{\rho_x} \stackrel{\tilde{a} \leq 1}{\approx} 8V\tilde{a}^2(\nu W_8 - W_6 - W_7), \quad \nu > 0. \quad (77)$$



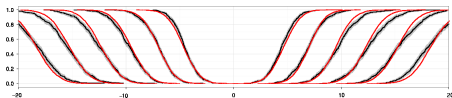
We are comparing this to lattice data right now!

Number of additional real modes

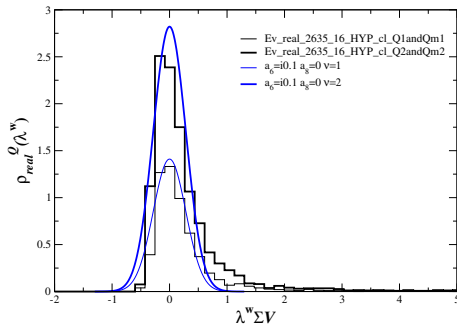


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Comparison with lattice data



- ▶ integrated distributions of individual eigenvalues of $D_5 = \gamma_5 D_W$
- ▶ Deuzman, Wenger, Wuiloid (2011)



- ▶ level density of real eigenvalues of D_W
- ▶ Damgaard, Heller, Splittorff (2012)

What has the future in store for us?



image from libertyscientist.com

In the 90's:

- ▶ chiral RMT in QCD
- ▶ Shuryak, Verbaarschot

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Future:

- ▶ broken chiral RMT in lattice QCD at finite chemical potential and temperature

Be ready for the next round in RMT for QCD!

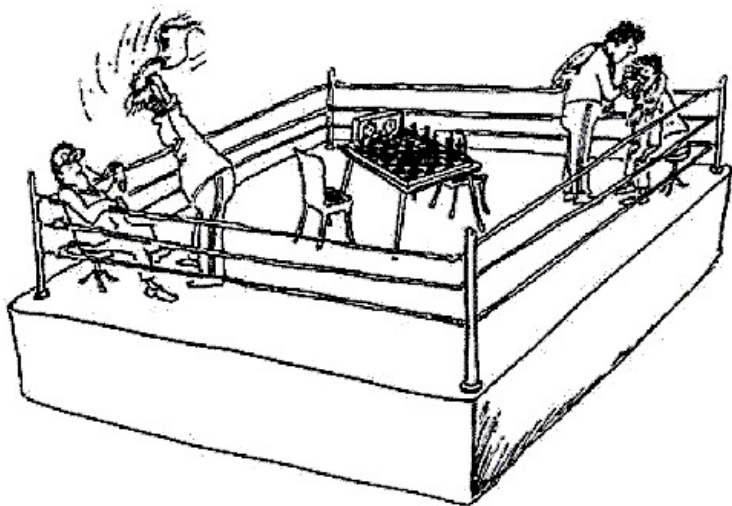


image from chessbase.de

Thank you for your attention!

Some papers:

- ▶ Kieburg, Verbaarschot, Zafeiropoulos: arXiv:1109.0656
- ▶ Kieburg, Verbaarschot, Zafeiropoulos: arXiv:1110.2690
- ▶ Kieburg, Splittorff, Verbaarschot: arXiv:1202.0620
- ▶ Kieburg: arXiv:1202.1768

Three papers are still in preparation.