Raising the temperature in random matrix theory

Neil O'Connell, University of Warwick

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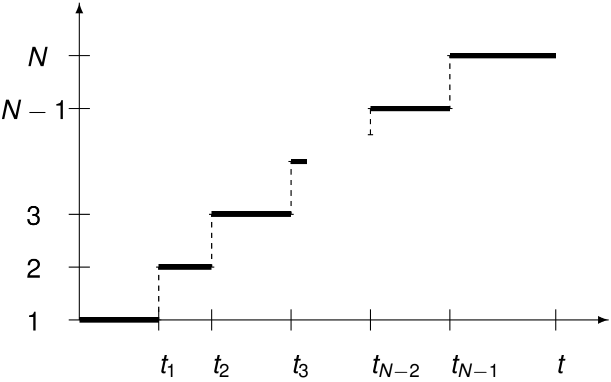
A semi-discrete random polymer model



A path $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$.

A semi-discrete random polymer model

The environment: $B_1, B_2, \ldots$ independent standard 1-dim Brownian motions.

For $\phi \equiv \{0 < t_1 < \ldots < t_{N-1} < t\}$, define

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{N-1}).$$

Boltzmann measure:

$$\mathbb{P}(d\phi) = Z_t^N(\beta)^{-1} e^{\beta E(\phi)} d\phi, \qquad Z_t^N(\beta) = \int e^{\beta E(\phi)} d\phi.$$

Scaling property

By Brownian scaling,

$$(Z_t^N(\beta), t \geq 0) \stackrel{d}{=} (\beta^{-2(N-1)} Z^N_{\beta^2 t}(1), t \geq 0).$$

Set $Z_t^N = Z_t^N(1)$.

As an interacting particle system

Set $X_t^N = \log Z_t^N$. Then

$$dX_t^N = e^{X_t^{N-1} - X_t^N} dt + dB_t^N.$$

This is a variant of TASEP, and can also be thought of as a discretisation of the Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \frac{1}{2}\partial_y^2 h + \frac{1}{2}(\partial_y h)^2 + \xi(t, y),$$

where $\xi(t, y)$ is space-time white noise.

The free energy density

Infinite particle system has product-form invariant measure for each given intensity, which allows computation of the free energy density:

Theorem (O'C-Yor '01, O'C-Moriarty '07)

Almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N^N(\beta) = \inf_{t>0} [t\beta^2 - \Psi(t)] - \log \beta^2 =: f(\beta),$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$*.*

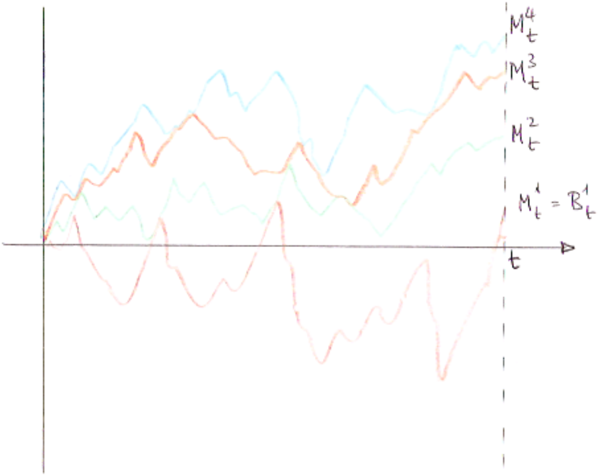
Zero-temperature limit

The law of $Z_t^N(\beta)$ is well-understood in the zero temperature limit. Define

$$\begin{aligned} M_t^N &= \lim_{\beta\to\infty} \frac{1}{\beta} \log Z_t^N(\beta) \\ &= \max_{0=t_0\le t_1\le\cdots\le t_{N-1}\le t_n=t} \sum_{i=1}^N B_i(t_i) - B_i(t_{i-1}). \end{aligned}$$

The process $(M_t^N,\ t\ge 0)$ is B^N ‘reflected off’ B^{N-1} ‘reflected off’ … ‘reflected off’ B^2 ‘reflected off’ B^1.

The reflected Brownian motion process



Connection with random matrices

By Brownian scaling, the law of $M_t^N/\sqrt{t}$ is independent of t.

Theorem (Baryshnikov '01, Gravner-Tracy-Widom '01)

The random variable M_1^N has the same law as the largest eigenvalue of a $N \times N$ GUE random matrix, that is

$$\mathbb{P}(M_1^N \leq y) = \int_{\max_{1 \leq i \leq N} x_i \leq y} c_N e^{-\sum_{i=1}^N x_i^2/2} h(x)^2 dx$$

where

$$h(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$$

and c_N is a normalisation constant.

Dyson's Brownian motion

Dyson (1960): The eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$

of a Brownian motion on $N \times N$ Hermitian matrices evolve according to the SDE

$$d\lambda_i = dB_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt.$$

Dyson's Brownian motion

Dyson's Brownian motion can be interpreted as an N-dimensional Brownian motion conditioned (in the sense of Doob) never to exit the Weyl chamber

$$C_N = \{x \in \mathbb{R}^N : \; x_1 > \cdots > x_N\}.$$

It is a diffusion process with generator

$$\frac{1}{2} h(x)^{-1} \Delta_{C_N} h(x) = \Delta/2 + \nabla \log h \cdot \nabla.$$

We can also think of it as a Brownian motion which is killed at the boundary of C_N and then conditioned to survive forever.

Extends to the level of processes

Theorem (Bougerol-Jeulin '02, O'C-Yor '02)

The process $(M_t^N,\ t \geq 0)$ has the same distribution as as the first coordinate of a Brownian motion in C_N started from 0.

The quantum Toda lattice

The quantum Toda lattice is a quantum integrable system with Hamiltonian

$$H = -\Delta + 2 \sum_{i=1}^{N-1} e^{x_{i+1} - x_i}.$$

There is a particular set of eigenfunctions ψ_λ of H, naturally indexed by $\lambda \in \iota \mathbb{R}^N$, given explicitly by an integral formula due to Givental (1997) and Joe-Kim (2003).

There is a (particular) positive eigenfunction ψ_0 with $H\psi_0 = 0$.

The process $\log Z_t^N$

Theorem (O'C 2009)

The stochastic process $\log Z_t^N,\ t > 0$ *has the same law as the first coordinate of a diffusion process in* $\mathbb{R}^N$ *with generator*

$$\mathcal{L} = -\frac{1}{2}\psi_0^{-1} H \psi_0 = \frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla$$

started from from a particular entrance law μ_t *from* '$-\infty'$.

Positive temperature Dyson Brownian motion

The diffusion process with generator

$$\mathcal{L} = -\frac{1}{2}\psi_0^{-1} H \psi_0 = \frac{1}{2}\Delta + \nabla \log \psi_0 \cdot \nabla$$

is the analogue of Dyson's Brownian motion in this setting.

It is a Brownian motion in $\mathbb{R}^N$ which is killed according to the potential $\sum_{i=1}^{N-1} e^{x_{i+1}-x_i}$, then conditioned to survive forever.

Whittaker functions

The eigenfunctions ψ_λ are $GL(N,\mathbb{R})$-Whittaker functions.

They are given by an integral formula due to Givental (1997) and Joe-Kim (2003), analogous to the combinatorial definition of Schur polynomials as generating functions for semistandard tableaux:

$$\psi_\lambda(x) = \int_{\Gamma(x)} e^{\mathcal{F}_\lambda(T)} \prod_{k=1}^{n-1} \prod_{i=1}^{k} dT_{k,i}, \qquad \lambda \in \mathbb{C}^N$$

$$\Gamma(x) = \{(T_{k,i},\ 1 \le i \le k \le n) : \ T_{n,i} = x_i,\ 1 \le i \le n\},$$

$$\mathcal{F}_\lambda(T) = \sum_{k=1}^{n} \lambda_k \left(\sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) - \sum_{k=1}^{n-1} \sum_{i=1}^{k} \left(e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}} \right).$$

Zero-temperature limit

We have:

$$\lim_{\beta\to\infty} \beta^{-N(N-1)/2}\psi_{\lambda/\beta}(\beta x) = h(\lambda)^{-1}\det(e^{\lambda_i x_j}),$$

where $h(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$. In particular,

$$\lim_{\beta\to\infty} \beta^{-N(N-1)/2}\psi_0(\beta x) = \left(\prod_{j=1}^{N-1} j!\right) h(x).$$

The functions $h(\lambda)^{-1}\det(e^{\lambda_i x_j})$ are eigenfunctions of the Dirichlet Laplacian on the Weyl chamber C_N.

Two Whittaker integral identities

For $\gamma, z \in \mathbb{C}$ and $\lambda, \nu \in \mathbb{C}^N$, $\epsilon(n) = 1_{n\ odd}$,

$$\int_{\mathbb{R}^N} e^{-e^{x_1 - z}} \psi_\lambda(x) \psi_\nu(x) dx = e^{z \sum(\lambda_i + \nu_i)} \prod_{i,j} \Gamma(\lambda_i + \nu_j)$$

$$\int_{\mathbb{R}^N} e^{-\gamma \sum_i (-1)^i x_i} e^{-e^{x_1 - z}} \psi_\lambda(x) dx = e^{z(\gamma\epsilon(n) + \sum \lambda_i)} \prod_i \Gamma(\gamma + \lambda_i) \prod_{i<j} \Gamma(\lambda_i + \lambda_j).$$

The first was conjectured by Bump (1989) and proved by Stade (2002); for this version, Gerasimov-Lebedev-Oblezin (2008).

The second was conjectured by Bump-Friedberg (1990) and proved by Stade (2001); for this version, O'C-Seppalainen-Zygouras (2012).

They are analogues of Selberg's integrals or Cauchy-Littlewood.

The entrance law

The entrance law μ_t from '$-\infty$' is given by

$$\mu_t(dx) = \psi_0(x) \int_{\iota\mathbb{R}^N} \exp\left(\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t\right) \psi_\lambda(x) s_N(\lambda) d\lambda,$$

where

$$s_N(\lambda) = \frac{1}{(2\pi\iota)^N N!} \prod_{j\neq k} \Gamma(\lambda_j - \lambda_k)^{-1}$$

is the *Sklyanin measure* - the Plancherel measure for the quantum Toda lattice [Sklyanin 1985, Wallach 1992, Semenov-Tian-Shanski 1994, Kharchev-Lebedev 1999].

The measure $\mu_t(dx)$ is the analogue of the GUE.

The law of the partition function

Using Stade's (2002) Whittaker integral identity we deduce:

Corollary (O'C 2009)

For $s > 0$*,*

$$Ee^{-sZ_t^N} = \int s^{-\sum \lambda_i} \prod_i \Gamma(\lambda_i)^N e^{\frac{1}{2}\sum_i \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where the integral is along vertical lines with $\Re\lambda_i > 0$ *for all* i*.*

The RHS is a Fredholm determinant.

Connection with random matrices

The probability measure on $\iota\mathbb{R}^N$ with density proportional to

$$e^{\sum_i \lambda_i^2 t/2} s_N(\lambda) \equiv \frac{1}{(2\pi\iota)^N N!} e^{\sum_i \lambda_i^2 t/2} \prod_{i>j} (\lambda_i - \lambda_j) \prod_{i<j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}$$

is (up to a factor of $\iota\pi$) the law, at time $1/t$, of the radial part of a Brownian motion in the symmetric space of positive definite Hermitian matrices. In particular, it is a determinantal point process, so $Ee^{-sZ_t^N}$ can be written as a Fredholm determinant.

Positive temperature GOE

Recall that

$$GUE_{N,t}(dx) = c_{N,t} e^{-\sum_{i=1}^N x_i^2/2t} h(x)^2 dx$$

where $h(x) = \prod_{1\le i<j\le N}(x_i - x_j)$. Informally,

$$GUE_{N,t}(dx) = \mathbb{P}_0(X_t \in dx \mid \tau = +\infty)$$

where X_t is a Brownian motion in $\mathbb{R}^N$ and

$$\tau = \inf\{t > 0 : \ X_t^i = X_t^j, \text{ some } i \neq j\}.$$

Similarly

$$GOE_{N,t}(dx) = c'_{N,t} e^{-\sum_{i=1}^N x_i^2/2t} h(x) dx = \mathbb{P}_0(X_t \in dx \mid \tau > t).$$

Positive temperature GOE

In the positive temperature setting, let τ be the lifetime of a Brownian motion X_t in $\mathbb{R}^N$ which is killed according to the potential $\sum_{i=1}^{N-1} e^{x_{i+1}-x_i}$. Then, informally,

$$\mu_t(dx) = \mathbb{P}_{-\infty}(X_t \in dx \mid \tau = +\infty),$$

and it is natural to define analogue of GOE informally by

$$\nu_t(dx) = \mathbb{P}_{-\infty}(X_t \in dx \mid \tau > t).$$

Then more precisely

$$\nu_t(dx) = d_{N,t} \int_{\iota\mathbb{R}^N} e^{\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t} \psi_\lambda(x) s_N(\lambda) d\lambda.$$

Positive temperature GOE

For the measure

$$\nu_t(dx) = d_{N,t} \int_{\iota\mathbb{R}^N} e^{\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t} \psi_\lambda(x) s_N(\lambda) d\lambda,$$

we can compute the generating function of ‘largest eigenvalue’ using

$$\int_{\mathbb{R}^N} e^{-e^{x_1-z}} \psi_\lambda(x) dx = e^{z\sum\lambda_i} \prod_i \Gamma(\lambda_i) \prod_{i<j} \Gamma(\lambda_i + \lambda_j)$$

to obtain

$$\int_{\mathbb{R}^N} e^{-e^{x_1-z}} \nu_t(dx) = d_{N,t} \int e^{z\sum\lambda_i} \prod_i \Gamma(\lambda_i) \prod_{i<j} \Gamma(\lambda_i+\lambda_j) e^{\frac{1}{2}\sum_{i=1}^N \lambda_i^2 t} s_N(\lambda) d\lambda,$$

where the integral is along vertical lines with $\Re\lambda_i > 0$ for all i.

Positive temperature LUE

Laguerre unitary ensemble / complex Wishart

Let $A = (a_{ij})$ be a $n \times n$ matrix with independent complex normal entries. Eigenvalues of AA^* have density

$$z_n^{-1} h(x)^2 \prod_i e^{-x_i} dx, \qquad x_1 > x_2 > \cdots > x_n > 0.$$

First marginal gives law of last passage time for directed percolation with exponential weights (Johansson 2000).

Positive temperature LUE

Positive temperature version:

$$L(dx) = \prod_{ij} \Gamma(a_i + b_j)^{-1} e^{-e^{-x_n}} \psi_{-a}(x) \psi_{-b}(x) dx, \qquad x \in \mathbb{R}^n$$

where $a_i, b_i > 0$.

First marginal gives law of logarithmic partition function for random polymer on lattice with log-gamma weights (Corwin-O'C-Seppalainen-Zygouras 2011).

Positive temperature LOE/LSE interpolating ensembles

LOE/LSE interpolating ensembles, Baik (2002)

$$k_n^{-1} e^{\delta \sum_i (-1)^i x_i} h(x) \prod_i e^{-x_i} dx, \qquad x_1 > x_2 > \cdots > x_n > 0.$$

First marginal gives law of last passage time for directed percolation with exponential weights constrained to be symmetric about main diagonal, with diminished weight on the diagonal (cf. Baik-Rains 2001).

Positive temperature LOE/LSE interpolating ensembles

Define

$$M(dx) = (k_n')^{-1} e^{\delta \sum_i (-1)^i x_i} e^{-e^{-x_n}} \psi_{-a}(x) dx, \qquad x \in \mathbb{R}^n$$

where $a_i > 0$ and

$$k_n' = \prod_i \Gamma(a_i + \delta) \prod_{i<j} \Gamma(a_i + a_j).$$

First marginal gives law of log partition function for random polymer with log-gamma weights constrained to be symmetric about main diagonal, with diminished weight on the diagonal (O'C-Seppalainen-Zygouras 2012).