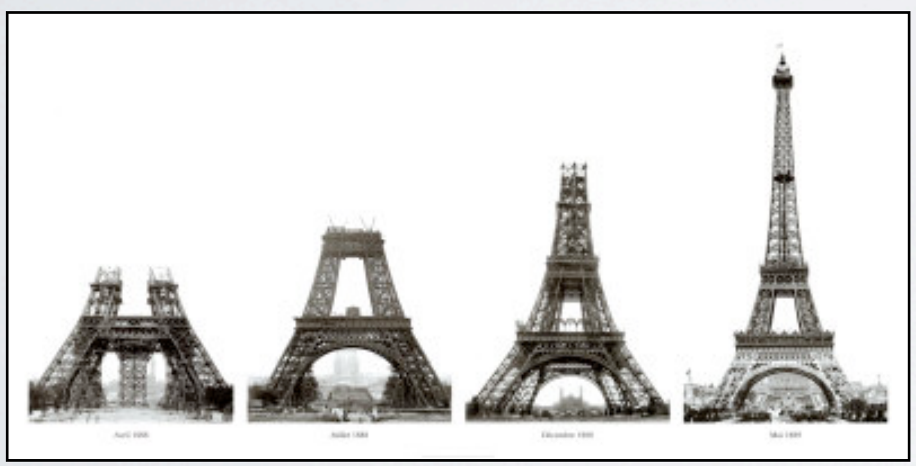
Invariant beta-Wishart ensembles and crossover densities



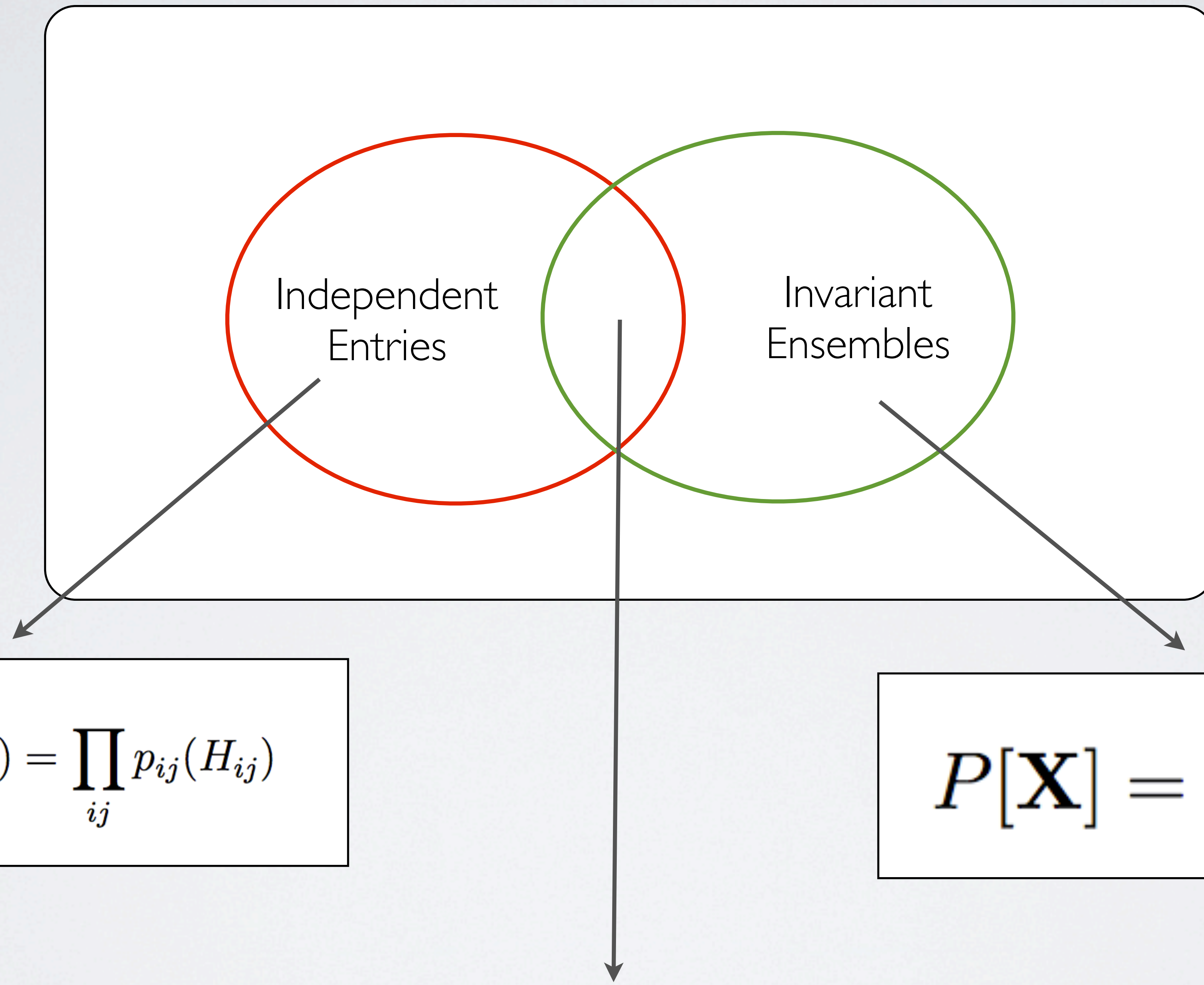
Pierpaolo Vivo

(LPTMS - Paris)

advancing the frontiers

in collaboration with R. Allez, J.-P. Bouchaud and S. N. Majumdar

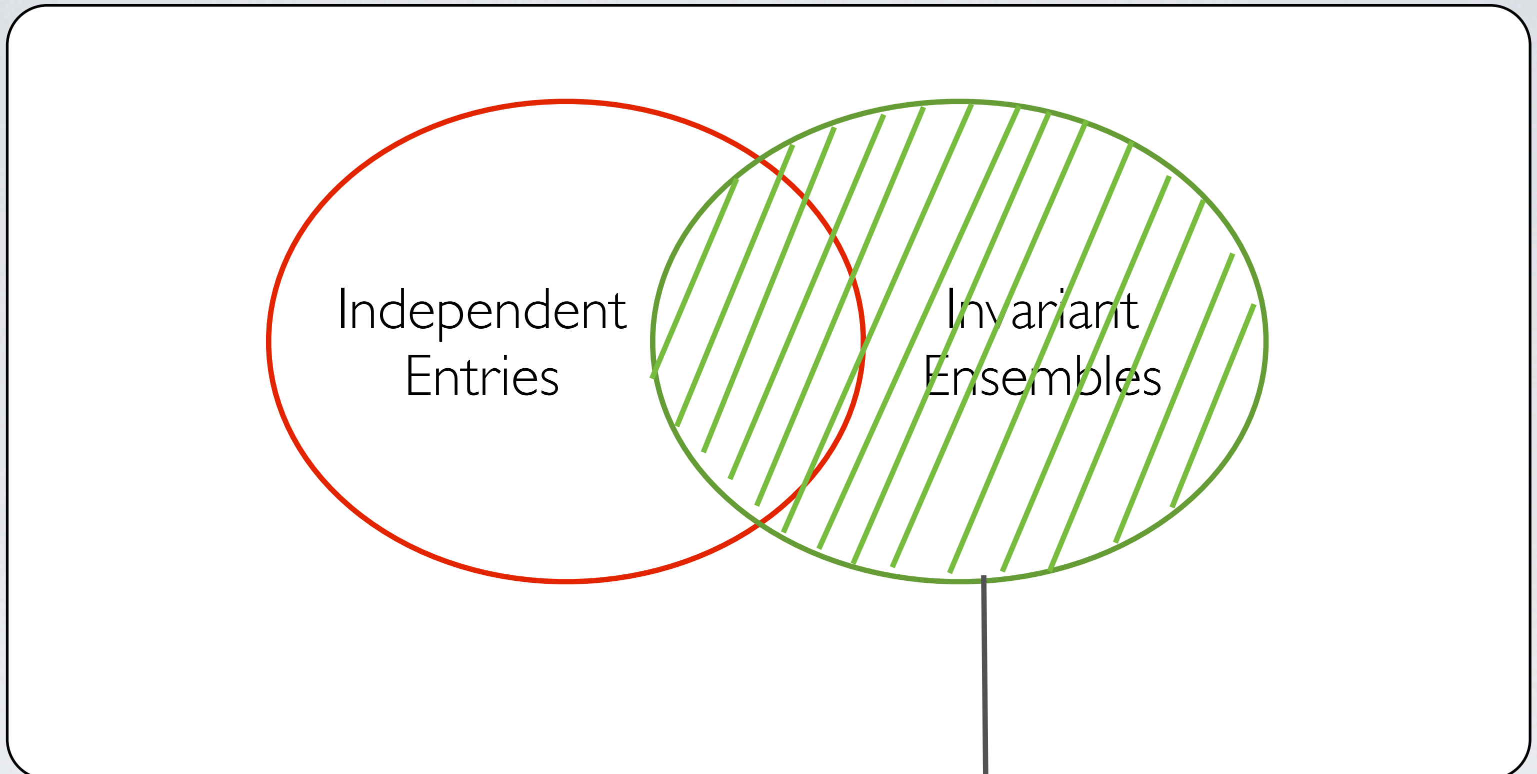
Matrices with real spectrum



$$P(X_{11}, \ldots, X_{NN}) = \prod_{ij} p_{ij}(H_{ij})$$

$$P[\mathbf{X}] = P[\mathbf{U}\mathbf{X}\mathbf{U}^{-1}]$$

Gaussian Ensembles

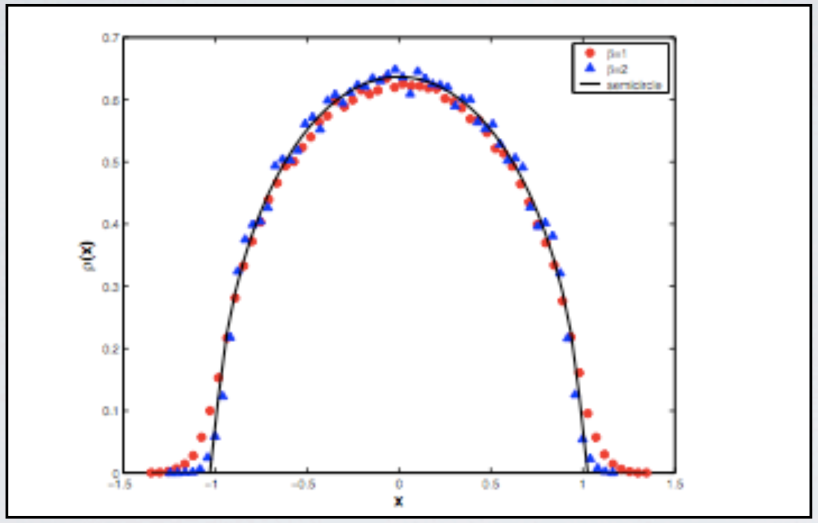


$$P(\lambda_1, \ldots, \lambda_N) = C_N \prod_{j<k} |\lambda_j - \lambda_k|^{\beta} e^{-\sum_{j=1}^{N} V(\lambda_j)}$$

Dyson's "threefold way"

N

N G



Gaussian

$\mathbf{G}_{ij} \sim \mathcal{N}(0,1)$

$V(x) = x^2$

0

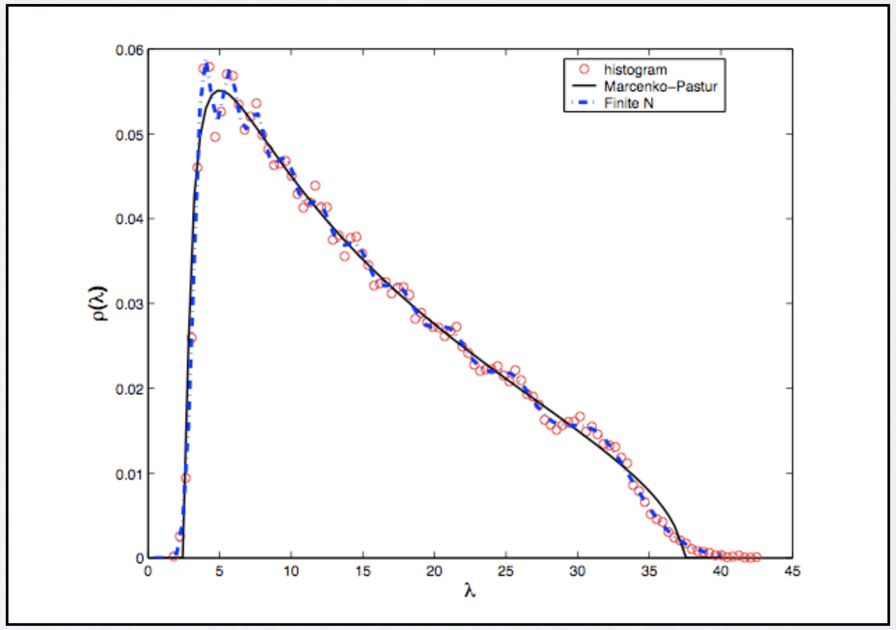
M

N G X G =N W

N

Wishart

$\mathbf{W} = \mathbf{G}\mathbf{G}^{\dagger}$



$V(x) = x - \alpha \log x$

0

2 questions

- Can we lift Dyson's quantization?

- If yes, is a continuous beta index compatible with rotational invariance?

2 questions

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YES!

Matrix Models for Beta Ensembles

Ioana Dumitriu* and Alan Edelman†

February 5, 2008

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Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations

R. B. Laughlin
Lawrence Livermore National Laboratory, University of California, Livermore, California 94550
(Received 22 February 1983)

This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.

nomial in z. The antisymmetry of ψ requires that f be odd. Conservation of angular momentum requires that $\prod_{j<k} f(z_j - z_k)$ be a homogeneous polynomial of degree M, where M is the total angular momentum. We have, therefore, $f(z) = z^m$, with m odd. To determine which m minimizes the energy, I write

$$|\psi_m|^2 = |\{\prod_{j<k}(z_j - z_k)^m\}\exp(-\tfrac{1}{4}\sum_l |z_l|^2)|^2 = e^{-\beta\Phi}, \quad (7)$$

where $\beta = 1/m$ and Φ is a classical potential energy given by

$$\Phi = -\sum_{j<k} 2m^2 \ln|z_j - z_k| + \tfrac{1}{2} m \sum_l |z_l|^2. \quad (8)$$

Φ describes a system of N identical particles of charge $Q = m$, interacting via logarithmic potentials and embedded in a uniform neutralizing background of charge density $\sigma = (2\pi a_0{}^2)^{-1}$. This is the classical one-component plasma (OCP), a system which has been studied in great detail.

Matrix Models for Beta Ensembles

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February 5, 2008

Obviously **non** invariant!

Hermite Matrix $n \in \mathbb{N}$	$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$
Laguerre Matrix $m \in \mathbb{N}$ $a \in \mathbb{R}$ $a > \frac{\beta}{2}(m-1)$	$L_\beta = B_\beta B_\beta^T$, where $B_\beta \sim \begin{pmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{pmatrix}$

HERMITE: $$f_\beta(\lambda) = c_H^\beta \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\sum_{i=1}^n \lambda_i^2/2} \,,$$

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On invariant 2×2 β-ensembles of random matrices

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Abstract

We introduce and solve exactly a family of invariant 2×2 random matrices, depending on one parameter η, and we show that rotational invariance and real Dyson index β are not incompatible properties. The probability density for the entries contains

$$\mathbf{P}^{\star}_{\eta} = \mathrm{C}_{\eta} \frac{e^{-\frac{1}{2}\mathrm{Tr}\mathcal{X}^2}}{[2\mathrm{Tr}\mathcal{X}^2 - (\mathrm{Tr}\mathcal{X})^2]^{\eta}}$$

Standard Dyson's Brownian motion construction for Gaussian real symmetric matrices (GOE)

One introduces a fictitious time t for the evolution of an $N \times N$ real symmetric matrix $\mathbf{M}(t)$. The evolution of the symmetric matrix is governed by the following stochastic differential equation (SDE):

$$\mathrm{d}\mathbf{M}(t) = -\frac{1}{2}\mathbf{M}(t)\mathrm{d}t + \mathrm{d}\mathbf{H}(t) \qquad (1)$$

Independent of **M**

where $\mathrm{d}\mathbf{H}(t)$ is a symmetric Brownian increment (i.e. a symmetric matrix whose entries above the diagonal are independent Brownian increments with variance $\langle \mathrm{d}\mathbf{H}_{ij}^2(t)\rangle = \frac{\sigma^2}{2}(1+\delta_{ij})\mathrm{d}t$). Standard second order perturbation theory allows one to write the evolution equation for the eigenvalues λ_i of the matrix $\mathbf{M}(t)$:

$$\mathrm{d}\lambda_i = -\frac{1}{2}\lambda_i \mathrm{d}t + \frac{\sigma^2}{2}\sum_{j\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_j} + \sigma \mathrm{d}b_i, \qquad (2)$$

Fixed number!

where $b_i(t)$ are independen… …ard Brownian motions.

$$P^*(\{\lambda_i\}) = Z\prod_{i<j}|\lambda_i - \lambda_j|^\beta \exp\left[-\frac{1}{2\sigma^2}\sum_i \lambda_i^2\right],$$

with beta = 1

PRL 109, 094102 (2012) PHYSICAL REVIEW LETTERS week ending 31 AUGUST 2012

Invariant Beta Ensembles and the Gauss-Wigner Crossover

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(Received 16 May 2012; published 29 August 2012)

More precisely, our model is defined as follows: we divide time into small intervals of length $1/n$ and for each interval $[k/n;(k+1)/n]$, we choose independently Bernoulli random variables ϵ_k^n, $k \in \mathbb{N}$ such that $\mathbb{P}[\epsilon_k^n = 1] = p = 1 - \mathbb{P}[\epsilon_k^n = 0]$. Then, setting $\epsilon_t^n = \epsilon_{[nt]}^n$, our diffusive matrix process simply evolves as:

$$\mathrm{d}\mathbf{M}_n(t) = -\frac{1}{2}\mathbf{M}_n(t)\mathrm{d}t + \epsilon_t^n \mathrm{d}\mathbf{H}(t) + (1-\epsilon_t^n)\mathrm{d}\mathbf{Y}(t) \quad (8)$$

where $\mathrm{d}\mathbf{H}(t)$ is a symmetric Brownian increment as above and where $\mathrm{d}\mathbf{Y}(t)$ is a symmetric matrix that is co-diagonalizable with $\mathbf{M}_n(t)$ (i.e. the two matrix have the same eigenvectors) but with a spectrum given by N independent Brownian increments of variance $\sigma^2\mathrm{d}t$.

New Dyson index

'Free' slice

'Commuting' slice

$$\mathrm{d}\lambda_i = -\frac{1}{2}\lambda_i\mathrm{d}t + p\frac{\sigma^2}{2}\sum_{j\neq i}\frac{\mathrm{d}t}{\lambda_i - \lambda_j} + \sigma\mathrm{d}b_i,$$

FEATURES

- Rotationally invariant by construction (both the “free” and the “commuting” part respect the invariance)
- Based on the alternative addition of the standard Brownian matrix (“free”) **or** a matrix that commutes with the original one (“commuting”)
- Whether to add one or the other depends on the probability **p**
- This probability **p** in turn becomes the continuous Dyson index in [0,1] of the ensemble

HOWEVER....

The spectrum for large N is disappointingly trivial!

$p = 0$	Gaussian
$p > 0$	Semicircle

How to make the spectrum interesting?

$$p = \frac{2c}{N}, \qquad c \sim \mathcal{O}(1)$$

The modified spectral density can be computed in two alternative ways

- from Ito's calculus
- from saddle point route

4

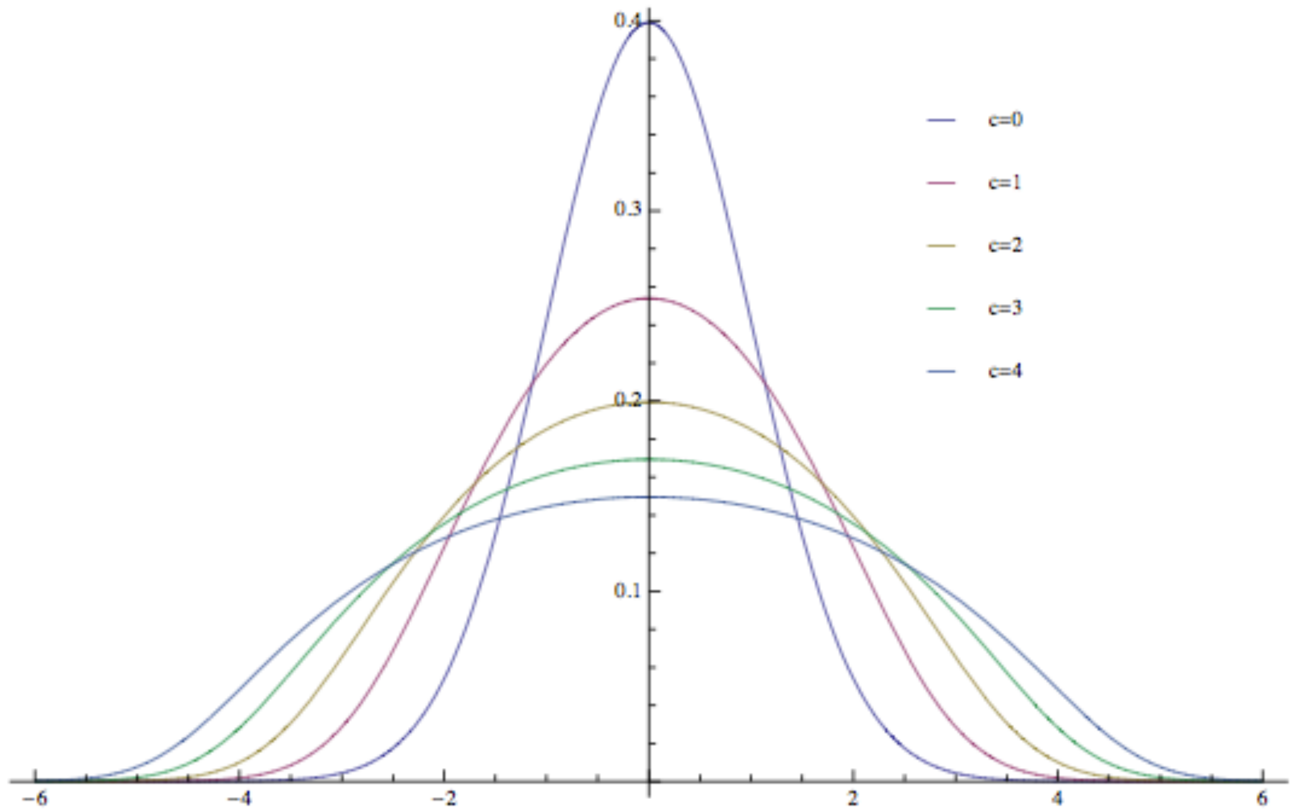


FIG. 2. Density $\rho_c(u)$ for $c = 0, 1, 2, 3, 4$, showing the progressive deformation of the Gaussian towards Wigner's semicircle.

$$\rho_c(\lambda) = \frac{1}{\sqrt{2\pi}\Gamma(1+c)} \frac{1}{|D_{-c}(i\lambda)|^2};$$
$$D_{-c}(z) = \frac{e^{-z^2/4}}{\Gamma(c)} \int_0^\infty \mathrm{d}x e^{-zx-\frac{x^2}{2}} x^{c-1}.$$

$c \to 0$ Gaussian

$c \to \infty$ Semicircle

SUMMARY

- Allez-Bouchaud-Guionnet construction: invariant Gaussian model with continuous beta-index
- Based on a variation of Dyson's Brownian motion construction
- Random alternation of 'free' and 'commuting' addition
- If the continuous Dyson index scales with **1/N**, we get a family of spectral densities interpolating between a Gaussian and the semicircle
- This result can be established in two alternative ways
  - from Ito's calculus
  - from saddle point route

Invariant β-Wishart ensembles, crossover densities and asymptotic corrections to the Marčenko-Pastur law

Romain Allez[1,2], Jean-Philippe Bouchaud[2], Satya N. Majumdar[3], and Pierpaolo Vivo[3]

Goal: to build a diffusive matrix model for the Wishart ensemble (in analogy with Gaussian case) [redacted] Choose a large value of n and an initial symmetric matrix $\mathbf{W}_0$. The construction is iterative. Suppose that the process is constructed until time k/n and let us explain how to compute the matrix $\mathbf{W}^n_{(k+1)/n}$ at the next discrete time of the grid, $(k+1)/n$.

1. Step 1. We first need to compute the matrix $\sqrt{\mathbf{W}^n_{k/n}}$. It suffices to compute the orthogonal matrix $\mathbf{O}^n_{k/n}$ such that

$$\mathbf{W}^n_{k/n} = \mathbf{O}^n_{k/n} \mathbf{\Sigma}^n_{k/n} \mathbf{O}^n_{k/n}{}^{\dagger}$$

$$\sqrt{\mathbf{W}^n_{k/n}} = \mathbf{O}^n_{k/n} \sqrt{\mathbf{\Sigma}^n_{k/n}} \mathbf{O}^n_{k/n}{}^{\dagger}$$

2. Step 2. We sample the Bernoulli random variable ϵ^n_k with $\mathbb{P}[\epsilon^n_k = 1] = p = 1 - \mathbb{P}[\epsilon^n_k = 0]$.

Continuous beta-index

3. Step 3. It depends on the value of ϵ^n_k:

- if $\epsilon^n_k = 1$, we sample a $N \times N$ matrix $\mathbf{G}_n$ filled with independent Gaussian variables with mean 0 and variance $1/n$ and then we compute the matrix $\mathbf{W}^n_{(k+1)/n}$ by the formula

$$\mathbf{W}^n_{(k+1)/n} = \left(1 - \frac{1}{n}\right) \mathbf{W}^n_{k/n} + \sqrt{\mathbf{W}^n_{k/n}}\, \mathbf{G}_n + \mathbf{G}_n^{\dagger} \sqrt{\mathbf{W}^n_{k/n}} + \frac{1}{n} M\, \mathbf{I}\,.$$

- if $\epsilon^n_k = 0$, we sample N independent Gaussian variables $(z_1, \cdots, z_N)$ with mean 0 and variance $1/n$. We then compute the matrix $\mathbf{Y}_n$, which is co diagonalizable with the matrix $\mathbf{W}^n_{k/n}$, defined as the product

$$\mathbf{Y}_n := \mathbf{O}^n_{k/n} \mathrm{Diag}\,(z_1, z_2, \ldots, z_N)\, \mathbf{O}^n_{k/n}{}^{\dagger}\,. \tag{B.2}$$

Finally we obtain the matrix $\mathbf{W}^n_{(k+1)/n}$ by

$$\mathbf{W}^n_{(k+1)/n} = \left(1 - \frac{1}{n}\right) \mathbf{W}^n_{k/n} + \sqrt{\mathbf{W}^n_{k/n}}\, \mathbf{Y}_n + \mathbf{Y}_n^{\dagger} \sqrt{\mathbf{W}^n_{k/n}} + \frac{1}{n} \delta\, \mathbf{I}\,.$$

$$d\lambda_i = -\lambda_i dt + 2\sqrt{\lambda_i}\, db_i + \left(pM + (1-p)\delta + p \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \right) dt\,.$$

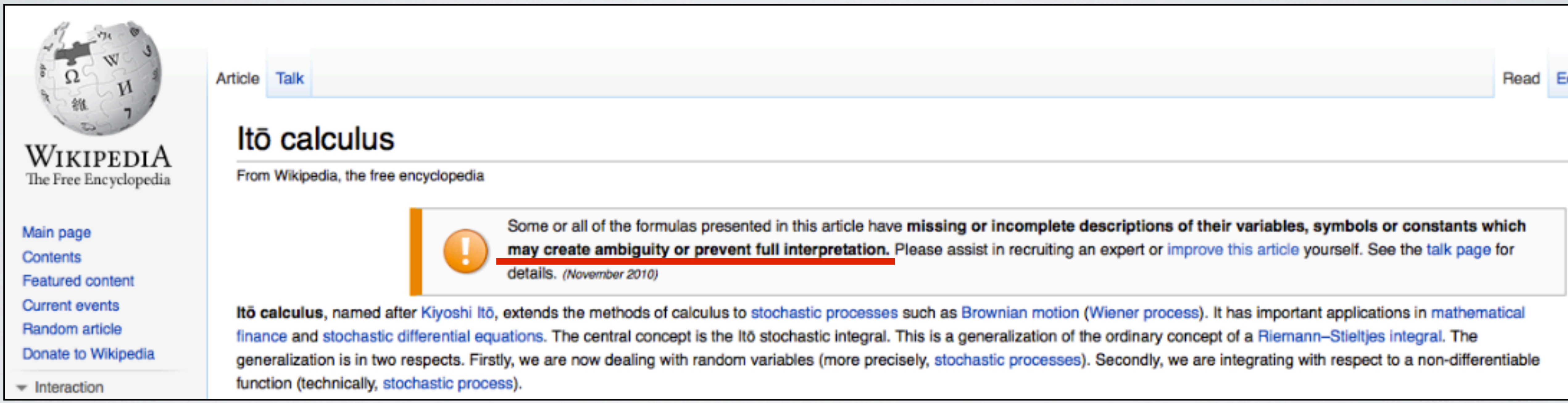
$$P^*(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} e^{-\frac{1}{2}\sum_{i=1}^{N} \lambda_i} \prod_{i=1}^{N} \lambda_i^{\frac{p}{2}(M-N+1-\delta)-(1-\frac{\delta}{2})} \prod_{i<j} |\lambda_i - \lambda_j|^{p}.$$

Wishart jpdf

Let’s scale **p** with **N** again....

2 Alternative routes

- Ito’s calculus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Itō calculus

From Wikipedia, the free encyclopedia

Some or all of the formulas presented in this article have **missing or incomplete descriptions of their variables, symbols or constants which may create ambiguity or prevent full interpretation.** Please assist in recruiting an expert or improve this article yourself. See the talk page for details. *(November 2010)*

Itō calculus, named after Kiyoshi Itō, extends the methods of calculus to stochastic processes such as Brownian motion (Wiener process). It has important applications in mathematical finance and stochastic differential equations. The central concept is the Itō stochastic integral. This is a generalization of the ordinary concept of a Riemann–Stieltjes integral. The generalization is in two respects. Firstly, we are now dealing with random variables (more precisely, stochastic processes). Secondly, we are integrating with respect to a non-differentiable function (technically, stochastic process).

- Saddle point calculation on Dyson’s Coulomb gas

$$Z = \int_{[0,\infty]^N} \prod_i d\lambda_i e^{-\frac{1}{2}\sum_i \lambda_i} \prod_{i<j} |\lambda_i - \lambda_j|^p \prod_i \lambda_i^{\frac{p}{2}(M-N+1-\delta)-(1-\delta/2)}$$
$$= \int_{[0,\infty]^N} \prod_i d\lambda_i e^{-E[\{\lambda_i\}]}$$

where the energy function $E[\{\lambda_i\}]$ is given by

$$E[\{\lambda_i\}] = \frac{1}{2}\sum_i \lambda_i - \left(\frac{p}{2}(M-N+1-\delta) - (1-\delta/2)\right)\sum_i \ln\lambda_i - \frac{p}{2}\sum_{i\neq j}\ln|\lambda_i - \lambda_j|.$$

Continuum limit

$$E[\rho(\lambda)] = \frac{N}{2}\int d\lambda\lambda\rho(\lambda) - \left[\frac{p}{2}\left((\frac{1}{q}-1)N+1-\delta\right) - \left(1-\frac{\delta}{2}\right)\right] N\int d\lambda\rho(\lambda)\ln\lambda$$
$$-\frac{p}{2}N^2\int\int d\lambda d\lambda'\rho(\lambda)\rho(\lambda')\ln|\lambda-\lambda'| + \frac{p}{2}N\int d\lambda\rho(\lambda)\ln\frac{1}{\rho(\lambda)} + C_1\left(\int d\lambda\rho(\lambda) - 1\right)$$

Dyson's self energy term

$$Z \approx \int \mathcal{D}[\rho]\, e^{-E[\rho(\lambda)]} J[\rho(\lambda)]$$

Jacobian

$$Z = \int \mathcal{D}[\rho] e^{-E[\rho(\lambda)]} e^{-N \int d\lambda \rho(\lambda) \ln \rho(\lambda)} = \int \mathcal{D}[\rho] e^{-NF[\rho(\lambda)]} \quad (3.16)$$

where the free energy $F[\rho(\lambda)]$ is given by:

$$F[\rho(\lambda)] = \frac{1}{2}\int d\lambda \lambda \rho(\lambda) - \left[\frac{p}{2}\left((\frac{1}{a}-1)N + 1 - \delta\right) - \left(1 - \frac{\delta}{2}\right)\right] \int d\lambda \rho(\lambda) \ln \lambda$$
$$- \frac{p}{2}N \int\int d\lambda d\lambda' \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'| + \left(1 - \frac{p}{2}\right) \int d\lambda \rho(\lambda) \ln \rho(\lambda) + C_1 \left(\int d\lambda \rho(\lambda) - 1\right) \quad (3.17)$$

Jacobian term has the same form of Dyson's self-energy, but opposite sign!

[Dean & Majumdar, PRE 2008]

If **p** scales as 1/**N**, the energy and the entropy become of the same order !

$$p = 2c/M = 2cq/N$$

$$F[\rho(\lambda)] = \frac{1}{2}\int d\lambda\lambda\rho(\lambda) - \left[cq\left(\frac{1}{q}-1\right) - \left(1-\frac{\delta}{2}\right)\right]\int d\lambda\rho(\lambda)\ln\lambda$$
$$- cq\int\int d\lambda d\lambda'\rho(\lambda)\rho(\lambda')\ln|\lambda-\lambda'| + \left(1-\frac{cq}{N}\right)\int d\lambda\rho(\lambda)\ln\rho(\lambda) + C_1\left(\int d\lambda\rho(\lambda)-1\right)$$

Saddle point equation

$$\frac{\lambda}{2} - a\ln\lambda - 2cq\int d\lambda'\rho^*(\lambda')\ln|\lambda-\lambda'| + \ln\rho^* + C_2 = 0$$

New unusual term, due to the entropic contribution

$$\frac{1}{2} - \frac{a}{\lambda} - 2cq\ \mathrm{Pr}\int\frac{\rho(\lambda')}{\lambda-\lambda'}d\lambda' + \frac{\rho'(\lambda)}{\rho(\lambda)} = 0$$

$$H(z) = \int \frac{\rho(\lambda)}{\lambda - z} d\lambda$$

Resolvent

$$\frac{1}{2} - \frac{a}{\lambda} - 2cq \,\mathrm{Pr} \int \frac{\rho(\lambda')}{\lambda - \lambda'} d\lambda' + \frac{\rho'(\lambda)}{\rho(\lambda)} = 0$$

Multiply by $\rho(\lambda)/(\lambda - z)$

and integrate over lambda

$$\frac{dH}{dz} + \gamma H^2 + \frac{1}{2}\left(1 + \frac{\alpha}{z}\right) H + \frac{1}{2z} = 0$$

$$\alpha = (2 - \delta) - 2c(1 - q), \quad \gamma = cq\,.$$

Eq. is no longer algebraic, but differential!

$$\boxed{\frac{dH}{dz} + \gamma H^2 + \frac{1}{2}\left(1+\frac{\alpha}{z}\right) H + \frac{1}{2z} = 0}$$

$$\alpha = (2-\delta) - 2c(1-q), \quad \gamma = cq\,.$$

$$\rho(\lambda) = \frac{1}{\pi}\mathrm{Im}[H(z \to \lambda)]$$

$$H(z) = \frac{1}{\gamma}\frac{u'(z)}{u(z)} = \frac{1}{\gamma}\,\partial_z \ln u(z)\,.$$

Vai alla pagina 12

$$u''(z) + \frac{1}{2}\left[1+\frac{\alpha}{z}\right]u'(z) + \frac{\gamma}{2z}u(z) = 0\,.$$

Jacopo Francesco Riccati (1676-1754)

Vai alla pagina 12

$$u''(z) + \frac{1}{2}\left[1 + \frac{\alpha}{z}\right] u'(z) + \frac{\gamma}{2z} u(z) = 0\,.$$

$$u(z) = \mathcal{C}_2\, e^{-z/4}\, z^{\alpha/4}\, W_{-\zeta,\mu}(-z/2)$$

Whittaker function

$$H(z) = \frac{1}{\gamma}\frac{u'(z)}{u(z)} = \frac{1}{\gamma}\,\partial_z \ln u(z)\,.$$

$$\rho(\lambda) = \frac{1}{\pi}\mathrm{Im}[H(z \to \lambda)]$$

$$\rho(\lambda) = \frac{A}{|W_{-\zeta,\mu}(-\lambda/2)|^2}\,.$$

Normalization Constant **A**

$$\frac{1}{A} = 2\int_0^{\infty} \frac{d\lambda}{|W_{-\zeta,\mu}(-\lambda)|^2}\,.$$

$$W_{\zeta,\mu}(z) = z^{\mu+1/2}\, e^{-z/2}\, U(\mu-\zeta+1/2, 1+2\mu; z)$$

$$\int_0^{\infty} \frac{dt\; e^{-t}\, t^{-b}}{z+t}\, |U(a,b;-t)|^{-2} = \Gamma(a)\Gamma(a-b+2)\,\frac{1}{z}\,\frac{U(a,b-1;z)}{U(a,b;z)}; \quad \text{for } a>0,\ 1<b<a+1$$

[M.E.H. Ismail & D.H. Kelker, SIAM J. Math. Anal. **10**, 884 (1979)]

$$\rho_c(\lambda) = \frac{1}{2\Gamma(\mu+\zeta+\frac{1}{2})\Gamma(\zeta-\mu+\frac{3}{2})}\,\frac{1}{|W_{-\zeta,\mu}(-\frac{\lambda}{2})|^2}$$

$c \to 0$	Gamma distribution
$c \to \infty$	Marčenko-Pastur

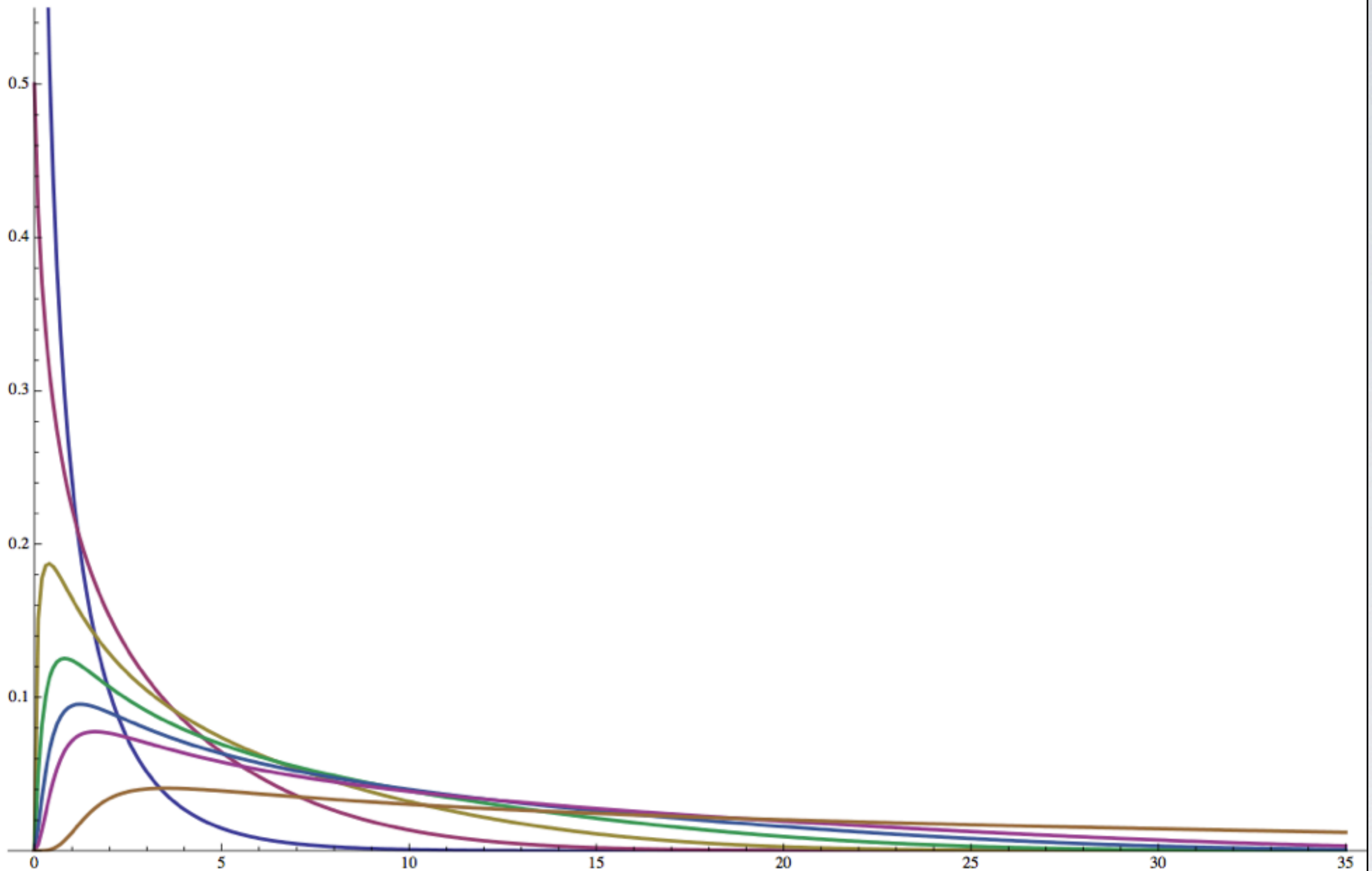

Figure 1: Density $\rho_c(\lambda)$ for $c = 0, 1, 2, 3, 4, 5, 10$ of Eq. (3.49) showing the progressive deformation of the Gamma distribution (3.2) with parameter $\delta = 1$ towards the Marčenko-Pastur distribution with parameter $q = 1/2$. The value $\rho_c(0)$ at the origin decreases when c increases.

FINAL SUMMARY

- Modified Allez-Bouchaud-Guionnet construction: invariant Wishart model with continuous beta-index
- Based on a variation of Dyson's Brownian motion construction
- Random alternation of 'free' and 'commuting' addition
- If the continuous Dyson index scales with 1/N, we get a family of spectral densities interpolating between a Gamma distribution and the Marcenko-Pastur
- This result can be established in two alternative ways
- The free energy of the Coulomb gas is no longer dominated by the energetic component (energy and entropy now scale in the same way!)