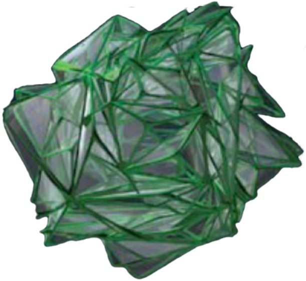
# Matrix Models in Causal Dynamical Triangulations



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VIII Brunel–Bielefeld Workshop
on Random Matrix Theory
December 15, 2012

Motivation and Background

- Motivation
- Introduction to CDT

Random Matrix Models for CDT

- Generalised CDT, String Field Theory and Matrix Model
- Matrix Models for discrete CDT and Combinatorial Interpretation
- Applications of CDT Matrix Model

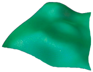
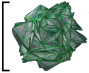
Summary and Final Considerations

# Motivation

- Causal Dynamical Triangulations (CDT) is a non-perturbative approach to quantum gravity
- CDT respects the Lorentzian nature of the path integral by disallowing *acausal* configurations in the sum over geometries
- Recent numerical simulations give evidence for interesting results in higher dimensions (emergence of de Sitter space-time, scale dependent spectral dimension)
- It is vital to extend the analytical techniques in two-dimensions by means of matrix model formulations

# Quantum Gravity via DT and CDT

Dynamical Triangulations (DT)[a] and Causal Dynamical Triangulations (CDT)[b] are non-perturbative approaches to define the gravitational path integral as a sum over geometries. Schematically,

$\hat{Z} = \int_{(\cdot)\in\mathcal{M}}$  $d\mu \exp\left(-\hat{S}\left[\ \right]\right)$  $\to Z = \sum_{(\cdot)\in\mathcal{T}}$  $\exp\left(-S\left[\ \right]\right)$ 

[a] see Ambjørn, Durhuus, Jonsson, *Quantum Geometry*, Cambridge

[b] introduzed by Ambjørn and Loll, hep-th/9805198 Nucl.Phys. B

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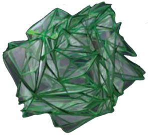
Once $Z$ is calculated, one takes the continuum limit:



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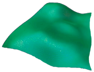

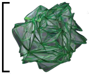

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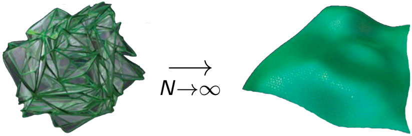


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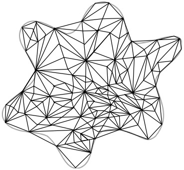
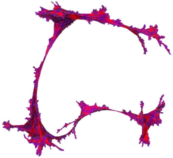
# The difference between DT and CDT; Lessons from 2D

In DT the partition function sums over all planar triangulations:



$\underset{N \to \infty}{\longrightarrow}$



$d_h = 4$

In CDT only triangulations with a strict time-slicing are allowed (no spatial topology changes):

$\underset{N \to \infty}{\longrightarrow}$

$d_h = 2$

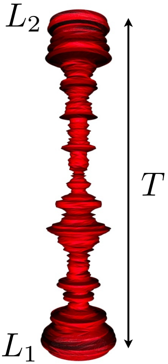
# Continuum Dynamics of Pure CDT

One can solve the above discretised path integral and obtain the continuum propagator

$$G_\Lambda^{(0)}(L_1, L_2; T) = < L_1 | e^{-H_0 T} | L_2 >$$

with Hamiltonian

$$H_0(L) = -L \frac{\partial^2}{\partial L^2} + \Lambda L$$

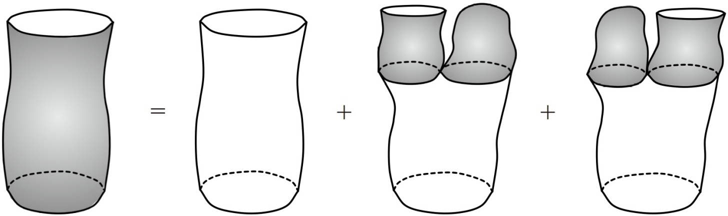



Later we will often use the Laplace transformed

$$G_\Lambda^{(0)}(X, Y; T) = \int \int dL_1 \, dL_2 \, e^{-L_1 X - L_2 Y} G_\Lambda^{(0)}(L_1, L_2; T)$$

# Generalised CDT

Introduce generalised CDT[a] by allowing spatial topology changes weighted by a coupling constant $g_s$. The Dyson-Schwinger (DS) equations for the propagator read

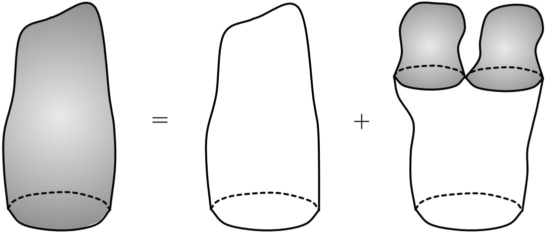


$$\frac{\partial}{\partial T} G_{\Lambda,g_s}(X,Y;T) = -\frac{\partial}{\partial X}\left[\left((X^2-\Lambda)+2g_s\, W_{\Lambda,g_s}(X)\right) G_{\Lambda,G_s}(X,Y;T)\right]$$

[a]Ambjørn, Loll, Westra, SZ, 0709.2784 JHEP

# Generalised CDT: Disc Function

Using a consistency relation, one can solve the DS equations for the disc function for all "tree-diagrams"



$$W_{\Lambda,g_s}(X) = \frac{-(X^2-\Lambda)+(X-C)\sqrt{(X+C)^2-2g_s/C}}{2g_s}$$

$$C = \tilde{C}\sqrt{\Lambda}, \quad \tilde{C}^3 - \tilde{C} + g_s/\lambda^{3/2} = 0$$

# Matrix Model Correspondence

The results of generalised CDT can be obtained from the following $N \times N$ Hermitian matrix model[a]

$$Z(\Lambda, g_s) = \int d\Phi \; e^{-\frac{N}{g_s}\mathrm{Tr} V(\Phi)}, \quad V(\Phi) = \Lambda\Phi - \frac{1}{3}\Phi^3$$

The disc function is then given by the standard resolvent

$$\left\langle \frac{1}{N} \mathrm{Tr} \left( \frac{1}{X - \Phi} \right) \right\rangle = W_{\Lambda, g_s}(X) + O(N^{-2})$$

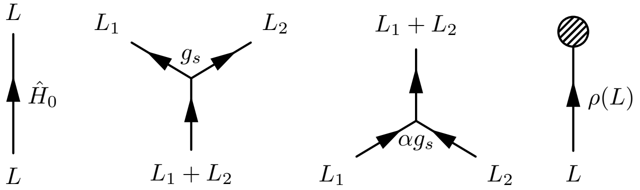
Note that this represents a matrix model for *continuum surfaces* similar to the Kontsevich model.[b]

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0804.0252 Phys. Lett. B

[b] Kontsevich, Funk. Anal. 25 (1991) 50

# String Field Theory

One can go beyond the tree diagrams by introducing a formalism of second quantisation, i.e. a string field theory (SFT)[a][b]




- One has a propagation, splitting, joining and tadpole term.

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0802.0719 JHEP
[b] for previous work in DT see Kawai et al. hep-th/9406207 Phys. Rev. D

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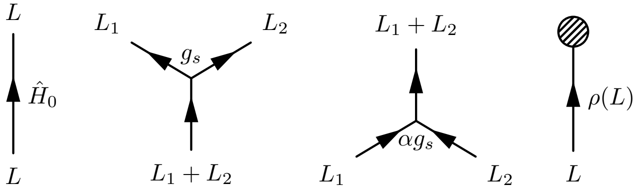


- One has a propagation, splitting, joining and tadpole term.
- The corresponding DS-equations give the "tree-diagrams" for $\alpha = 0$ and match to the matrix model loop equations for $\alpha = 1/N^2$.

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0802.0719 JHEP

[b] for previous work in DT see Kawai et al. hep-th/9406207 Phys. Rev. D

# String Field Theory: Some Details

The detailed Hamiltonian reads

$$
\begin{aligned}
\hat{H} \; = \; & \int \frac{dL}{L}\, \Psi^\dagger(L)(-L\frac{\partial^2}{\partial L^2} + \Lambda L)\Psi(L), \\
& -g_s \int dL_1 \int dL_2 \Psi^\dagger(L_1)\Psi^\dagger(L_2)\Psi(L_1+L_2) \\
& -\alpha g_s \int dL_1 \int dL_2 \Psi^\dagger(L_1+L_2)\Psi(L_2)\Psi(L_1) - \int \frac{dL}{L}\, \delta(L)\Psi(L)
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\end{aligned}
$$

Introducing the partition function with source term

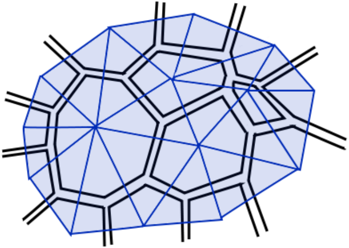
$$
Z(J) = \lim_{T\to\infty} \langle 0|\, e^{-T\hat{H}}\, e^{\int dL\, J(L)\Psi^\dagger(L)}\, |0\rangle
$$

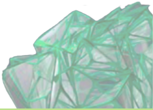
one can obtain the DS-equations by taking suitable derivatives w.r.t. $J(L)$.

# Matrix models for DT: Counting planar maps

The dual fat-graphs of maps are counted by matrix models of $N \times N$ Hermitian matrices

$$Z(\{t_n\}) = \int [d\phi] e^{-N \mathrm{Tr} V(\phi)}, \quad V(\phi) = \frac{1}{2}\phi^2 - \sum_{n \geq 3} \frac{t_n}{n} \phi^n.$$





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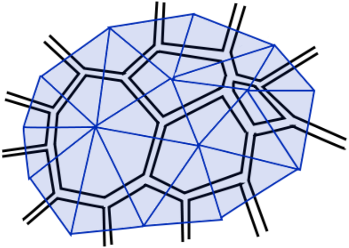
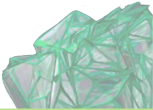


- The propagator in the potential represents double lines in the fat-graph
- Terms of order $t_n \phi^n$ correspond to vertices of order $n$ in the fat-graph
- $N \to \infty$ corresponds to the planar limit

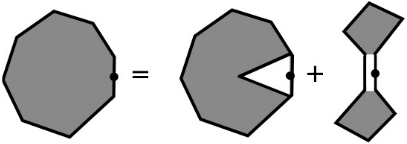


# Combinatorial Interpretation of Loop Equations

In the planar limit one obtains the following loop equations for the resolvent for a matrix model with potential $V(\phi) = \phi^2/2 - g\phi^3/3$:

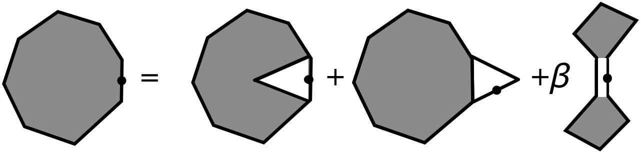
$$w(z) = zgw(z) + \frac{1}{z}Q(z,g) + \frac{1}{z}w^2(z), \quad w(z) = \left\langle \frac{1}{N}\,\mathrm{Tr}\left(\frac{1}{z-\phi}\right)\right\rangle$$

Their combinatorial interpretation is given through Tutte's equation, pictorially



# Loop Equations for CDT

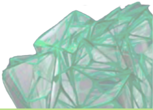
In generalised CDT we assign a coupling constant $\beta$ to spatial topology changes. One can obtain the following loop equations[a]




$$w(z) = zgw(z) + \frac{g}{z}w(z) + \frac{1}{z}Q(z,g) + \beta\frac{1}{z}w^2(z)$$

- The second term on the rhs is essential when $\beta \to 0$
- For $\beta \to 0$ one obtains the exact combinatorial expressions which count pure CDT.

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0810.2503 Acta Phys. Pol. B



# Matrix Model for discrete CDT and continuum limit

The loop equations for discrete CDT correspond to a $N \times N$ Hermitian matrix model

$$Z(g,\beta) = \int [d\phi] e^{-\frac{N}{\beta}\mathrm{Tr}\, V(\phi)}, \quad V(\phi) = -g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3.$$

The linear term corresponds to the new term in the loop equation.

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The linear term corresponds to the new term in the loop equation.

Performing the usual CDT scaling limit on the level of the action

$$\phi = z_c + a\Phi \quad g = g_c - a^2\Lambda \quad \beta = a^3 g_s$$

one obtains back the matrix model for continuum CDT surfaces:

$$Z(\Lambda, g_s) = \int d\Phi \; e^{-\frac{N}{g_s}\mathrm{Tr}\,V(\Phi)}, \quad V(\Phi) = \Lambda\Phi - \frac{1}{3}\Phi^3$$

# CDT as new continuum limit

One can understand the new continuum limit[a] in terms of the eigenvalue distribution

$$Z(g,\beta) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\lambda_i \prod_{j \neq i} (\lambda_j - \lambda_i)^2 \exp\left\{ -\frac{N}{\beta} \sum_{i=1}^{N} V(\lambda_i, g) \right\}$$

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0810.2408 Phys. Lett. B

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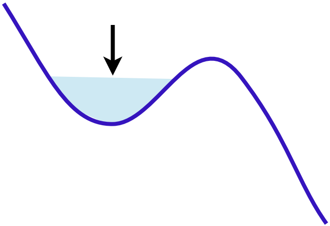


- At the critical point one has $V'(\lambda_c,g_c)=V''(\lambda_c,g_c)=0$
- One flattens the potential while at the same time taking the eigenvalue repulsion to zero

[a] Ambjørn, Loll, Watabiki, Westra, SZ, 0810.2408 Phys. Lett. B

# Application: Multi-Critical Points and Mater Coupling

- One can investigate the $m = 3$ multi-critical point of a higher order potential with [a]

$$V'(\{t_{n,cr}\}) = V''(\{t_{n,cr}\}) = V'''(\{t_{n,cr}\}) = 0$$

- It can be seen that this model corresponds to the scaling limit of hard dimers, a (2,5)-CFT, coupled to CDT [b]
- One can generalise this to the $m$-th multi-critical point which corresponds to a $(2, 2m-1)$-CFT coupled to CDT [c]

[a] Ambjørn et al. 1202.4435 Phys. Lett. B
[b] Atkin, SZ: 1202.4322 Phys. Lett. B
[c] Atkin, SZ: 1203.5034 JHEP

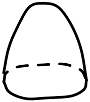
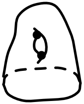
# Application: Sum over Topologies

One can obtain the sum over topologies from the ordinary integral $(N = 1)$[a]

$$Z(\kappa, \Lambda) = \int dx \exp\left[-\frac{1}{g_s}\left(\Lambda x - \frac{1}{3}x^3\right)\right], \quad \kappa = \frac{g_s}{\Lambda^{3/2}}$$

The disc function is then given in terms of Airy functions

$$W(L) = \frac{\mathrm{Bi}\left(\frac{\Lambda}{g_s^{2/3}} - g_s^{1/3} L\right)}{\mathrm{Bi}\left(\frac{\Lambda}{g_s^{2/3}}\right)}$$

$=$  $+ \kappa$  $+ \kappa^2 \Big($  $+$  $\Big) + \mathcal{O}(\kappa^3)$.

[a] Ambjørn, Loll, Westra, SZ, 0905.2108 PLB, 0908.4224 PLB

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- We gave an overview of recent developments for matrix model techniques in CDT.
- Interestingly, the continuum surface model of generalised CDT is again described by a matrix model.
- We gave a combinatorial interpretation of the loop equations of the discretized model.
- We discussed several application of these techniques to describe higher multi-critical points as well as the sum over topologies.
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Thank you very much!