STONY BROOK UNIVERSITY

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Applications of Random Matrix Theory in Lattice QCD

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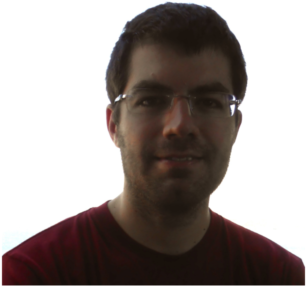
SUNY Stony Brook (NY, USA)

Brunel (West London), December 15th, 2012

Our group



our PhD student
Savvas Zafeiropoulos

Jacobus Verbaarschot
& myself

Outline

- Introduction to Lattice QCD
- Two-dimensional naive Discretization
- Wilson RMT
- What has the future in store for us?

Introduction to Lattice QCD

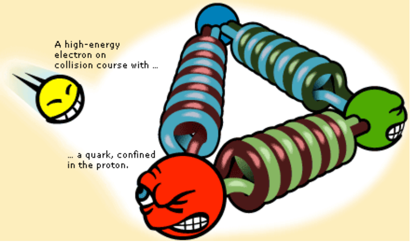


image from www.nobelprize.org (2004)

Action of continuum QCD

The partition function of N_{f} fermionic flavors

$$Z = \int \exp\left[-S_{\mathrm{YM}}(A) - \sum_{j=1}^{N_{\mathrm{f}}} \int \bar{\psi}_j (\imath D(A) - m_j) \psi_j d^4x \right] \mathrm{D}[A, \psi]$$

The Yang-Mills action of $\mathrm{SU}(3)$

$$S_{\mathrm{YM}}(A) = \frac{1}{4g^2} \int \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} d^4x$$

with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g\,[A_\mu, A_\nu]$$

The four components of the vector potential $A_\mu \in \mathrm{su}(3)$ are 3×3 matrix valued functions.

The continuum Dirac-operator

Fermionic fields ψ_j are Grassmann variables

$$\Rightarrow Z = \int \prod_{j=1}^{N_\mathrm{f}} \det\left(\imath D(A) - m_j\right) \exp\left[-S_{\mathrm{YM}}(A)\right] \mathrm{D}[A]$$

The Dirac operator

$$D(A) = \gamma^\mu\left(\frac{1}{\imath}\partial_\mu + gA_\mu\right)$$

Index-theorem:
number of zero modes (**i**ndex)=topological charge

$$\nu = \frac{1}{32\pi^2}\int \varepsilon^{\mu\nu\lambda\kappa}\mathrm{tr}\, F_{\mu\nu}F_{\lambda\kappa}d^4x$$

QCD in Continuum $\Longrightarrow$ **Lattice QCD**

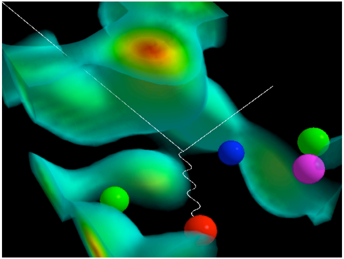


image by Derek Leinweber (CSSM)

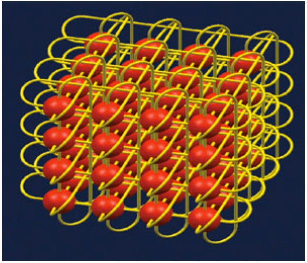
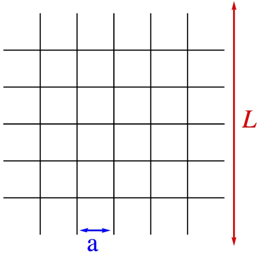


image from www.llnl.gov

Lattice QCD

- space-time V becomes discrete with lattice spacing a
- vector field $A_\mu \in \mathrm{su}(3)$ replaced by $U_\mu \in \mathrm{SU}(3)$



volume of space-time: $V = L^4$

Big question:

How do we perform the limits $a \to 0$ and $V \to \infty$ such that we obtain continuum QCD?

Fundamental problem on the lattice

Energy in continuum:

$$E^2 = k_\mu k^\mu + M_0^2$$

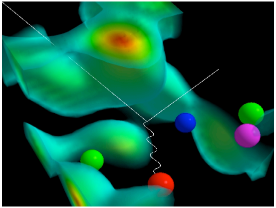


image by Derek Leinweber (CSSM)

Energy on lattice:

$$E^2 = \frac{\sin^2(k_\mu a)}{a^2} + M_0^2$$

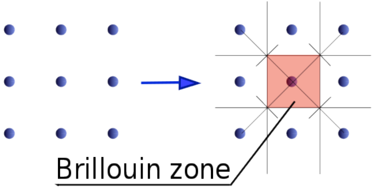


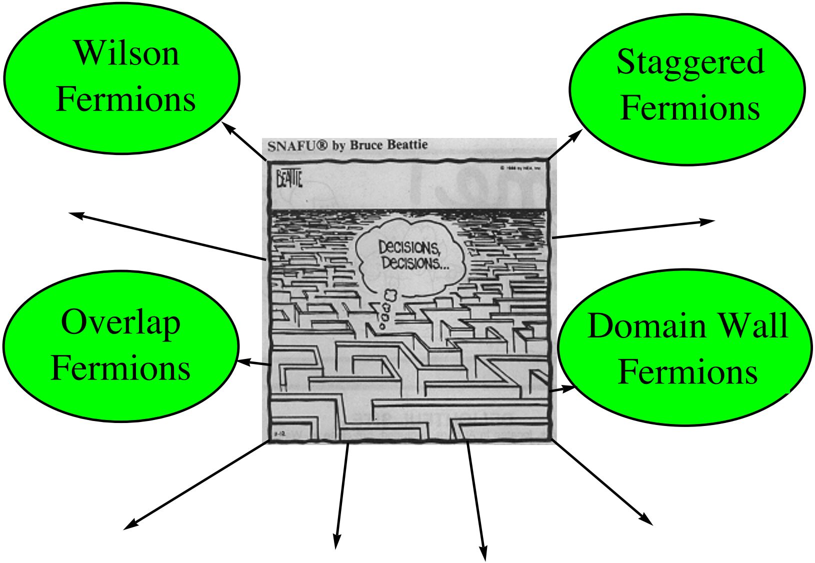
image from Wikipedia.org

Doubler Problem:

$$k_\mu \to \begin{cases} k_\mu \\ \frac{\pi}{a} - k_\mu \end{cases}$$

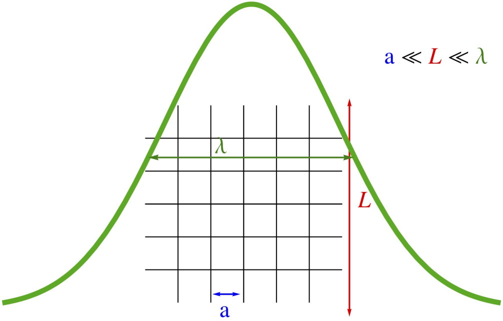
one momentum=$(2^4 = 16)$ **particles**

Many ways and no guide



The ϵ-regime of QCD

- infrared limit of QCD
- large Compton wavelength of Goldstone bosons $\gg$ box size $V^{1/4} = L$
- lattice volume (space-time volume) $V \to \infty$



Saddlepoint approximation:

- spontaneous breaking of chiral symmetry
- global Goldstone bosons = Mesons

e.g. $N_{\mathrm{f}} = 2$, $\mathrm{SU}(2)$-integral = zero momentum modes of the three pions

Partition function in the ϵ-regime for N_f flavors

$$
\begin{aligned}
Z &\propto \int_{\mathrm{SU}(N_f)} \exp[\mathcal{L}(U)] d\mu(U) \\
&\propto \sum_{\nu\in\mathbb{Z}} \int_{\mathrm{U}(N_f)} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)
\end{aligned}
$$

Lagrangian of the Goldstone bosons:

$$
\mathcal{L}(U) = \frac{\Sigma V}{2} \mathrm{tr}\, M(U + U^{\dagger}) + \mathcal{L}_{\mathrm{correction}}(V, a, U)
$$

- index of the Dirac operator: ν
- masses of the quarks: M
- low energy constants: $\Sigma, \ldots$

. . . and the same for RMT

Model: $D_{\rm QCD} \longrightarrow D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} + D_{\rm correction}(a)$

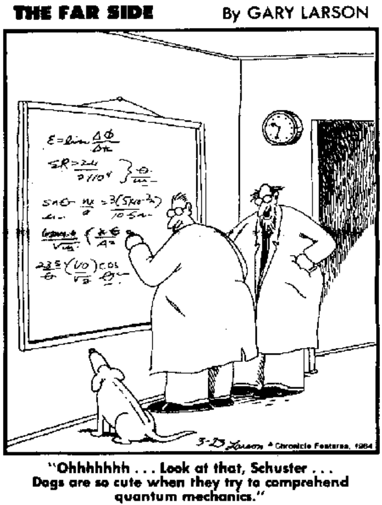
$$Z \propto \sum_{\nu\in\mathbb{Z}} \int_{\mathrm{U}(N_f)} \exp[\mathcal{L}(U)]\det{}^{\nu} U d\mu(U)$$

Lagrangian of the Goldstone bosons:

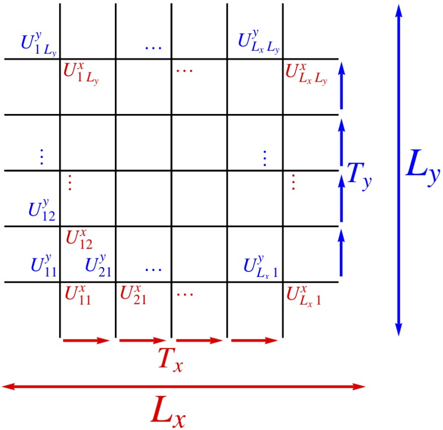
$$\mathcal{L}(U) = \frac{\Sigma V}{2} \mathrm{tr}\, M(U + U^\dagger) + \mathcal{L}_{\rm correction}(V, a, U)$$

Derived by Shuryak and Verbaarschot (90's)!

Two-dimensional naive Discretization



The naive Dirac Operator



Naive Dirac operator:

$$D_{\text{naive}} = \frac{1}{2a}\gamma^{\mu}(T_{\mu} - T_{\mu}^{-1})$$

Translation operator:

$$T_{\mu} = T_{\mu}(U^{\mu}_{ij})$$

- has the doubler problem
- but is the starting point for constructing staggered fermions

Why naive fermions?

- starting point for deriving staggered fermions
- same universality class as staggered fermions
- RMT model for staggered fermions by Osborn (2004), immensely complicated

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Why 2-D?

- simpler to understand
- our group has no supercomputer $\longrightarrow$ numerically cheaper

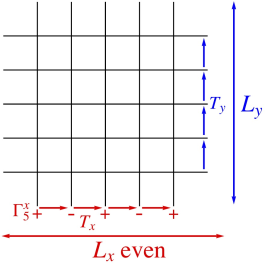
Artificial chiral structure

General RMT model: $D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix}$

Original Classification (Verbaarschot, 90's):

$$W \text{ is } \begin{cases} \text{real,} & \beta = 1 \\ \text{complex,} & \beta = 2 \\ \text{quaternion,} & \beta = 4 \end{cases}$$

Artificial chiral structure



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Yesterday

Reasons:

other dimensions = other universality classes

+

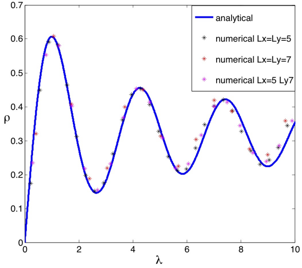
Artificial symmetry: $\Gamma_5^x T_x \Gamma_5^x = -T_x, \ \Gamma_5^x T_y \Gamma_5^x = T_y$

$\Rightarrow$ change of the universality class

Extensions into Altland-Zirnbauer classification!

Similar to the classification of topological insulators (Schnyder, Ryu, Furusaki, Ludwig (2008))

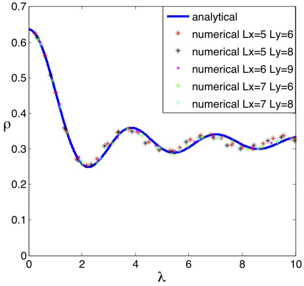
Comparison: Lattice Data $\leftrightarrow$ RMT
Three colors ($SU(3)$) & adjoint representation



odd-odd

=

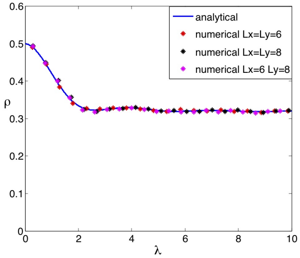
2-d cont. QCD



odd-even

=

level density of anti-symm. matrices



even-even

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staggered fermions in 2-d $\beta = 1$ in 4-d cont. QCD

Drastic change of the universality class!

Wilson RMT

Kenneth G. Wilson

The Wilson Dirac Operator

Main idea to solve the doubler problem:

- Make 15 particles infinitely heavy in the continuum limit ($a \to 0$).

$\Rightarrow$ too inertial, decouple from the system

- Wilson-Dirac operator

$$D_{\mathrm{W}} = D_{\mathrm{naive}} + a\Delta$$

$$\propto \gamma^{\mu} \sin(k_{\mu} a) + \frac{\sin^2(k_{\mu} a/2)}{a}$$

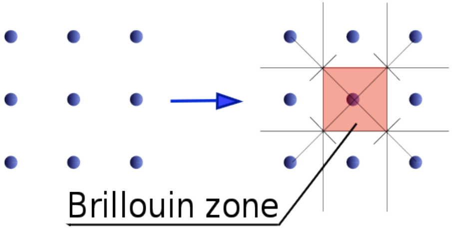


image from Wikipedia.org

- Laplace operator Δ
  - additional effective mass
  - explicitly breaks chiral symmetry
  - Dirac operator γ_5-Hermitian: $D_5 = \gamma_5 D_{\mathrm{W}}$ is Hermitian

Wilson RMT

Dirac operator:

$$D_{\mathrm{QCD}} \to D_{\mathrm{W}} = \begin{pmatrix} aA & W \\ -W^{\dagger} & aB \end{pmatrix} + am_6 \mathbb{1} + a\lambda_7\gamma_5$$

Weight:

$$\exp[-S_{\mathrm{YM}}] \to P(D_{\mathrm{W}}) : \text{ Gaussian}$$

- Hermitian random matrices A ($n \times n$), B ($(n+\nu) \times (n+\nu)$) and scalar random variables m_6, λ_7 are the Wilson-terms $\Rightarrow$ breaking of chiral symmetry
- complex W ($n \times (n+\nu)$) matrix

Damgaard, Splittorff, Verbaarschot (2010)

Partition Function of N_f flavors

$$Z \propto \int_{\mathrm{U}(N_f)} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)$$

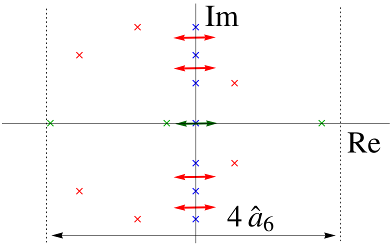
Lagrangian of the Goldstone bosons:

$$\begin{aligned}\mathcal{L}(U) &= \frac{V\Sigma}{2} \mathrm{tr}\, M(U + U^\dagger) \\ &+ a^2 V W_6 \mathrm{tr}^2(U + U^\dagger) + a^2 V W_7 \mathrm{tr}^2(U - U^\dagger) + a^2 V W_8 \mathrm{tr}\,(U^2 + U^{\dagger 2})\end{aligned}$$

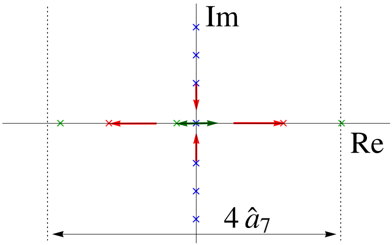
Damgaard, Splittorff, Verbaarschot (2010)

What are the low energy constants Σ and $W_{6/7/8}$?

Effect of the low energy constants

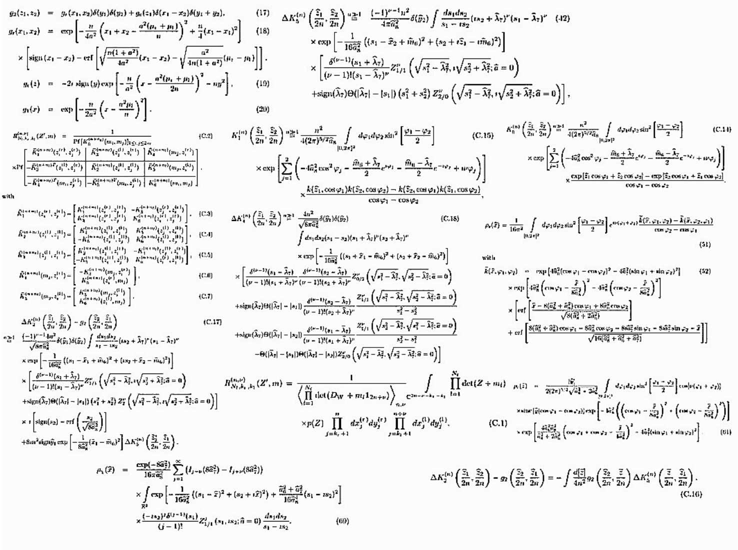


effect of W_6



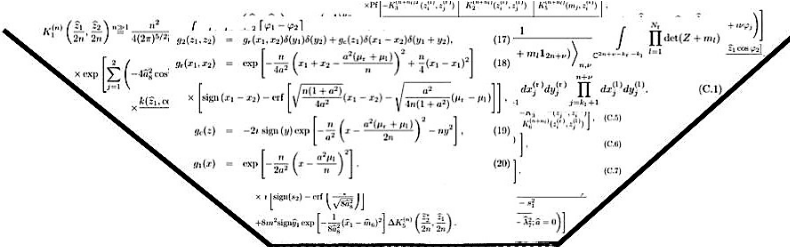
effect of W_7

Please do not read this now!



Will published soon!

Read this instead!



- the average number of the additional real modes for the lowest index:

$$N_{\rm add}^{\nu=0} \overset{\tilde{a}\ll 1}{=} 2V\tilde{a}^2(W_8 - 2W_7), \qquad (75)$$

- the width of the Gaussian shaped strip of complex eigenvalues:

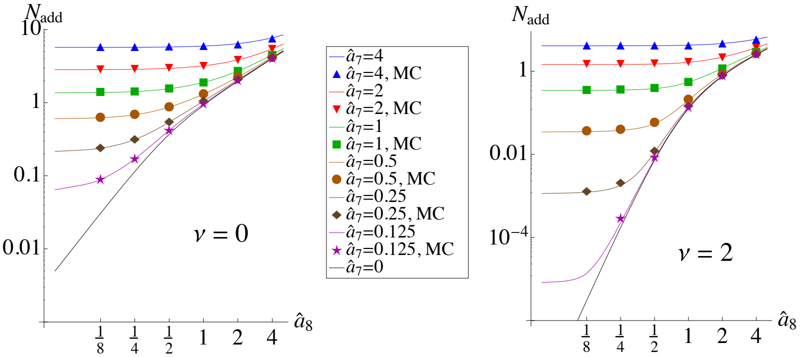
$$2\sigma \overset{\tilde{a}\ll 1}{=} 4\tilde{a}\sqrt{\frac{W_8 - 2W_6}{V\Sigma^2}}, \qquad (76)$$

- the variance of the distribution of chirality over the real eigenvalues:

$$\langle (V\Sigma\tilde{x})^2\rangle_{\rho_\chi} \overset{\tilde{a}\ll 1}{=} 8V\tilde{a}^2(\nu W_8 - W_6 - W_7),\ \nu > 0. \qquad (77)$$

We are comparing this to lattice data right now!

Number of additional real modes



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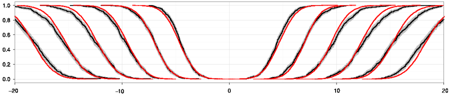
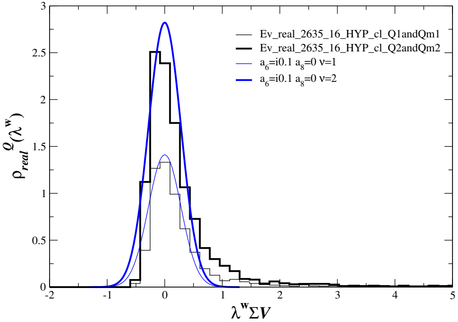
Comparison with lattice data



- integrated distributions of individual eigenvalues of $D_5 = \gamma_5 D_\mathrm{W}$
- Deuzman, Wenger, Wuilloid (2011)



- level density of real eigenvalues of D_W
- Damgaard, Heller, Splittorff (2012)

What has the future in store for us?

image from libertyscientist.com

In the 90's:

- chiral RMT in QCD
- Shuryak, Verbaarschot

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- chiral Ginibre RMT in QCD at finite chemical potential and temperature
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In the 10's:

- broken chiral RMT in lattice QCD
- Akemann, Damgaard, Kieburg, Osborn, Splittorff, Verbaarschot, Zafeiropoulos

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In the 10's:

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Future:

- broken chiral RMT in lattice QCD at finite chemical potential and temperature

Be ready for the next round in RMT for QCD!

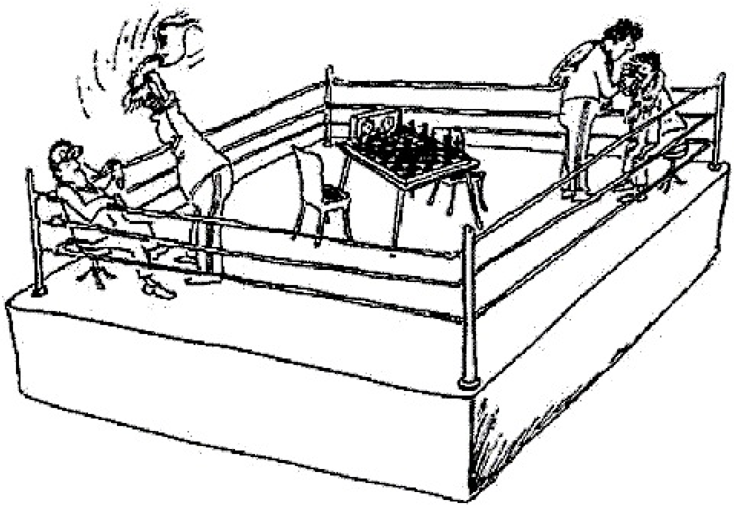

image from chessbase.de

Thank you for your attention!

Some papers:

- Kieburg, Verbaarschot, Zafeiropoulos: arXiv:1109.0656
- Kieburg, Verbaarschot, Zafeiropoulos: arXiv:1110.2690
- Kieburg, Splittorff, Verbaarschot: arXiv:1202.0620
- Kieburg: arXiv:1202.1768

Three papers are still in preparation.