Number of Minima of a Random Landscape

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Outline

1. Introduction : the model (random Gaussian surface)
2. Mean number of minima (of the random surface) and RMT (GOE)
3. Glass-like transition : analysis of the two phases using large deviations
4. Close to the transition point : Tracy-Widom

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Hamiltonian (energy)

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$V(\mathbf{x})$: random Gaussian-distributed field such that :

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- $\mu \to \infty$: harmonic well $\Rightarrow \langle \mathcal{N}_m \rangle \to 1$.
- $\mu \to 0$: random potential $\Rightarrow \langle \mathcal{N}_m \rangle \gg 1$.

Glass transition for single particle at zero temperature

$$\boxed{\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}\sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{x}_{\mathbf{k}}^{\mathbf{2}} + \mathbf{V}(\mathbf{x_1}, ..., \mathbf{x_N})}$$: random energy surface (N dimensions)

Replica trick analysis (temperature T) ⇒ critical value $\boxed{\mu_c = \sqrt{f''(0)}}$.
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$\boxed{\mu < \mu_c}$
broken replica symmetry
GLASSY PHASE

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$\boxed{\langle \mathcal{N}_m \rangle \propto \exp\left(N\Sigma_m(\mu)\right)}$ as $N \rightarrow \infty$

with $\Sigma_m(\mu) > 0$ (=complexity)

$\boxed{\mu > \mu_c}$

replica symmetric phase

$\langle \mathcal{N}_m \rangle$ subexponential

$\langle \mathcal{N}_m \rangle \rightarrow 1$ as $\mu \rightarrow \infty$

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Computing the mean number of minima $\langle \mathcal{N}_m \rangle$

$\boxed{\langle \mathcal{N}_m \rangle = \int \rho_m(\mathbf{x}) d^N \mathbf{x}}$ with $\rho_m(\mathbf{x}) =$**density of minima**.

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Kac-Rice expression for ρ_m :

$$\rho_m(\mathbf{x}) = \left\langle \left| \det \left(\partial^2_{i,j} \mathcal{H} \right) \right| \, \theta \left(\partial^2_{i,j} \mathcal{H} \right) \, \prod_{k=1}^{N} \delta \left(\partial_k \mathcal{H} \right) \right\rangle_V$$

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Heaviside : $\theta(A) = \begin{cases} 1 & \text{if } A \text{ positive definite matrix} \\ 0 & \text{otherwise} \end{cases}$

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Translational invariant covariance structure of V

$\Rightarrow \langle \partial_k V \; \partial_i \partial_j V \rangle = 0$ (at same $\mathbf{x}$) and $\langle \partial_j V \partial_k V \rangle = -\delta_{j,k} f'(0) \equiv \delta_{j,k}\sigma^2$.

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Thus

$$\boxed{\rho_m(\mathbf{x}) = \frac{1}{(\sqrt{2\pi\sigma^2})^N} \, e^{-\frac{\mu^2 \mathbf{x}^2}{2\sigma^2}} \; \langle |\det(\mu\,\mathrm{Id} - M)| \; \theta(\mu\,\mathrm{Id} - M) \rangle_M}$$

Hessian $= \partial^2_{i,j}\mathcal{H} = \mu\,\mathrm{Id} - M$ with $\boxed{M_{i,j} = -\partial_i \partial_j V}$

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M = **Gaussian random matrix**, law independent of $\mathbf{x}$:

$$\begin{aligned} \langle M_{i,j} \rangle &= 0 \\ \langle M_{k,l} M_{i,j} \rangle &= \frac{\mu_c^2}{N} \left(\delta_{i,k} \delta_{j,l} + \delta_{i,l} \delta_{j,k} + \delta_{i,j} \delta_{k,l} \right) \end{aligned} \quad \text{with } \mu_c^2 = f''(0).$$

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Link with standard Gaussian ensembles of matrices

$$\langle \mathcal{N}_m \rangle = \frac{1}{\mu^N} \left\langle \left| \det\left(\mu - M\right) \right| \, \theta\left(\mu - M\right) \right\rangle_M \quad \text{with} \quad \mathcal{P}(M) \propto e^{-\frac{N}{4\mu_c^2}\left[\mathrm{tr} M^2 - \frac{(\mathrm{tr} M)^2}{N+2}\right]}$$

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Introduce additional Gaussian integration :

$$\langle \mathcal{N}_m \rangle = \frac{1}{\mu^N} \int_{-\infty}^{+\infty} dt \; \sqrt{\frac{N}{2\pi}} \, e^{-N\frac{t^2}{2}} K_N(z_t) \qquad \text{with} \quad z_t = \mu + \mu_c t$$

$$\boxed{K_N(z) = \left\langle \, |\det(z - M_0)| \; \theta(z - M_0) \, \right\rangle_{M_0}}$$

Gaussian Orthogonal Ensemble (GOE) :

$$\boxed{P(M_0) = C_N \; \exp\left\{ -\frac{N}{4\mu_c^2} \, \mathrm{tr} {M_0}^2 \right\}}$$

From the matrix to the eigenvalues

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- $O(N)$ invariance of **GOE** measure

  $\Rightarrow$ introduce the **eigenvalues** λ_i of M_0

$$\boxed{K_N(z) = \frac{1}{z_N} \int_{-\infty}^{z} d\lambda_1 ... \int_{-\infty}^{z} d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} |z - \lambda_i| \; e^{-\frac{N}{4\mu_c^2} \lambda_i^2}}$$

$\prod_{i<j} |\lambda_i - \lambda_j|$: Vandermonde determinant (Jacobian).

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- **Change of variable** :

  $K_N(z) = z_N^{-1} \, (2\mu_c^2/N)^{\frac{N(N-1)}{4}+N} \;\; \tilde{\kappa}_N(y)$ with $z = y\sqrt{\frac{2\mu_c^2}{N}}$

$$\boxed{\tilde{\kappa}_N(y) = \int_{-\infty}^{y} d\lambda_1 \ldots \int_{-\infty}^{y} d\lambda_N \, \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} (y - \lambda_i) \, e^{-\frac{\lambda_i^2}{2}}}$$

Relation with maximal eigenvalue of GOE

- We want to compute :

$$\tilde{\kappa}_N(y) = \int^y d\lambda_1 ... \int^y d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{N} (y - \lambda_i) \; e^{-\frac{\lambda_i^2}{2}}$$

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- For **standard GOE**, ie $P(M_{\rm GOE}) \propto \; e^{-\frac{1}{2}\,{\rm tr} M_{\rm GOE}^2}$:

$$P(\lambda_1, ..., \lambda_N) = D_N \; \prod_{i<j} |\lambda_i - \lambda_j| \; \prod_i e^{-\frac{\lambda_i^2}{2}}$$

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Cumulative distribution of maximal eigenvalue for GOE :

$$\mathbb{P}_N(\lambda_{\rm max} \leq y) = \frac{Z_N(y)}{Z_N(\infty)} \quad \text{with} \quad Z_N(y) = \int^y \prod_i d\lambda_i \prod_{i<j} |\lambda_i - \lambda_j| \prod_i e^{-\frac{\lambda_i^2}{2}}$$

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- Relation with our problem :

$$\boxed{\frac{dZ_N(y)}{dy} = N\, e^{-\frac{y^2}{2}} \;\; \tilde{\kappa}_{N-1}(y)}$$

Number of Minima and GOE eigenvalues

$$\mathcal{H} = \frac{\mu}{2}\sum_{k=1}^{N} x_k^2 + V$$

Finally ,we get :

$$\boxed{\begin{gathered}\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu}\right)^N \; B_N \;\; I_N(\mu/\mu_c) \\[2em] \text{with} \;\; I_N(\mu/\mu_c) = \int_{-\infty}^{+\infty} dy \; e^{\frac{y^2}{2} - \frac{N}{2}\left(y\sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c}\right)^2} \; \frac{d}{dy}\left[\mathbb{P}_{N+1}(\lambda_{\max} \leq y)\right]\end{gathered}}$$

Number of Minima and GOE eigenvalues

$\mathcal{H} = \frac{\mu}{2} \sum_{k=1}^{N} x_k^2 + V$

Finally ,we get :

$$\boxed{\begin{aligned} &\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu} \right)^N \; B_N \;\; I_N(\mu/\mu_c) \\ &\text{number of minima} \\ &\text{with} \;\; I_N(\mu/\mu_c) = \int_{-\infty}^{+\infty} dy \; e^{\frac{y^2}{2} - \frac{N}{2} \left(y \sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c} \right)^2} \frac{d}{dy} \left[\mathbb{P}_{N+1}(\lambda_{\max} \leq y) \right] \end{aligned}}$$

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Finally ,we get :

$$\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu}\right)^N \; B_N \;\; I_N(\mu/\mu_c)$$

number of minima

μ : curvature (harmonic part)

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$\mu_c = \sqrt{f''(0)}$

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cumulative distribution of $\lambda_{\max}$ for the **GOE**

Outline

1. Introduction : the model (random Gaussian surface)

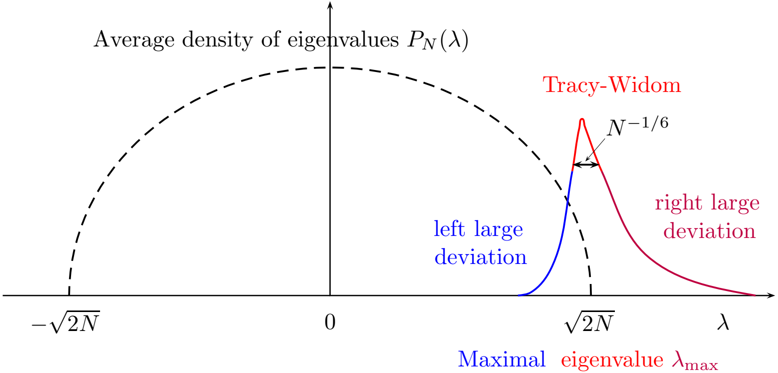
2. Mean number of minima (of the random surface) and RMT (GOE)

3. Glass-like transition : analysis of the two phases using large deviations

4. Close to the transition point : Tracy-Widom

Maximal eigenvalue of GOE : large deviations as $N \to \infty$

Standard GOE : $P(M_{\mathrm{GOE}}) \propto \; e^{-\frac{1}{2}\,\mathrm{tr} M_{\mathrm{GOE}}^2}$

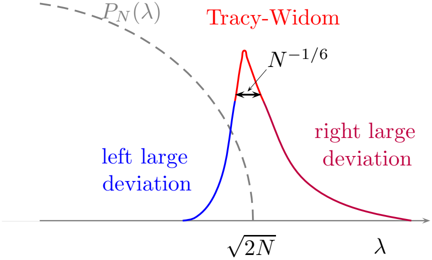


Maximal eigenvalue of GOE : large deviations as $N \to \infty$

Large deviations of pdf of $\lambda_{\max}$ for GOE

[Dean, Majumdar (2006)]

[Majumdar, Vergassola (2009)]



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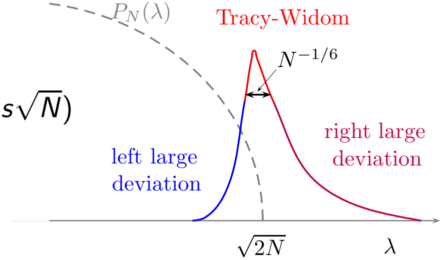


$$\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{\max} \le y)\right] = P_N(\lambda_{\max} = y = s\sqrt{N})$$

$$\approx \begin{cases} e^{-N^2\psi_-(s)} & \text{for } s < \sqrt{2} \\ e^{-N\psi_+(s)} & \text{for } s > \sqrt{2} \end{cases}$$

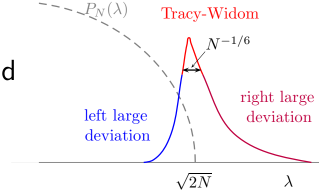
$$\psi_-(s) = \frac{s^2}{3} - \frac{s^4}{108} - \left(\frac{s^3}{108} + \frac{5s}{36}\right)\sqrt{s^2+6} - \frac{1}{2}\ln\left[\frac{s+\sqrt{s^2+6}}{3\sqrt{2}}\right]$$

$$\psi_+(s) = \frac{s^2}{2}\sqrt{1-\frac{2}{s^2}} + \ln\left[\sqrt{\frac{s^2}{2}} - \sqrt{\frac{s^2}{2}-1}\right]$$

Correspondance between $\lambda_{\max}$ and our glass transition

$$\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu} \right)^N \; B_N \; \int dy \; e^{\frac{y^2}{2} - \frac{N}{2} \left(y \sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c} \right)^2} \; \frac{d}{dy} \left[\mathbb{P}_{N+1}(\lambda_{\max} \leq y) \right]$$

Use large deviations of $\lambda_{\max}$ + saddle point method as $N \to \infty$



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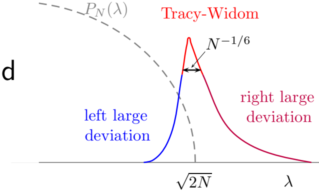

$\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{\max} \leq y)\right]$ for **GOE**	$\langle \mathcal{N}_m \rangle$ for **disordered system**
$y = \sqrt{2N} = \langle \lambda_{\max} \rangle$	$\mu = \mu_c = \sqrt{f''(0)}$

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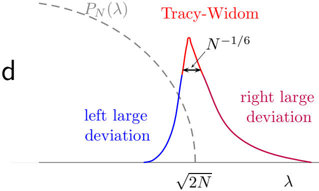

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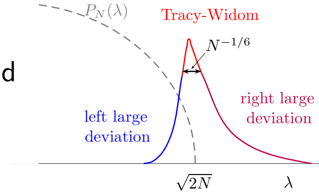

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$y < \sqrt{2N}$: left tail	$\mu < \mu_c$: glassy phase
$y > \sqrt{2N}$: right tail	$\mu > \mu_c$

Glassy phase (logarithmic equivalent for $N \to \infty$)

$$\langle \mathcal{N}_m \rangle = \left(\frac{\mu_c}{\mu} \right)^N \; B_N \quad \int dy \; e^{\frac{y^2}{2} - \frac{N}{2}\left(y\sqrt{\frac{2}{N}} - \frac{\mu}{\mu_c} \right)^2} \; \frac{d}{dy} \left[\mathbb{P}_{N+1}(\lambda_{\max} \leq y) \right]$$

Use large deviations of $\lambda_{\max}$ + saddle point method

$$\tfrac{d}{dy} \left[\mathbb{P}_N(\lambda_{\max} \leq y) \right] \approx \begin{cases} e^{-N^2 \psi_-(s)} & \text{for } s < \sqrt{2} \\ e^{-N \psi_+(s)} & \text{for } s > \sqrt{2} \end{cases}$$

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$\Rightarrow$ recover the result of [Fyodorov, Williams (2007)] :

$\mu < \mu_c$

$$\boxed{\langle \mathcal{N}_m \rangle \approx e^{N\Sigma\left(\frac{\mu}{\mu_c}\right)}}$$

$$\Sigma(m) = -\ln(m) - \frac{m^2}{2} + 2m - \frac{3}{2}$$

glassy phase : random part (V) dominates

$\mu > \mu_c$

$$\boxed{\langle \mathcal{N}_m \rangle \approx O(1)}$$

harmonic part dominates

critical point $\mu_c = \sqrt{f''(0)}$

A step further (1) : more detailed large deviation results

Large deviations of $\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{\max} \leq y)\right]$ for **GOE** : $y = s\sqrt{N}$, large N

- **Left tail** : $y < \sqrt{2N}$, ie $s < \sqrt{2}$

$$\boxed{\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{\max} \leq y)\right] \sim e^{-N^2\psi_-(s)+N\Phi_1(s)-\phi_1 \ln N-\Phi_2(s)}}$$

cf [Borot, Eynard, Majumdar, Nadal (2011)] : left large deviations of $\lambda_{\max}$ for GβE (here $\beta = 1$) using loop equations.

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- **Right tail** : $y > \sqrt{2N}$, ie $s > \sqrt{2}$

$$\boxed{\frac{d}{dy}\left[\mathbb{P}_N(\lambda_{\max} \leq y)\right] \sim \frac{e^{-N\psi_+(s)}}{2\sqrt{\pi}\,(-2+s^2)^{1/4}\sqrt{s+\sqrt{-2+s^2}}}}$$

cf [Borot, Nadal (2012)] : right large deviations for GβE (loop equations).

A step further (2) : exact equivalent for both phases

Using previous results and saddle point method, get large N equivalents :

- **Phase where the harmonic potential dominates :** $\mu > \mu_{\mathbf{c}}$

$$\boxed{\langle \mathcal{N}_m \rangle \sim 1} \quad \text{for large } N$$

one single minimum not only for $\mu \gg \mu_c$, but for all $\mu > \mu_c$.

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- **Glassy phase (where the random potential V dominates) :** $\mu < \mu_c$

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$$\boxed{S_N\left(m\right) = N\,\Sigma\left(m\right) + \sqrt{N}\Sigma_1\left(m\right) + N^{1/4}\,\Sigma_2\left(m\right)\, + \Sigma_3\left(m\right)}$$

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$$\Sigma_1(m) = \frac{4}{3}\sqrt{2}(1-m)^{3/2} \quad , \quad \Sigma_2(m) = -\frac{2}{3}2^{3/4}(1-m)^{3/4}$$

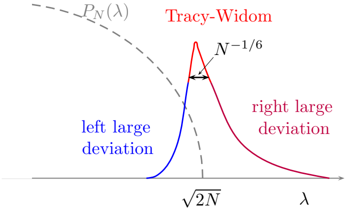
$$\Sigma_3(m) = -2 + 2m + \frac{1}{4}m^2 + \frac{137\ln 2}{96} + \frac{\ln\pi}{2} + \frac{23}{32}\ln\left([1-m]N^{1/3}\right) + \frac{1}{2}\zeta'(-1)$$

Outline

1. Introduction : the model (random Gaussian surface)
2. Mean number of minima (of the random surface) and RMT (GOE)
3. Glass-like transition : analysis of the two phases using large deviations
4. Close to the transition point : Tracy-Widom

Close to transition : Tracy-Widom (1)

GOE, fluctuations of $\lambda_{\max}$ **close to mean value**, ie $\lambda_{\max} - \sqrt{2N} = O(N^{-1/6})$:



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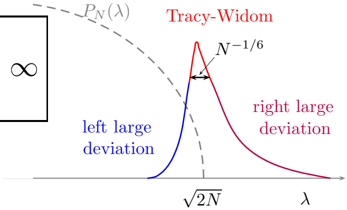
$$\boxed{\mathbb{P}_N\left(\frac{\lambda_{\max} - \sqrt{2N}}{N^{-\frac{1}{6}}/\sqrt{2}} \leq x\right) \sim \mathcal{F}_1(x) \quad \text{as } N \to \infty}$$

with $\mathcal{F}_1(x)$ the **Tracy-Widom** law for $\beta = 1$.



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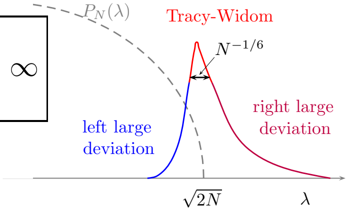

$$\mathcal{F}_1(x) = F(x)E(x)$$

$$F(x) = \exp\left[-\frac{1}{2}\int_x^{\infty}(z-x)q^2(z)dz\right] \quad , \quad E(x) = \exp\left[-\frac{1}{2}\int_x^{\infty} q(z)dz\right]$$

Close to transition : Tracy-Widom (1)

GOE, fluctuations of $\lambda_{\max}$ **close to mean value**, ie $\lambda_{\max} - \sqrt{2N} = O(N^{-1/6})$:

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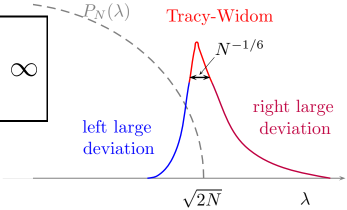

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$q(x)$ solution of **Painlevé II** :

$$\boxed{q''(x) = 2q^3(x) + xq(x)}$$

such that $q(x) \sim \mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}$ as $x \to \infty$.

Close to transition : Tracy-Widom (2)

Correspondance between **GOE** and our **disordered system** :

	$\lambda_{\max}$ for GOE	Curvature (disordered system)
Critical point	$y_c = \sqrt{2N}$	$\mu_c = \sqrt{f''(0)}$

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Transition regime :

$$\langle \mathcal{N}_m \rangle \sim \mathcal{N}(\delta) \qquad \text{for} \;\; \frac{\mu}{\mu_c} = 1 + \delta N^{-1/3} + ...$$

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with x^* the solution of $\quad -\dfrac{d}{dx}\left[\ln \mathcal{F}_1'(x^*)\right] = \delta$

Phases in disordered system $\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}\sum_{\mathbf{k}=1}^{\mathbf{N}} \mathbf{x}_\mathbf{k}^2 + \mathbf{V}$

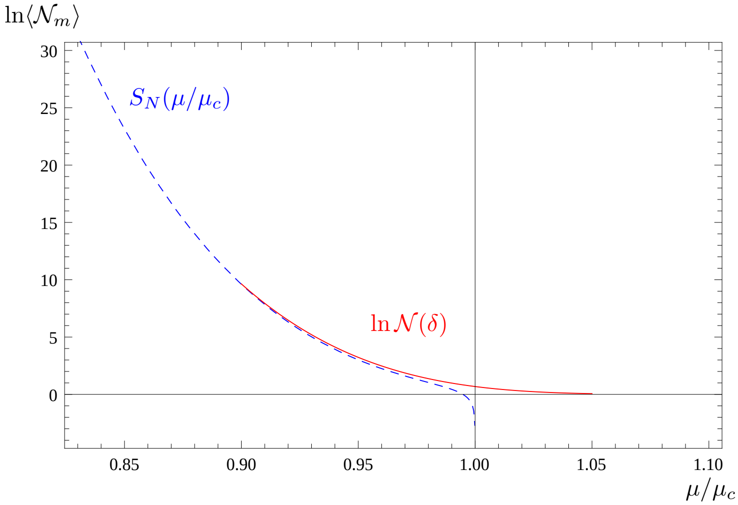

FIGURE: Log of the mean number of minima $\langle\mathcal{N}_m\rangle$ (for $N = 10000$)

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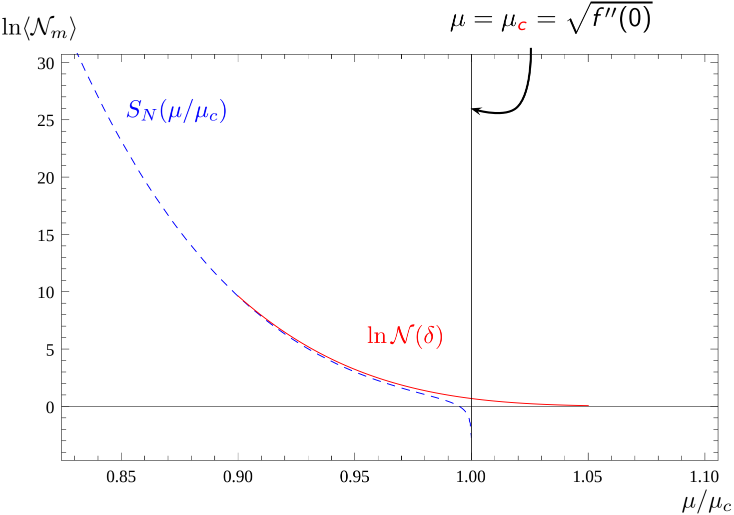

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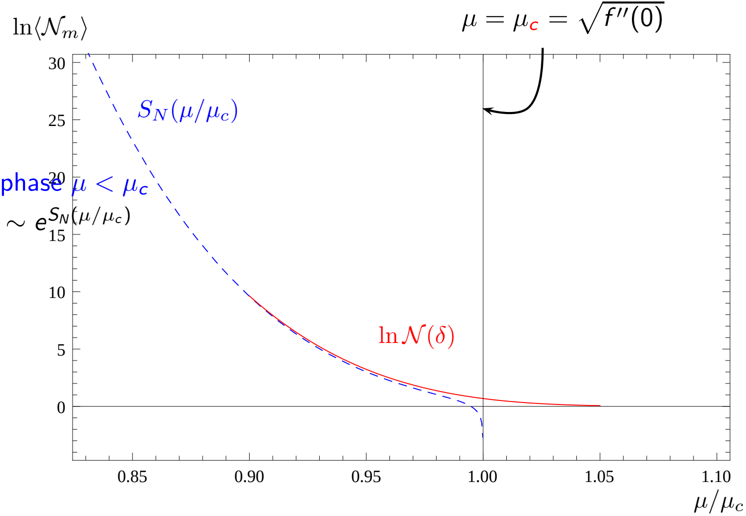
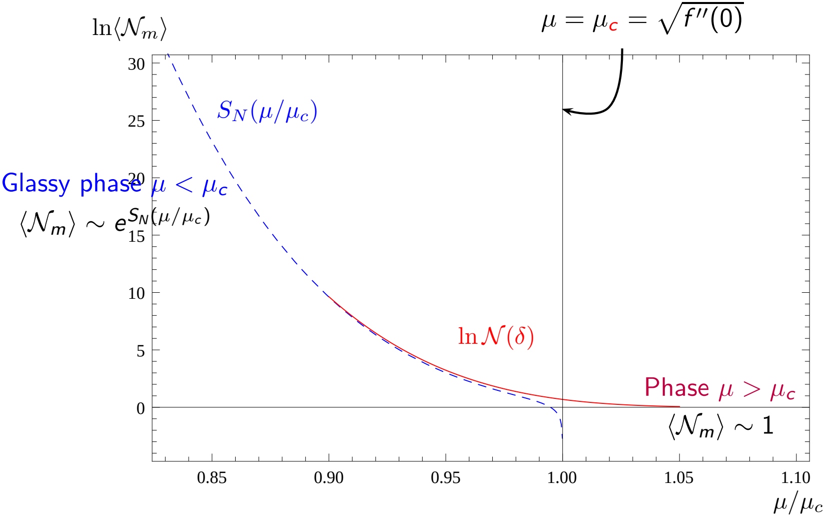


Glassy phase $\mu < \mu_c$

$\langle\mathcal{N}_m\rangle \sim e^{S_N(\mu/\mu_c)}$

FIGURE: Log of the mean number of minima $\langle\mathcal{N}_m\rangle$ (for $N = 10000$)

Phases in disordered system $\mathcal{H}(\mathbf{x}) = \frac{\mu}{2}\sum_{\mathbf{k}=1}^{\mathbf{N}} \mathbf{x}_{\mathbf{k}}^2 + \mathbf{V}$



$\mu = \mu_c = \sqrt{f''(0)}$

Glassy phase $\mu < \mu_c$

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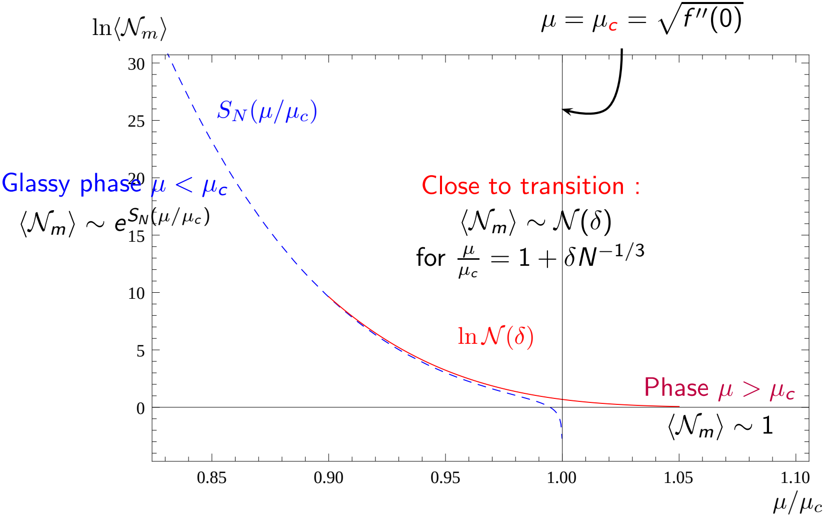

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Close to transition :

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for $\frac{\mu}{\mu_c} = 1 + \delta N^{-1/3}$

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Find $x^* \sim -2\sqrt{-2\delta}$ and

$$\boxed{\langle \mathcal{N}_m \rangle \sim 2\, 2^{\frac{21}{32}}\, \sqrt{\pi}|\delta|^{\frac{23}{32}}\, \tau_1\, e^{\frac{|\delta|^3}{3} + \frac{4\sqrt{2}}{3}|\delta|^{\frac{3}{2}} - \frac{2^{7/4}}{3}|\delta|^{\frac{3}{4}} + \frac{1}{4}}}$$

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Same result when we plug the scaling $\frac{\mu}{\mu_c} = 1 + \delta N^{-\frac{1}{3}}$ in glassy phase $\mu < \mu_c$: $\langle \mathcal{N}_m \rangle \sim e^{S_N(\mu/\mu_c)}$ (and expand for large N).

Conclusion

Study of the **mean number of minima** $\langle \mathcal{N}_m \rangle$ of a **random energy surface** :

$$\mathcal{H}(\mathbf{x}) = \frac{\mu}{2} \sum_{\mathbf{k=1}}^{\mathbf{N}} \mathbf{x}_\mathbf{k}^2 + \mathbf{V}(\mathbf{x_1}, ..., \mathbf{x_N})$$

Two phases :

- Glassy phase $\mu < \mu_c$ where $\langle \mathcal{N}_m \rangle \sim e^{S_N(\mu/\mu_c)}$ for large N.
- Phase $\mu > \mu_c$ where $\langle \mathcal{N}_m \rangle \sim 1$ for large N.

Close to transition point (intermediate regime) :

$$\langle \mathcal{N}_m \rangle \sim \mathcal{N}(\delta) \quad \text{for} \quad \frac{\mu}{\mu_c} = 1 + \delta N^{-1/3}$$

Thank you !

Reference :

Yan V. Fyodorov and Céline Nadal,

“Critical Behavior of the Number of Minima of a Random Landscape at the Glass Transition Point and the Tracy-Widom Distribution”,

Phys. Rev. Lett., **109**, 167203 (2012).