Vicious Walkers, Random Matrices and 2-d Yang-Mills theory

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- J. Randon-Furling (Univ. Paris 1, Paris)
- G. Schehr (LPTMS, Orsay)

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References:

- G. Schehr, S. M., A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- C. Nadal, S. M., Phys. Rev. E **79**, 061117 (2009)
- P. J. Forrester, S. M., G. Schehr, Nucl. Phys. B **844**, 500 (2011)
- G. Schehr, S. M., A. Comtet, P. J. Forrester, arXiv: **1210.4438** (to appear in J. Stat. Phys. (2013))

Plan:

- **Part I: Nonintersecting Brownian Motions**

  N nonintersecting Brownian excursions $\Longleftrightarrow$ half-watermelons

  Two questions:

  (i) joint distribution of the positions at fixed time

  (ii) distribution of the maximal height $\rightarrow$ Exact formula

  $\Longrightarrow$ large N asymptotics via YM_2

  $\Longrightarrow$ 3-rd order phase transition

- **Part II: Yang-Mills gauge theory in 2-d**

- Summary and Conclusions

I : Nonintersecting Brownian Motions

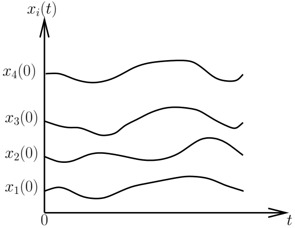
Non-intersecting Brownian motions in $1d$

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ , \ \langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{i,j}\delta(t-t')$$
$$x_1(0) < x_2(0) < \dots < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \dots < x_N(t)\ , \quad \forall t \geq 0$$



P.-G. De Gennes, 1968, D. A. Huse and M. E. Fisher, 1984, ...

Fibrous Polymers

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

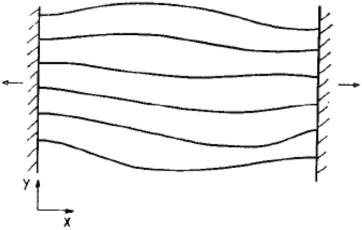


FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

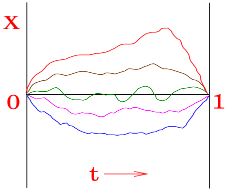
Non-intersecting Brownian bridges in 1d

- N Brownian bridges in one-dimension

$$\dot{x}_i(t) = \zeta_i(t)\ , \quad \langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{i,j}\delta(t-t')\ , \quad 0 \le t \le 1$$
$$x_i(0) = x_i(t=1) = 0$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \ldots < x_N(t)$$
$$0 < \forall t < 1$$



watermelon

Reunion probability $\to$ Comm.-Incomm. phase transition

Huse & Fisher, Fisher (1984), ..., Johansson (2002), Ferrari & Praehofer (2006),...

Non-intersecting Brownian excursions in $1d$

- N Brownian excursions in one-dimension

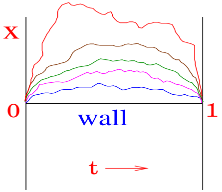
$$\dot{x}_i(t) = \zeta_i(t)\ , \quad \langle \zeta_i(t)\zeta_j(t')\rangle = \delta_{i,j}\delta(t-t')\ , \quad 0 \le t \le 1$$

$$x_i(0) = x_i(t=1) = 0 \qquad x_i(t) > 0 \quad for \quad 0 < t < 1$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \ldots < x_N(t)$$

$$0 < \forall t < 1$$



half–watermelon

watermelon "with a wall"

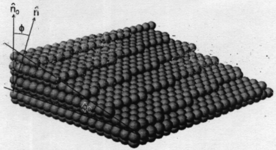
Katori & Tanemura (2004), Tracy & Widom (2007), ...

Vicious Walkers in Physics

- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- D. A. Huse and M. E. Fisher, *Commensurate melting, domain walls, and dislocations (1984)*; M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- B. Duplantier *Statistical Mechanics of Polymer Networks of Any Topology* (1989)
- J. W. Essam, A. J. Guttmann, *Vicious walkers and directed polymer networks in general dimensions* (1995)
- H. Spohn, M. Praehofer, P. L. Ferrari et al. *Stochastic growth models* (2006)
- T. L. Einstein et. al., *Fluctuating step edges on vicinal surfaces* (2004–)
- ...

Connection between Vicious Walkers and Random Matrix Theory

Fluctuating step edges: Maryland group



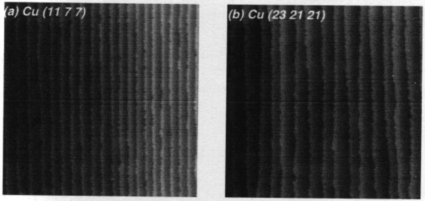
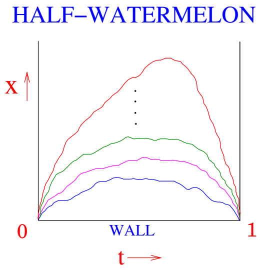


Fig. 1. STM images of (a) Cu (11 7 7) at $T=296$ K and (b) Cu (23 21 21) at $T=303$ K. The scan widths are 240 Å and 760 Å respectively. The monatomic steps run from top to bottom, and the surface height increases from left to right.

Brownian excursions and Dyck paths

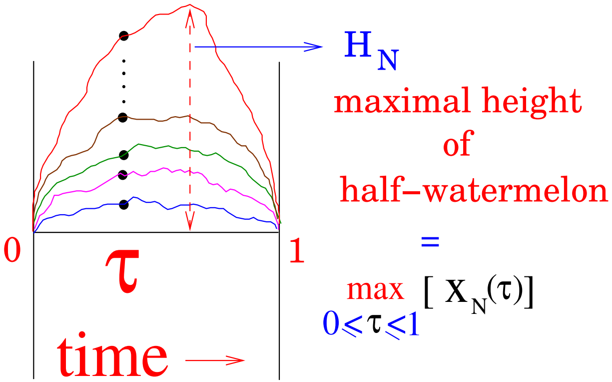


In presence of a hard wall at the origin $\rightarrow$ half-watermelons

Continuous space-time: Non-intersecting Brownian Excursions

Discrete space-time: Dyck paths (combinatorial objects)

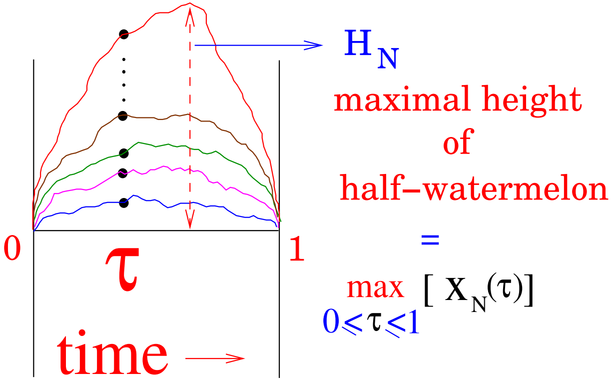
(Cardy, Katori, Tanemura, Krattenthaler, Fulmek, Feierl, Guttmann, Viennot, Tracy-Widom ...)



Q1: At fixed $0 < \tau < 1$, what is the joint distribution of positions $P_{\text{joint}}(\{x_i\}|\tau)$?

Q2: What is the probability distribution of the global maximal height $\text{Prob.}[H_N \leq M, N] = F_N(M)$?

Two questions



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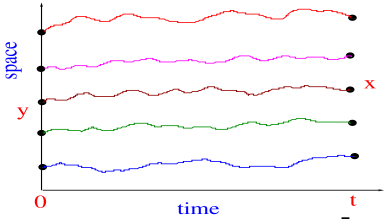
Method: Path Integral for free fermions



Propagator from $\vec{y}$ at $\tau = 0$ to $\vec{x}$ at $\tau = t$

$$G(\vec{x},\vec{y},t) = \int_{\vec{y}}^{\vec{x}} \mathcal{D}\vec{x}(\tau)\, \exp\left[-\frac{1}{2}\int_0^t \sum_i \dot{x}_i^2(\tau)d\tau\right] \mathbf{1}_{x_1(\tau)<x_2(\tau)<\cdots<x_N(\tau)}$$

$$= \langle \vec{x}|e^{-\hat{H}t}|\vec{y}\rangle; \text{ where } \hat{H} \equiv -\tfrac{1}{2}\sum_i \partial_{x_i}^2$$

$$= \sum_E \psi_E(\vec{x})\psi_E(\vec{y})\, e^{-E\,t}$$

$\psi_E(\vec{x}) \equiv \det[\phi_{n_i}(x_j)] \rightarrow$ Slater determinant ($N \times N$)

Alternative methods:

- Lindstrom-Gessel-Viennot method (discrete lattice paths)
- Karlin-Mcgregor formula (continuous paths)

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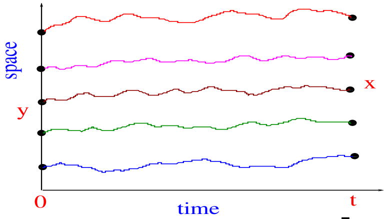

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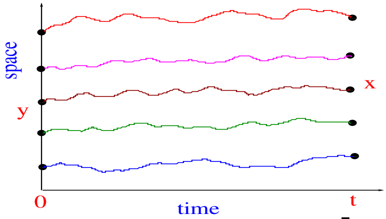

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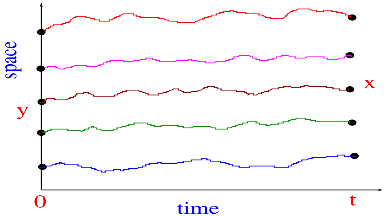

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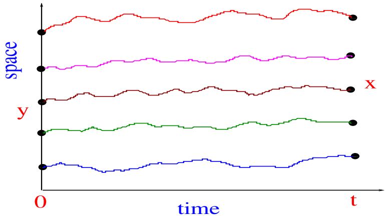

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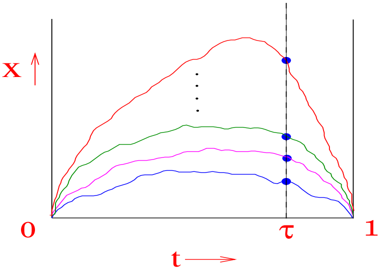
Q1: Joint distribution $\rightarrow$ Wishart eigenvalues



At fixed time $0 < \tau < 1$, let

$\{x_1, x_2, \ldots, x_N\} \rightarrow$ positions of walkers

$$\boxed{P_{\text{joint}}(\{x_i\}|\tau) \propto \prod_{i=1}^{N} x_i^2 \prod_{j<k} (x_j^2 - x_k^2)^2 \exp\left[-\frac{1}{2\tau(1-\tau)} \sum_i x_i^2\right]}$$

(Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008))

- $x_i^2 = \lambda_i \rightarrow$ eigenvalues of the Wishart matrix W

  $W = X^\dagger X$ where $X \rightarrow$ Gaussian random matrix (GUE)

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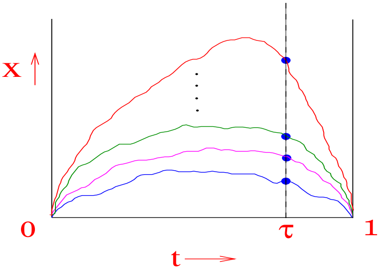

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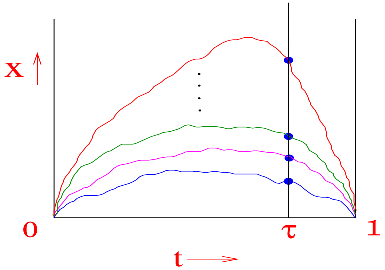

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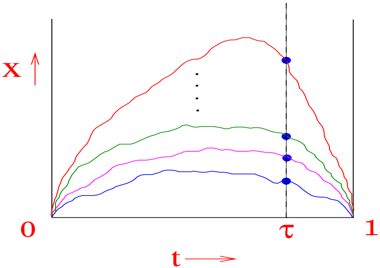
Top curve at fixed time: Tracy-Widom (GUE)



topmost curve at fixed time τ: $x_N^2(\tau) \to$ largest eigenvalue of Wishart matrices

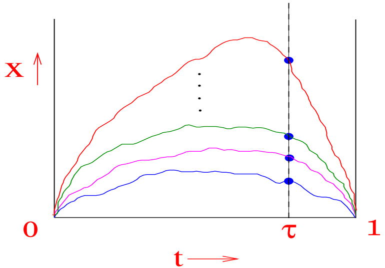
- largest eigenvalue of the Wishart GUE matrix (properly scaled for large N) is distributed via the Tracy-Widom GUE law (Johansson 2000, Johnstone, 2001)

- This shows that the top position $x_N(\tau)$ typically fluctuates for large N as

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} = 2\sqrt{N} + 2^{-2/3} N^{-1/6} \chi_2$$

where $\Pr[\chi_2 \le \xi] = \mathcal{F}_2(\xi) \to$ Tracy-Widom (GUE)

Top curve at fixed time: Tracy-Widom (GUE)



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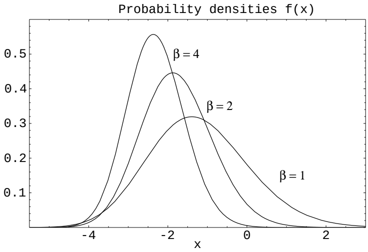
Tracy-Widom distributions (1994)

The scaling function $\mathcal{F}_\beta(x)$ has the expression:

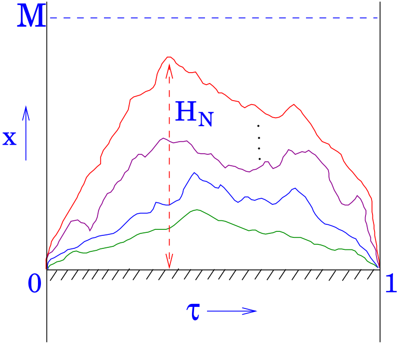
- $\beta = 1$: $\mathcal{F}_1(x) = \exp\left[-\frac{1}{2}\int_x^\infty \left[(y-x)q^2(y) + q(y)\right]\, dy\right]$
- $\beta = 2$: $\mathcal{F}_2(x) = \exp\left[-\int_x^\infty (y-x)q^2(y)\, dy\right]$

$\frac{d^2q}{dy^2} = 2\,q^3(y) + yq(y)$ with $q(y) \to \mathrm{Ai}(y)$ as $y \to \infty$ $\to$ Painlevé-II

Probability density: $f_\beta(x) = d\mathcal{F}_\beta(x)/dx$



Q2: Maximal height of a watermelon with a wall



$H_N \to$ random variable

Q: What is its distribution

$\mathrm{Prob}[H_N \le M, N] = F_N(M)$?

$N = 1$: $F_1(M) = \frac{\sqrt{2}\pi^{5/2}}{M^3} \sum_{k=1}^{\infty} k^2\, e^{-\pi^2 k^2/2\, M^2}$ (Chung '75, Kennedy '76)

$N = 2$, $F_2(M) \to$ complicated (Katori et. al., 2008)

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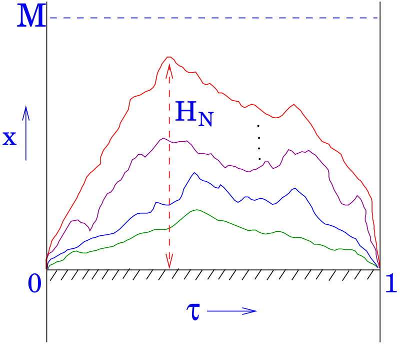

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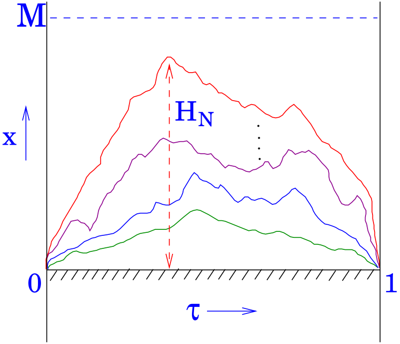

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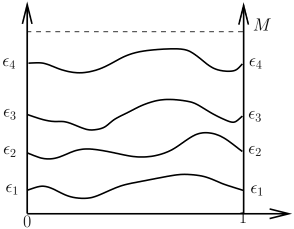
Exact result for all N via Fermionic path integral



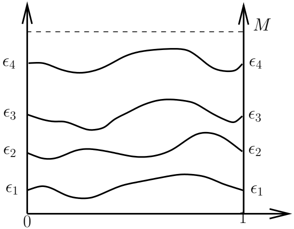
$$F_N(M) = \mathrm{Proba}[x_N(\tau) \leq M,\ \forall \tau \in [0,1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

$R_M(1) \equiv$ proba. that N walkers return to their initial positions at $\tau = 1$

- $R_M(1) = \langle \vec{\epsilon} | e^{-\hat{H}} | \vec{\epsilon} \rangle = = \sum_E |\psi_E(\vec{\epsilon})|^2 \, e^{-E}$
- $\hat{H} \equiv \sum_i \left[-\frac{1}{2} \partial_{x_i}^2 + V(x_i) \right]$
- potential $V(x) = 0$ for $0 < x < M$
  $= \infty$ for $x = 0,\ M$ (Absorbing b.c.)
- $\psi_E(\vec{\epsilon}) \equiv \det\left[\sin\left(n_i \pi \epsilon_j / M \right) \right] \rightarrow$ Slater determinant $(N \times N)$
- Energy $E = \frac{\pi^2}{2M^2} \left(n_1^2 + n_2^2 + \ldots + n_N^2 \right)$

Exact result for all N via Fermionic path integral



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  $= \infty$ for $x = 0,\ M$ (Absorbing b.c.)
- $\psi_E(\vec{\epsilon}) \equiv \det\left[\sin\left(n_i \pi \epsilon_j / M\right)\right] \to$ Slater determinant ($N \times N$)
- Energy $E = \frac{\pi^2}{2M^2}\left(n_1^2 + n_2^2 + \ldots + n_N^2\right)$

Exact result for all N via Fermionic path integral

Using Fermionic path-integral techniques we derived the full Prob. Dist. of H_N for all N exactly

Cumul. distr: $F_N(M) = \mathrm{Prob}[H_N \leq M]$

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[Schehr, S.M., Comtet, Randon-Furling, PRL, 101, 150601 (2008)]

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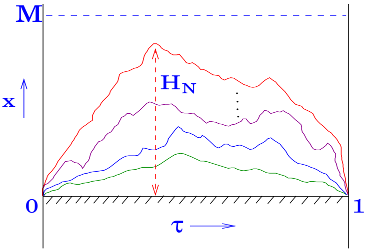
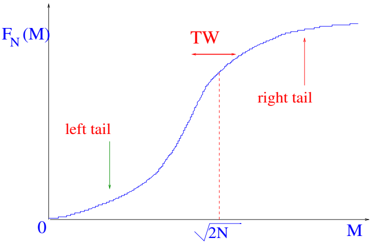
Asymptotic large N results:





Cum. distr. $F_N(M) = \mathrm{Prob.}[H_N \leq M]$ behaves, for large N as:

$$\sim \quad \exp\left[-N^2\,\Phi_-\left(\frac{M}{\sqrt{2N}}\right)\right] \qquad \text{for} \quad \sqrt{2N} - M \sim O(\sqrt{N})$$

$$\sim \quad \mathcal{F}_1\left[2^{11/6}\,N^{1/6}\,(M - \sqrt{2N})\right] \qquad \text{for} \quad |M - \sqrt{2N}| \sim O(N^{-1/6})$$

$$\sim \quad 1 - B\,\exp\left[-\beta N\,\Phi_+\left(\frac{M}{\sqrt{2N}}\right)\right] \qquad \text{for} \quad M - \sqrt{2N} \sim O(\sqrt{N})$$

Asymptotic large N results:

- where $\mathcal{F}_1(x) \to$ Tracy-Widom GOE
- $\phi_{\pm}(x) \to$ left and right rate functions $\Longrightarrow$ explicitly computable

(Schehr, S.M., Comtet, Forrester, 2011/2012)

Right rate function:

$$\phi_+(x) = 4\,x\,\sqrt{x^2-1} - 2\ln\left[2x\left(\sqrt{x^2-1}+x\right)-1\right]$$

Left rate function:

$\phi_-(x) \to$ can be expressed in terms of elliptic functions

- In particular,

$$\begin{aligned} \phi_+(x) &\simeq \frac{2^{9/2}}{3}(x-1)^{3/2} \quad \text{as} \quad x \to 1^+ \\ \phi_-(x) &\simeq \frac{16}{3}(1-x)^3 \quad \text{as} \quad x \to 1^{-1} \end{aligned}$$

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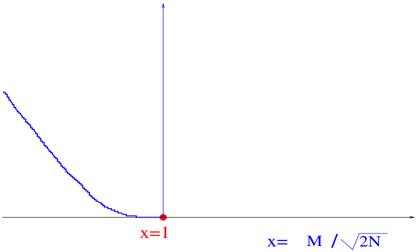
3-rd order phase transition



$$\lim_{N\to\infty} -\frac{1}{N^2}\ln F_N\left(M=\sqrt{2N}\,x\right) = \begin{cases} \phi_-(x)\,, & x<1 \\ 0\,, & x>1\,. \end{cases}$$

Since, $\phi_-(x) \sim (1-x)^3 \Rightarrow$ 3-rd order phase transition

$\Longrightarrow$ similar to the Douglas-Kazakov transition in large-N 2-d gauge theory

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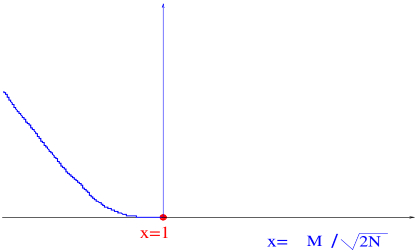

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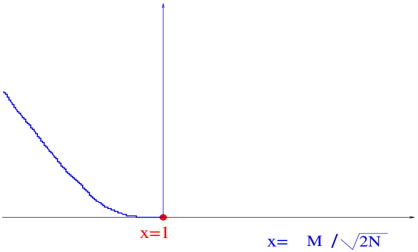

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II : Yang-Mills gauge theory in 2-d

Partition function of Yang-Mills theory in 2d

- Consider a 2-d manifold $\mathcal{M}$. At each point x: a pair of $N \times N$ matrix

$$A_\mu(x) \ (\mu = 1, 2) \rightarrow \text{gauge field}$$

Partition function: $\mathcal{Z}_\mathcal{M} = \int [\mathcal{D}A_\mu] e^{-\frac{1}{4\lambda^2}\int \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] d^2x}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \rightarrow \text{field strength}$$

$$\lambda \rightarrow \text{coupling strength}$$

- Under a local gauge transformation:

$$A_\mu \rightarrow S^{-1}(x) A_\mu S(x) - i\, S^{-1}(x) \partial_\mu S(x)$$

where $S(x) \rightarrow N \times N$ matrix that depends on the underlying gauge group G

Field strengths transform as $F_{\mu\nu} \rightarrow S^{-1}(x) F_{\mu\nu}(x) S(x)$ that keeps the action gauge invariant.

Ex: $G \equiv U(1)$: electrodynamics
$G \equiv SU(2)$: electro-weak interact^o
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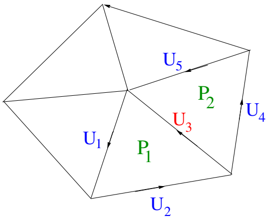
Lattice Regularization:

Consider, for instance, the $U(N)$ gauge theory

Regularization on the lattice:

$$\mathcal{Z}_{\mathcal{M}} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette}} U_L$$



$Z_P \to$ plaquette partition function

(Wilson, '74, Migdal, '75)

Heat-kernel action

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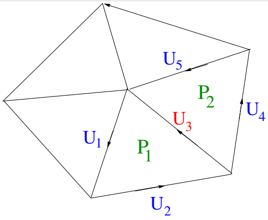

Wilson'74

- A common choice : Wilson's action

$$Z_P(U_P) = \exp\left[b\, N\, \mathrm{Tr}(U_P + U_P^\dagger)\right]$$

Exact solution of the Partition Function: (Gross & Witten, Wadia, '80)

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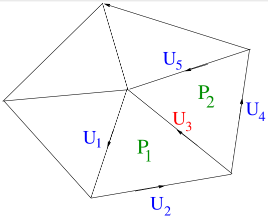

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Exact solution of the Partition Function: (Gross & Witten, Wadia, '80)

- Alternative choice : invariance under decimation $\Rightarrow$ Migdal's recursion relation

$$\int dU_3\, Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^{\dagger}) = Z_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

$$Z_P(U_P) = \sum_R d_R \chi_R(U_P) \exp\left[-\frac{A_P}{2N} C_2(R)\right]$$

Migdal'75, Rusakov'90

Partition function of Yang-Mills theory on the 2d-sphere

- Partition funct^{o} on $\mathcal{M}$, of genus g, computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_{R} d_R^{2-2g} \exp\left[-\frac{A}{2N} C_2(R)\right]$$

Partition function of Yang-Mills theory on the $2d$-sphere

- Partition funct^{o} on the sphere computed with the heat-kernel action

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- Irreducible representations R of G are labelled by the lengths of the Young diagrams:
  - If $G = \mathrm{U}(N)$

$$\mathcal{Z}_{\mathcal{M}} = c_N \, e^{-A\frac{N^2-1}{24}} \sum_{n_1,\ldots,n_N=0}^{\infty} \prod_{i<j} (n_i - n_j)^2 e^{-\frac{A}{2N} \sum_{j=1}^N n_j^2}$$

  - If $G = \mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N \, e^{A\,(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{j=1}^N n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}$$

Correspondence between YM_2 on the sphere and watermelons

- Partition function of YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \mathcal{Z}(A; \mathrm{Sp}(2N))$$

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- Cumulative distribution of the maximal height of watermelons with a wall

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \left(\prod_{j=1}^{N} n_j^2\right) \prod_{i<j} (n_i^2 - n_j^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{j=1}^{N} n_j^2}$$

$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

(Forrester, S. M., Schehr, '11)

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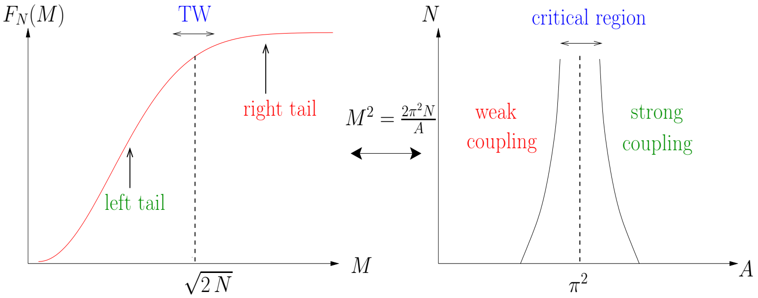
Large N limit of YM_2 and consequences for $F_N(M)$

Weak-strong coupling transition (3-rd order) in YM_2, Douglas-Kazakov '93



Critical point $A = A_c = \pi^2$ corresponds (using $A = \frac{2\pi^2 N}{M^2}$):

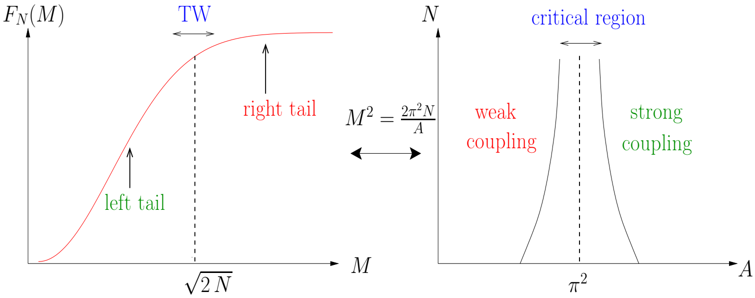
$$\boxed{M = M_c = \sqrt{2N}}$$

$A > A_c$ (Strong Coupling) $\longleftrightarrow$ $M < M_c = \sqrt{2N}$ (left tail of H_N)

$A < A_c$ (Weak Coupling) $\longleftrightarrow$ $M > M_c = \sqrt{2N}$ (right tail of H_N)

Large N limit of YM_2 and consequences for $F_N(M)$

Weak-strong coupling transition (3-rd order) in YM_2, Douglas-Kazakov '93



Critical point $A = A_c = \pi^2$ corresponds (using $A = \frac{2\pi^2 N}{M^2}$):

$$\boxed{M = M_c = \sqrt{2N}}$$

$A > A_c$ (Strong Coupling) $\longleftrightarrow$ $M < M_c = \sqrt{2N}$ (left tail of H_N)

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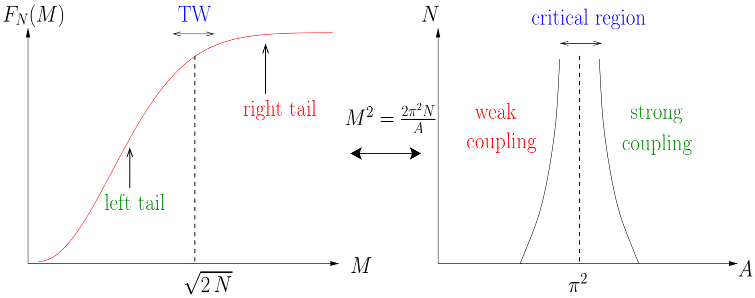

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Large N limit of YM_2 and consequences for $F_N(M)$

- In the critical regime, "double-scaling limit", the method of orthogonal polynomials (Gross-Matytsin '94, Crescimanno-Naculich-Schnitzer '96) shows

$$\frac{d^2}{dt^2}\log F_N\left(\sqrt{2N}(1+t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t)-q'(t)\right)$$

$$q''(t) = 2q^3(t) + t\,q(t)\ ,\ q(t) \sim \mathrm{Ai}(t)\ ,\ t\to\infty$$

$$\begin{aligned} F_N(M) &\to \mathcal{F}_1\left(2^{11/6}N^{1/6}\left|M-\sqrt{2N}\right|\right) \\ \mathcal{F}_1(t) &= \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\,q^2(s)+q(s)\right)\,ds\right) \\ &\equiv \text{Tracy-Widom distribution for } \beta = 1 \end{aligned}$$

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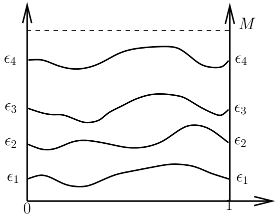
- double scaling regime $[A\sim A_c] \longleftrightarrow$ Tracy-Widom $[M\sim\sqrt{2N}]$

Forrester, S. M., Schehr, '11

Absorbing boundary condition $\rightarrow SP(2N)$

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with absorbing boundary conditions



$$F_N(M) = \mathrm{Proba}[x_N(\tau) \leq M, \forall \tau \in [0,1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

$R_M(1) \equiv$ proba. that N walkers return to their initial positions at $\tau = 1$

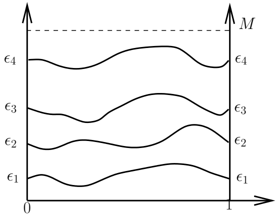
Related to YM_2 on the sphere with gauge group $\mathrm{Sp}(2N)$

$$F_N(M) \propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \mathrm{Sp}(2N)\right)$$

limiting form of $F_N(M)$: $\mathcal{F}_1 \rightarrow$ Tracy-Widom (GOE)

Periodic boundary condition $\to U(N)$

- Ratio of reunion probabilities for N vicious walkers on the segment $[0, M]$ with periodic boundary conditions



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Related to YM_2 on the sphere with gauge group $\mathrm{U(N)}$

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Reflecting boundary condition $\to SO(2N)$

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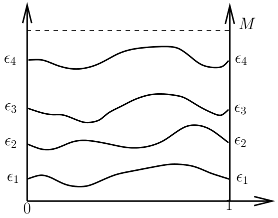

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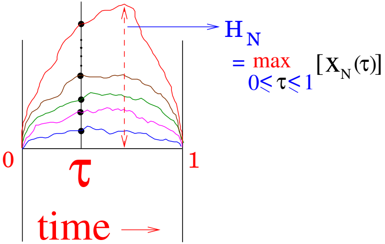

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$x_N(\tau) \rightarrow$ trajectory of the top path
$x_N(\tau)$ (centered and scaled)
$\rightarrow$ Airy_2 process minus a parabola

Prähofer & Spohn, '00

- At fixed time τ, the marginal $x_N(\tau)$ (centered and scaled)

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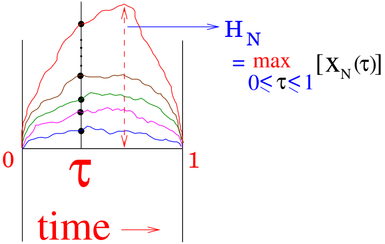
Summary



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Open Questions and related issues:

- boundary conditions $\Longleftrightarrow$ gauge groups

  deeper understanding needed

- Other interesting observables:

- Joint distribution of the maximal height $H_N = \max_{0 \le \tau \le 1}[x_N(\tau)]$ and the time τ_M at which it occurs: $P_N(H_N = M, \tau_M)$

  $\Longrightarrow$ Interesting relation to KPZ interfaces and $(1+1)$-d directed polymers

Rambeau & Schehr '11, Flores et. al. '12, Schehr '12, Quastel & Remenik, '12, Baik, Liechty, Schehr, '12

- distribution of the maximal height $H_1(N) = \max_{0 \le \tau \le 1}[x_1(\tau)] \to$ of the first (lowest) walker ?

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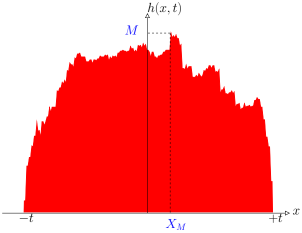
Consequences for curved stochastic growth



- Distribution of the height field $h(0,t)$ (Prähofer & Spohn, '00)

$$\lim_{t\to\infty} P\left(\frac{h(0,t)-2t}{t^{1/3}} \le s\right) = \mathcal{F}_2(s)$$

$$\mathcal{F}_2(s) \equiv \text{Tracy} - \text{Widom} \text{ distribution for } \beta = 2$$

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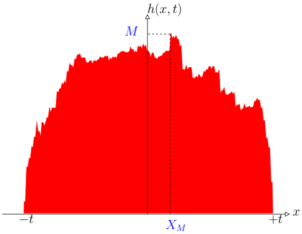

- Maximum $M \equiv \max_{-t \leq x \leq t} h(x,t)$ (Forrester, S.M. and Schehr, NPB '11)

$$\lim_{t\to\infty} P\left(\frac{M-2t}{t^{1/3}} \leq s\right) = \mathcal{F}_1(s)$$

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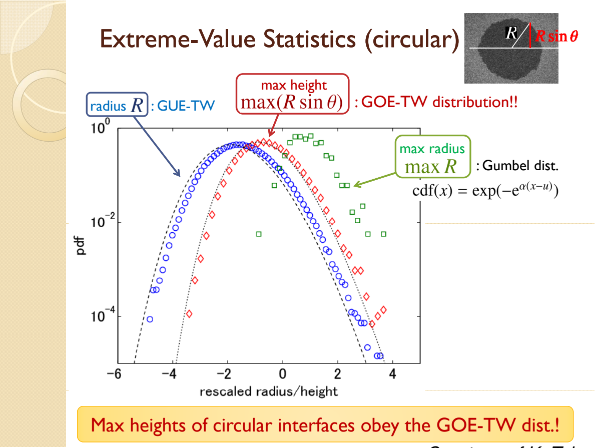
see also

- Krug *et al.* '92, Johansson '03 (indirect proof),
- G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716

Experiments on nematic liquid crystals

K. A. Takeuchi, M. Sano, Phys. Rev. Lett. **104**, 230601 (2010)



Courtesy of K. Takeuchi