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Random matrix theory and elliptic curves

December 15th, 2012

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Excised random matrix model

$$A \in \mathrm{SO}(2N)$$

Characteristic polynomial:

$$\Lambda_A(e^{i\theta}, N) := \det(I - e^{i\theta} A^{-1}) = \prod_{k=1}^{N} (1 - e^{i(\theta - \theta_k)})(1 - e^{i(\theta + \theta_k)})$$

$$|\Lambda_A(1, N)| \geq \exp \mathcal{X}$$

Call this set of matrices $T_{\mathcal{X}}$

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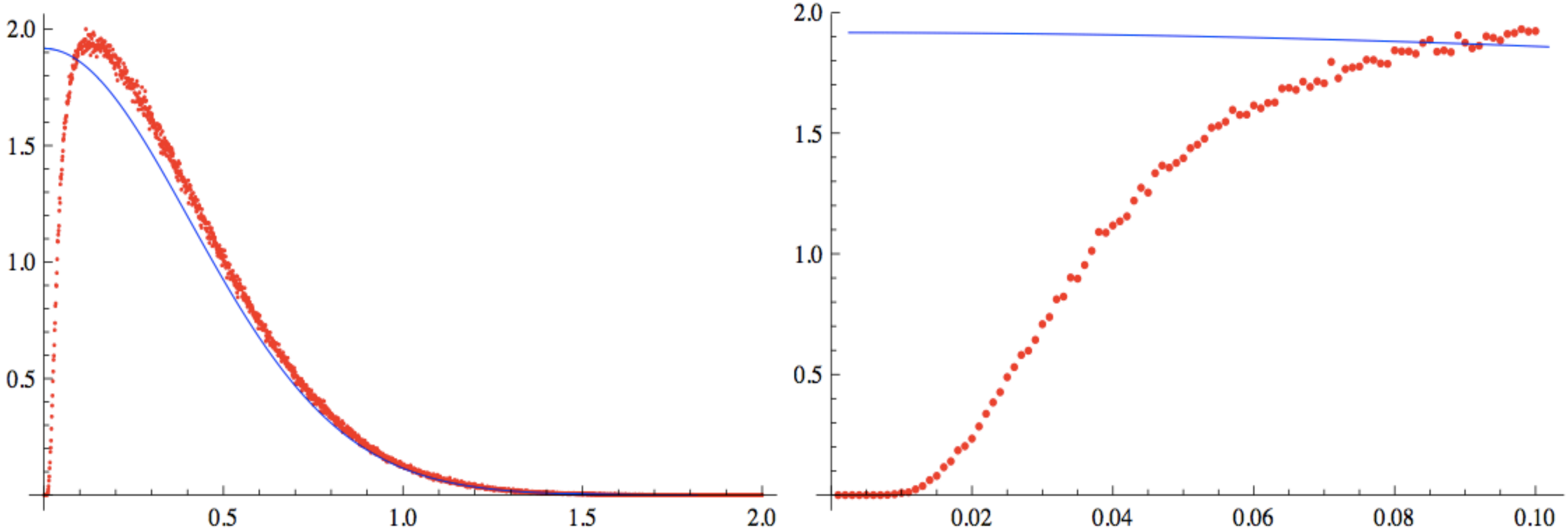
Distribution of the first eigenvalue from SO(24) (blue) and of the subset $T_{\mathcal{X}}$ with $\exp(\mathcal{X}) \approx 0.005$ (red)



Theorem 1.1. *The 1-level density $R_1^{T_\mathcal{X}}$ for the set $T_\mathcal{X}$ of matrices $A \in SO(2N)$ with characteristic polynomial $\Lambda_A(e^{i\theta}, N)$ satisfying $\log|\Lambda_A(1,N)| \geq \mathcal{X}$ is given by*

$$R_1^{T_\mathcal{X}}(\theta_1) = \frac{C_\mathcal{X}}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{Nr}\frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r-1/2, -1/2)dr$$

where $C_\mathcal{X}$ is a normalization constant

$$R_1^{J_N}(\theta_1; r-1/2, -1/2) = N\int_0^\pi \cdots \int_0^\pi \prod_{j=1}^N w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k}(\cos\theta_j - \cos\theta_k)^2 d\theta_2\cdots d\theta_N$$

is the 1-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r-1/2 \text{ and } \beta = -1/2.$$

Theorem 1.2. *The 1-level density $R_1^{T\chi}$ from Theorem 1.1 satisfies*

$$R_1^{T\chi}(\theta) = \frac{C_\chi}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(-r\mathcal{X})}{r}2^{N^2+2Nr-N}\times$$

$$\times\prod_{j=0}^{N-1}\frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)}\times$$

$$\times(1-\cos\theta)^r\frac{2^{1-r}}{2N+r-1}\frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)}P(N,r,\theta)\,dr$$

Theorem 1.3. *The expansion of $R_1^{T_\mathcal{X}}$ simplifies to*

$$R_1^{T_\mathcal{X}}(\theta) = \begin{cases} 0 & \text{for } d(\theta,\mathcal{X}) < 0 \\ R_1^{\mathrm{SO}(2N)}(\theta) + C_\mathcal{X} \sum_{k=0}^{\infty} b_k(\theta) \exp((k+1/2)\mathcal{X}) & \text{for } d(\theta,\mathcal{X}) \geq 0, \end{cases}$$

where $d(\theta,\mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ *and* b_k *are coefficients arising from the residues.*

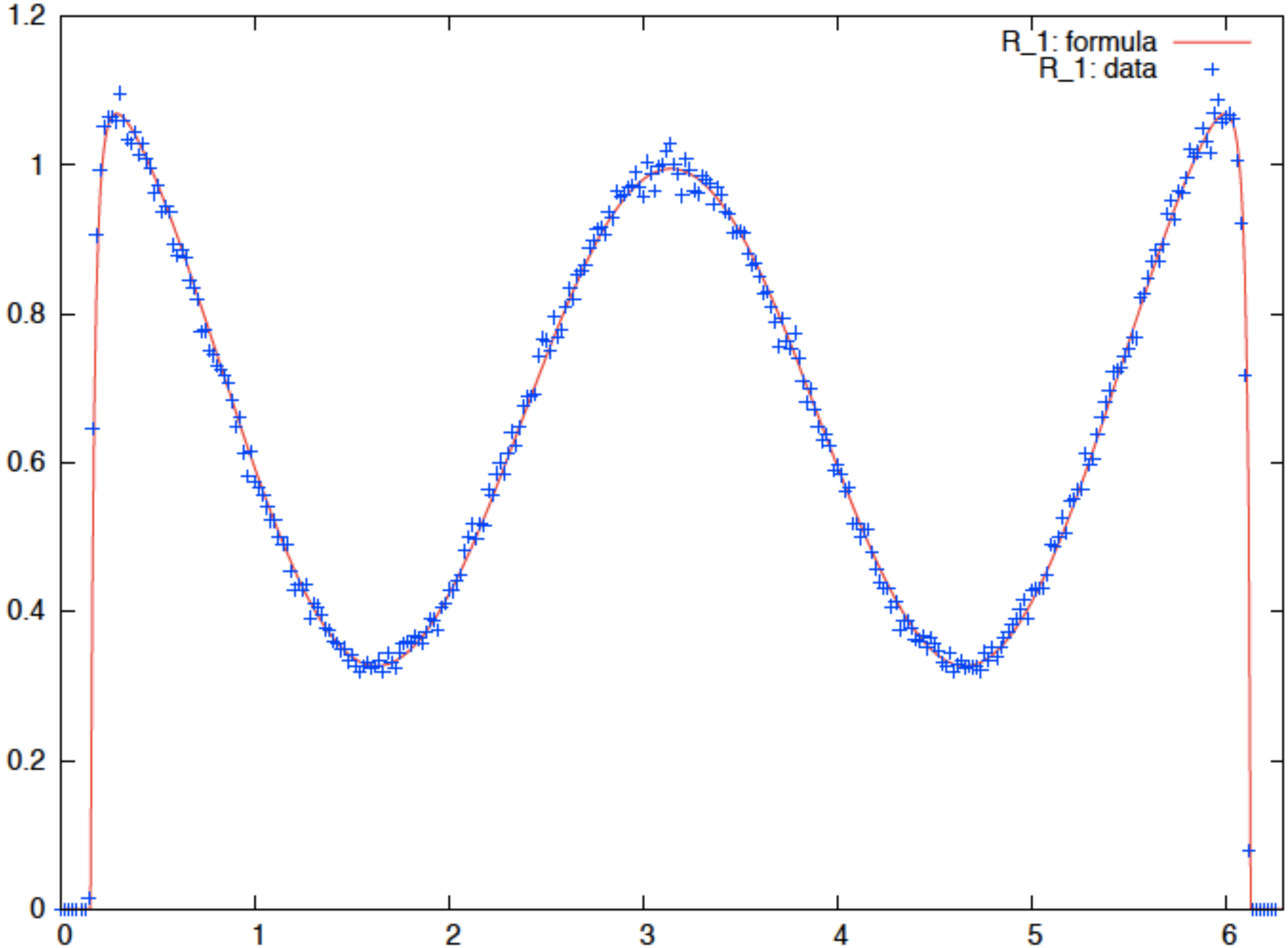
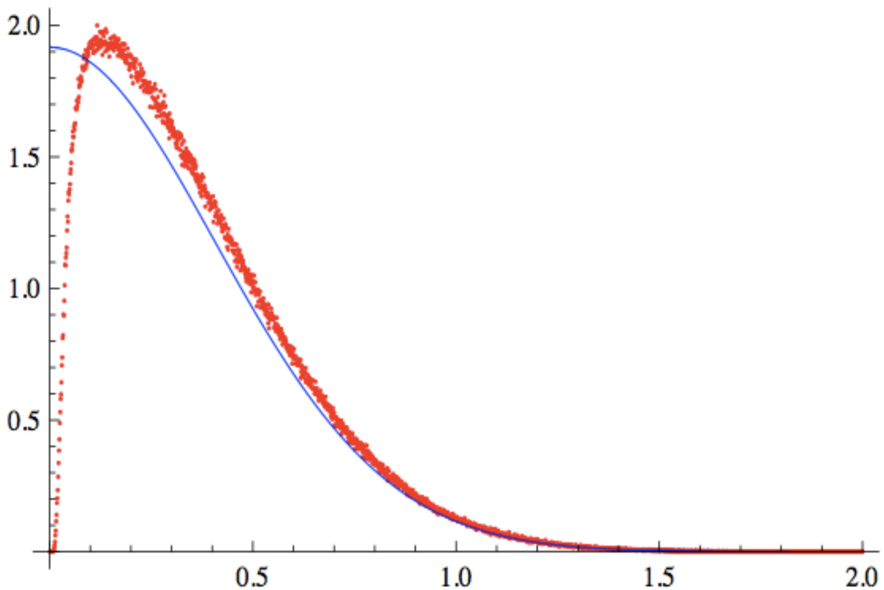
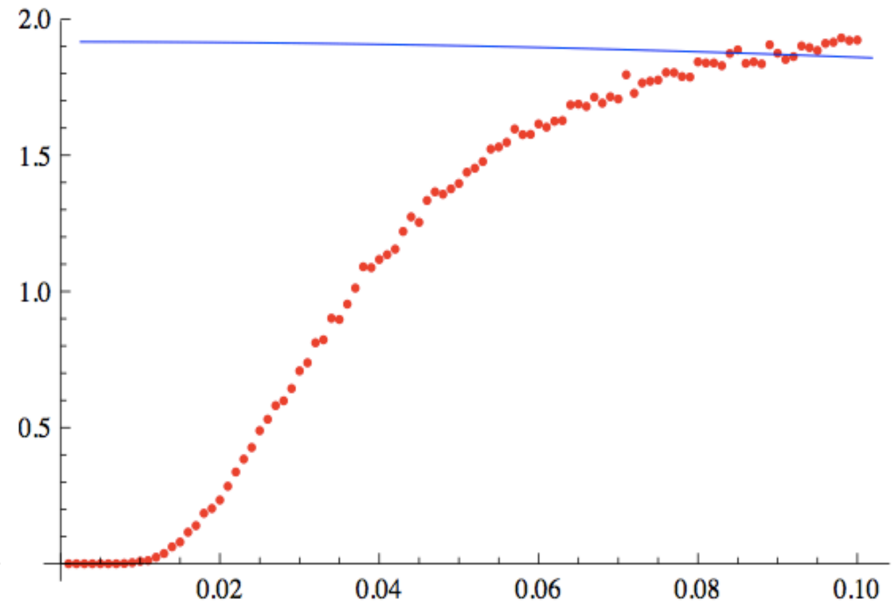


FIGURE 13. 1-level density of excised $\mathrm{SO}(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The *red curve* uses our formulæ from Theorems 1.2 and 1.3. The *blue crosses* give the empirical 1-level density of 200,000 numerically generated matrices.

Distribution of the first eigenvalue from SO(24) (blue) and of the subset $T_{\mathcal{X}}$ with $\exp(\mathcal{X}) \approx 0.005$ (red)





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Hard Gap

$$\Lambda_A(1, N) = 2^N \prod_{n=1}^{N} (1 - \cos\theta_n)$$

2N-2 eigenvalues at -1 and a pair at $e^{\pm i\theta}$

$$\log \Lambda_A(1, N) =$$

$$(2N-1)\log 2 + \log(1-\cos\theta) > \mathcal{X}$$

$$\theta > \theta_{\inf} := \cos^{-1}(1 - 2^{-(2N-1)} e^{\mathcal{X}})$$

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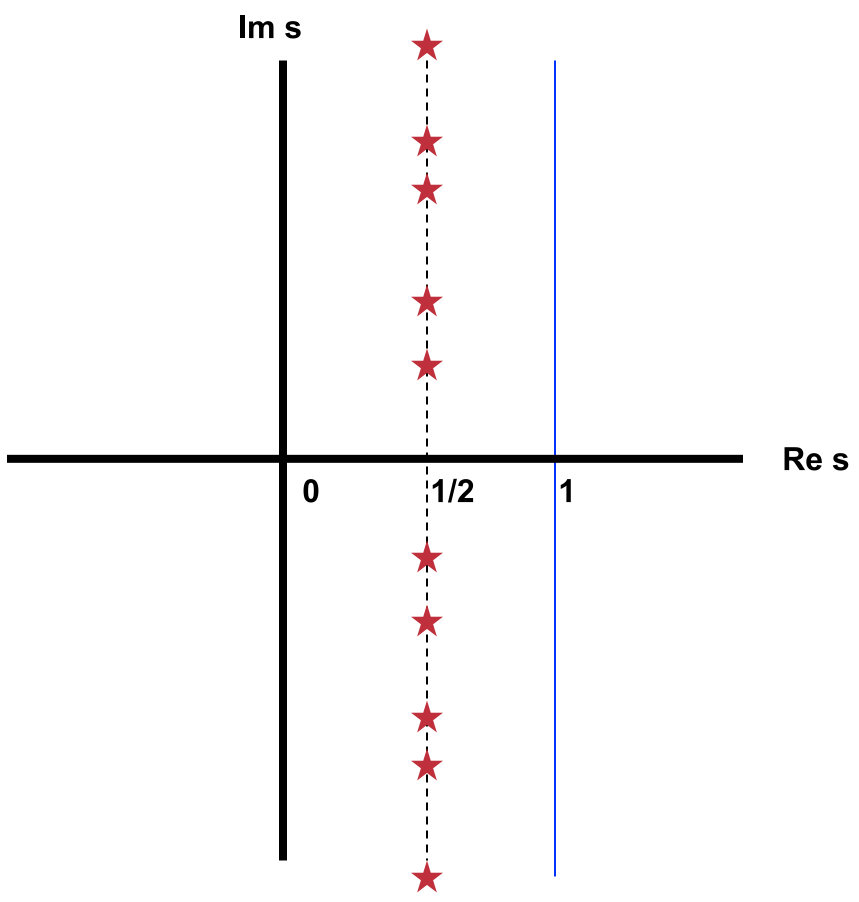
L- functions:



Families of L-functions:

- vary parameter(s) to obtain different L-functions
- family ordered by the parameter
- look at statistics of zeros averaged over the family

Katz and Sarnak (1999):

Averaged over a family of L-functions, zeros close to $s = 1/2$ show statistics like ONE of

$$U(N), O(N), USp(2N),$$

depending on the family, when the ordering parameter becomes large.

Elliptic curve L-functions:

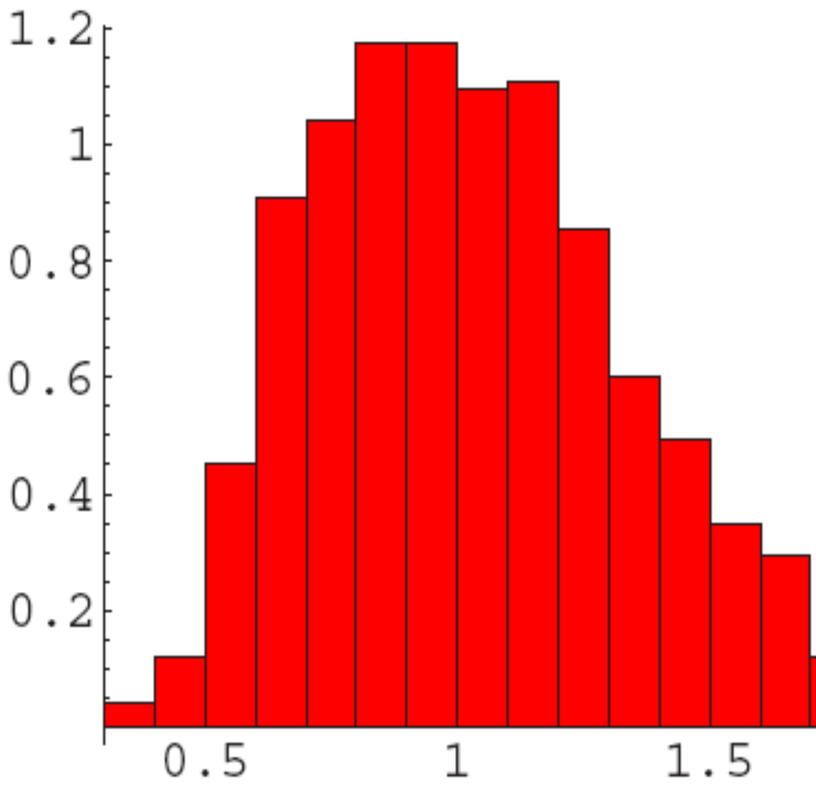
eg.

$$E_{11}:\ y^2 = 4x^3 - 4x^2 - 40x - 79$$

L-function:

$$L_{E_{11}}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

a_n determined by E_{11}



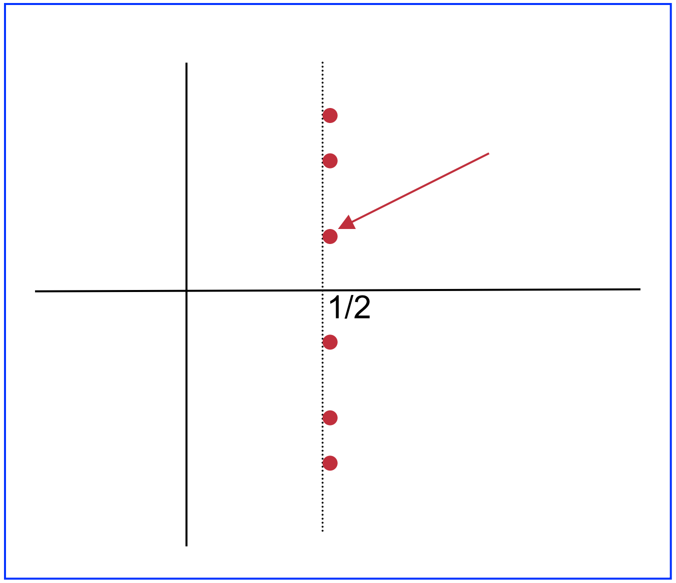


Figure 3: First normalized zero above the central point: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, standard deviation about the mean $= .32$

From: **Miller SJ,** Investigations of zeros near the central point of elliptic curve L-functions EXPERIMENTAL MATHEMATICS 15(3):257-279 2006

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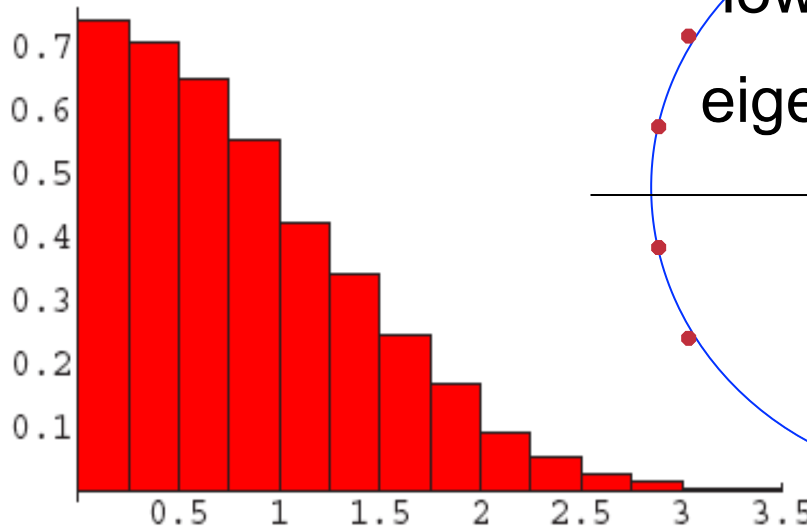


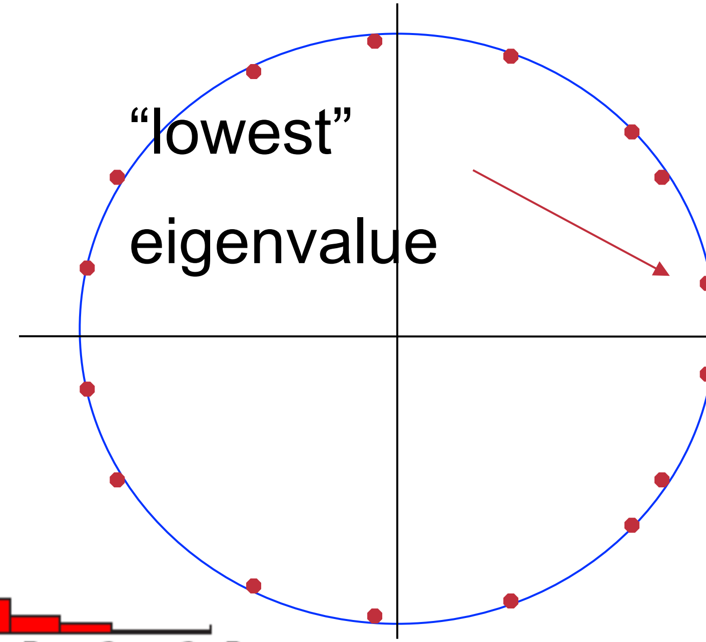


Figure 1a: First normalized eigenangle above 1: 23,040 SO(4) matrices
Mean = .709, Standard Deviation about the Mean = .601, Median = .709

From: **Miller SJ,** Investigations of zeros near the central point of elliptic curve L-functions

EXPERIMENTAL MATHEMATICS 15(3):257-279 2006

L-values discretised

$$L_E(1/2, \chi_d) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{d}},$$

(Waldspurger,Shimura,Kohnen-Zagier)

where the $c_E(|d|)$ are integers; the fourier coefficients of a half-integral weight form.

So, if

$$L_E(1/2, \chi_d) < \frac{\kappa_E}{\sqrt{d}}$$

then

$$L_E(1/2, \chi_d) = 0.$$

Hypothesis:

Discretisation causes apparent repulsion from the symmetry point of the zeros?

Resolution:

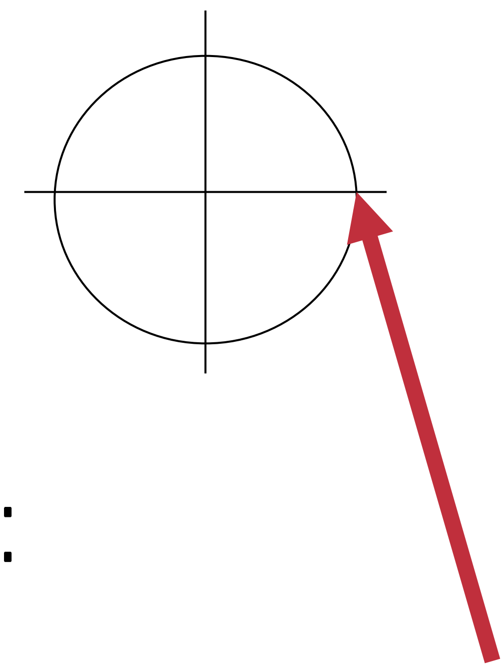
Discretise the values of the characteristic polynomial of the random matrices

$$\Lambda(s) = \prod_{j=1}^{N} (1 - se^{i\theta_j})(1 - se^{-i\theta_j})$$

$A \in SO(2N)$

Characteristic polynomial:

$$\Lambda_A(s) = \prod_{j=1}^{N} (1 - s e^{i\theta_j})(1 - s e^{-i\theta_j})$$

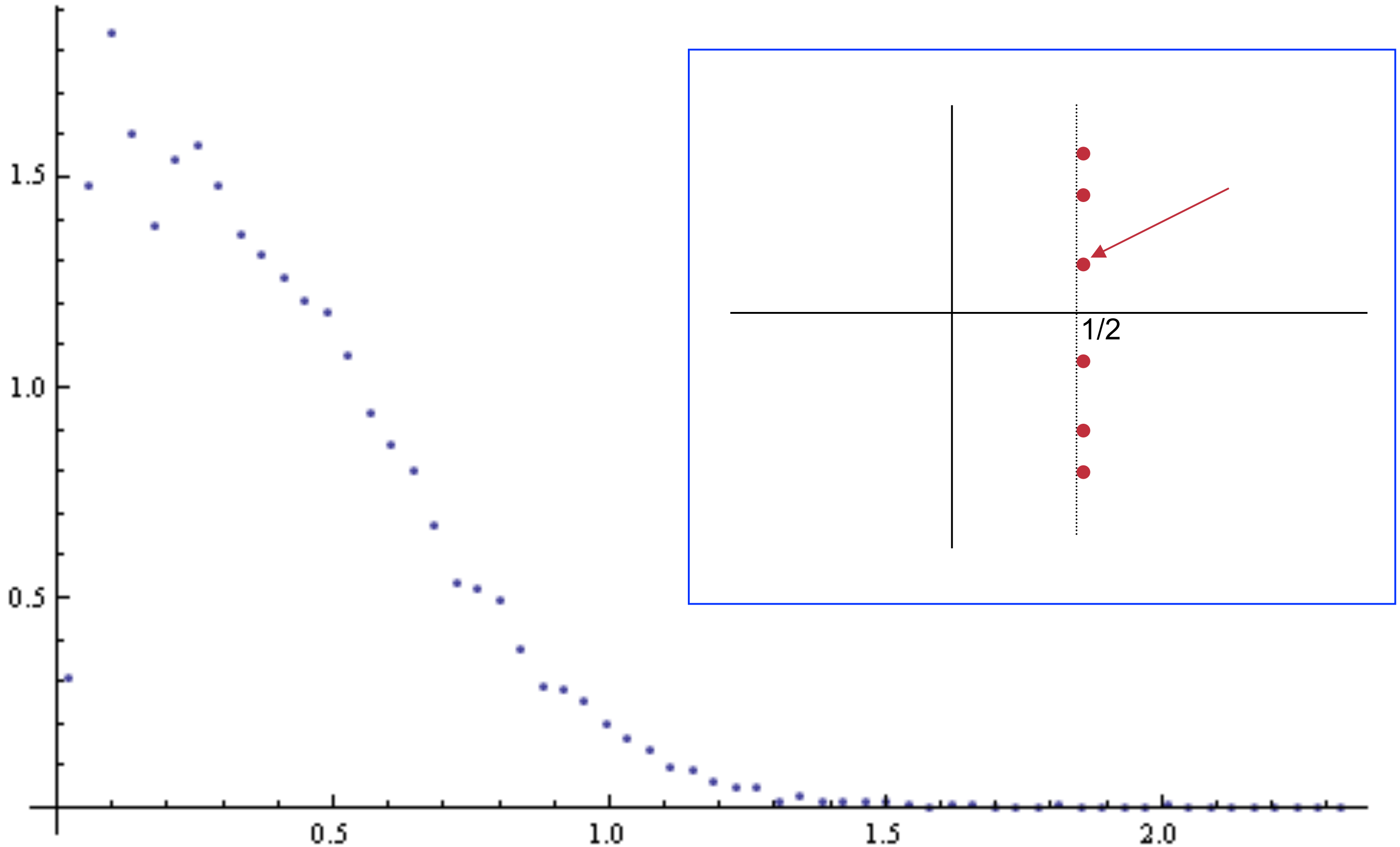


Subset of $SO(2N)$:

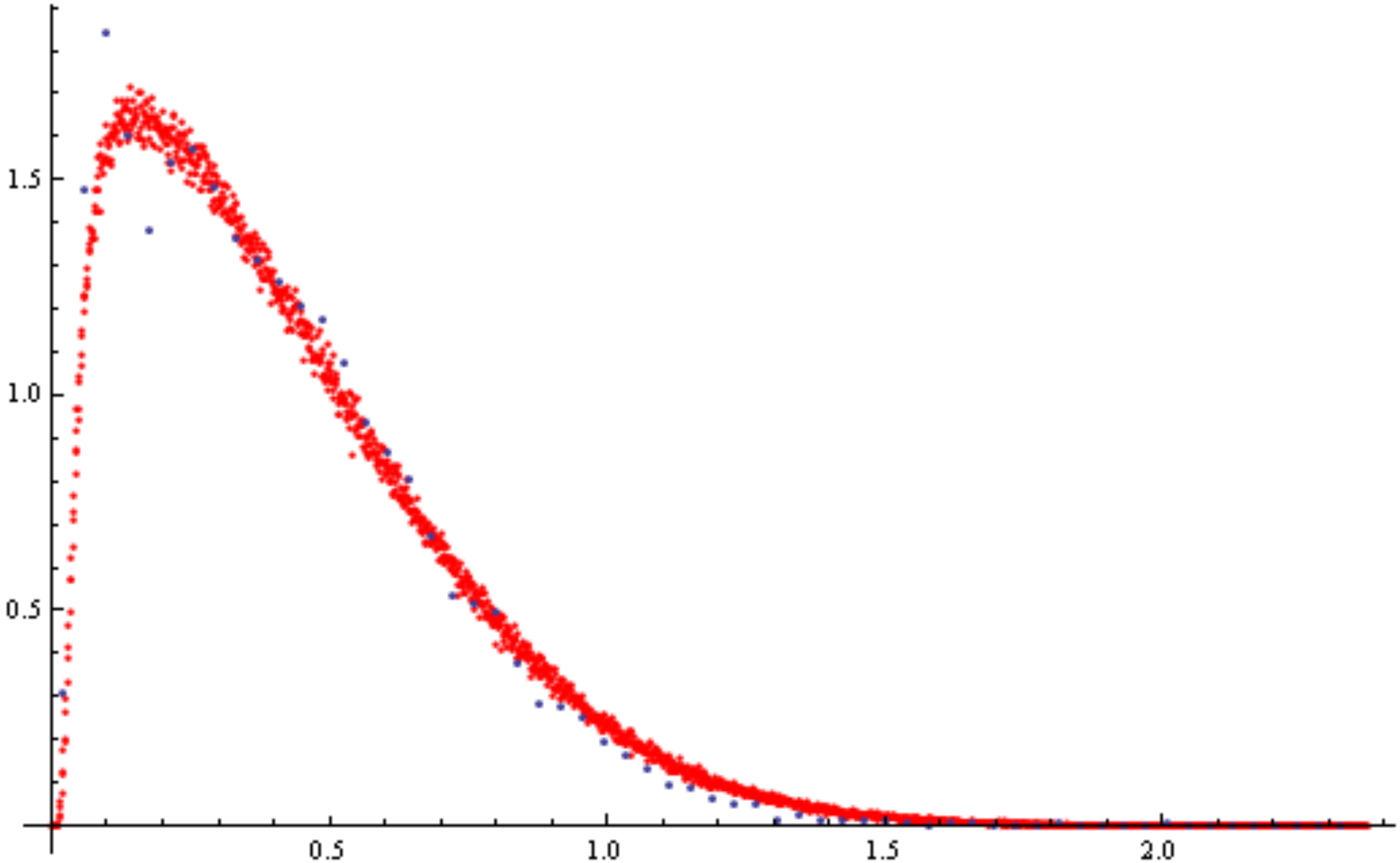
$\log d \sim N$

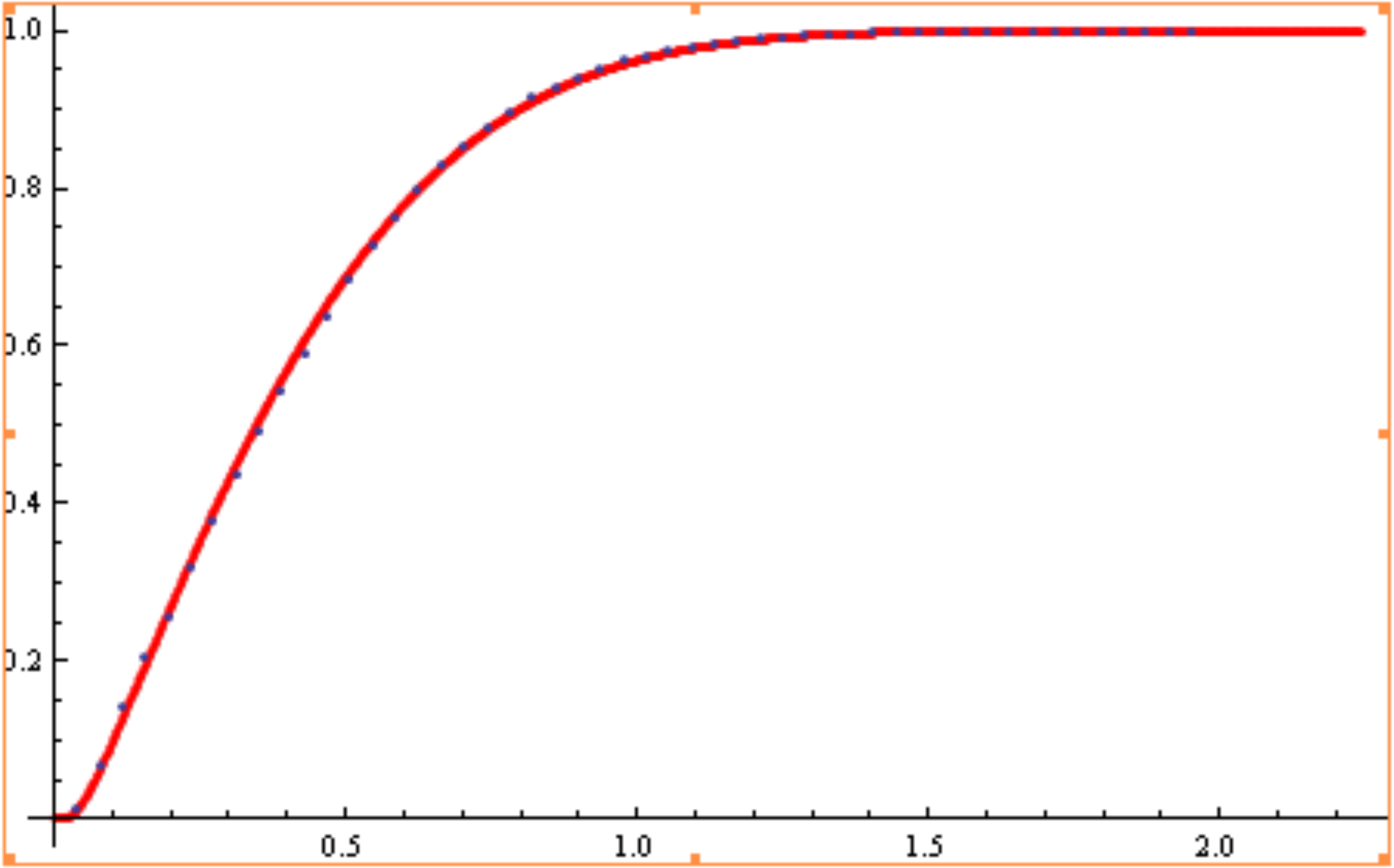
$\Lambda_A(1) \geq a \times \exp(-N/2)$

Histogram of first zero above critical point



Histogram of first zero with RMT model





Cummulative distribution of lowest zero (blue) compared with random matrix model (red) for a=2.188