From Classical to Quantum Integrable Systems

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The Ordinary Differential Equation / Integrable Model Correspondence

The
eigenvalues
of certain ordinary differential equations
are the Bethe roots of certain quantum integrable models
IM
ODE

[Bazhanov, Dorey, Dunning, Lukyanov, Suzuki, Tateo . . .]
Review article: J. Phys. A40 2007 R205 Dorey, TCD and Tateo

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The
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IM
ODE

Main result

The
eigenvalues
of linear systems of certain partial differential equations
are the Bethe roots of certain massive quantum integrable models
IM
ODE

Lukyanov, Zamolodchikov; Dorey, Faldella, Negro, Tateo;
Adamopoulou, TCD; Ito, Locke

Ordinary Differential Equations

The eigenvalues $\{E_j\}$ of

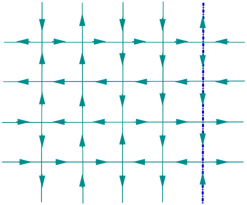
$$\left(-\frac{d^2}{dx^2} + (x^{2M} - E) + \frac{l(l+1)}{x^2}\right)\psi(x, E, l) = 0 \qquad \psi \in L^2(\mathbb{R}^+)$$

satisfy Bethe ansatz equations

$$\prod_{k=1}^{\infty} \frac{E_k - \omega^2 E_j}{E_k - \omega^{-2} E_j} = -e^{\frac{i\pi(2l+1)}{M+1}} \qquad j = 1, 2, \ldots, \infty$$

where $\omega = e^{\frac{i\pi}{M+1}}$

6-vertex model



The same Bethe ansatz equations encode the ground state of the 6-vertex model with twisted boundary conditions in the thermodynamic limit

Spectral parameter	ν	$\sim$	E
Anisotropy	η	$=$	$\frac{\pi}{2}\frac{M}{M+1}$
Twist	ϕ	$=$	$\frac{(2l+1)\pi}{2M+2}$

XXZ model

The same Bethe ansatz equations encode the ground state of the XXZ model

$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - \cos 2\eta\,\sigma_j^z\sigma_{j+1}^z\right)$$

with twisted boundary conditions

$$\sigma_{N+1}^z = \sigma_1^z \quad , \quad \sigma_{N+1}^x \pm i\sigma_{N+1}^y = e^{\pm i2\phi}\left(\sigma_1^x \pm i\sigma_1^y\right)$$

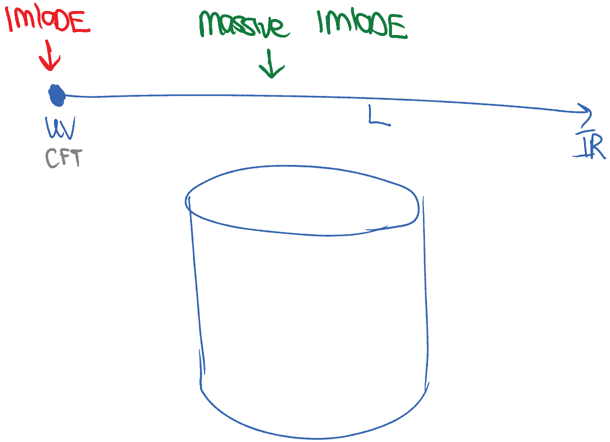
in the thermodynamic limit

Massless quantum field theory

The same Bethe ansatz equations encode the primary fields of a $c \leq 1$ conformal field theory or equivalently the massless limit of the quantum sine-Gordon model

Central charge	$c(M) = 1 - \frac{6M^2}{M+1}$	
Highest weight	$\Delta(M,l) = \frac{(2l+1)^2 - 4M^2}{16(M+1)}$	
Vacuum parameter	$p = \frac{2l+1}{4M+4}$	

Massive sine-Gordon model



Massive model is defined on a cylinder with radius s

A_1 example: sinh-Gordon / sine-Gordon models

The sinh-Gordon equation in light cone coordinates

$$\boxed{\partial_w \partial_{\bar{w}} \eta(w,\bar{w}) - e^{2\eta(w,\bar{w})} + e^{-2\eta(w,\bar{w})} = 0}$$

With the variable transformation

$$dw = p(z)^{1/2}\, dz \text{ and } d\bar{w} = p(\bar{z})^{1/2}\, d\bar{z}$$

introducing the potential

$$p(t) = t^{2M} - s^{2M}$$

and

$$\eta(w,\bar{w}) = \eta(z,\bar{z}) - \log(p(z)p(\bar{z})/4$$

the sinh-Gordon equation becomes

$$\boxed{\partial_z \partial_{\bar{z}} \eta(z,\bar{z}) - e^{2\eta(z,\bar{z})} + p(z)p(\bar{z})\, e^{-2\eta(z,\bar{z})} = 0}$$

A specific solution of the modified sinh-Gordon equation

Require a unique, real solution of

$$\partial_z \partial_{\bar{z}} \eta(z, \bar{z}) - e^{2\eta(z,\bar{z})} + p(z) p(\bar{z})\, e^{-2\eta(z,\bar{z})} = 0$$

that is finite everywhere except at $\rho = 0$ where

$z = \rho e^{i\phi}, \bar{z} = \rho e^{-i\phi}$ with

- Periodicity
$$\eta_i(\rho, \phi + \pi/M) = \eta_i(\rho, \phi)$$
- Asymptotic behaviour as $\rho \to 0$:
$$\eta(\rho, \phi) = l \ln \rho + \eta_0 + O(1)$$
- Asymptotic behaviour as $\rho \to \infty$
$$\eta(\rho, \phi) = M/2 \ln \rho + o(1)$$

Classical integrability: the Lax pair

Introduce the Lax matrices

$$U(z,\bar{z},\lambda) = \begin{pmatrix} \frac{1}{2}\partial_z\eta & -\lambda e^{\eta} \\ -\lambda p(z) e^{-\eta} & \frac{1}{2}\partial_z\eta \end{pmatrix}$$

and

$$V(z,\bar{z},\lambda) = \begin{pmatrix} -\frac{1}{2}\partial_z\eta & -\lambda^{-1} p(\bar{z}) e^{-\eta} \\ -\lambda^{-1} e^{\eta} & \frac{1}{2}\partial_{\bar{z}}\eta \end{pmatrix}$$

with $\lambda = \exp(\theta)$ the spectral parameter.

The zero curvature condition

$$[\partial_z + U, \partial\bar{z} + V] = 0$$

is equivalent to

$$\partial_z\partial_{\bar{z}}\eta(z,\bar{z}) - e^{2\eta(z,\bar{z})} + p(z)p(\bar{z})\, e^{-2\eta(z,\bar{z})} = 0$$

The associated linear problem

is

$$\big(\partial_z + U(z,\bar z,\lambda)\big)\mathbf{\Psi} = 0\,, \qquad \big(\partial_{\bar z} + V(z,\bar z,\lambda)\big)\mathbf{\Psi} = 0$$

where $\mathbf{\Psi} = (\psi_1, \psi_2)^{\mathrm{T}}$

The solution to the linear system is

$$\mathbf{\Psi}(z,\bar z,\lambda) = \begin{pmatrix} \lambda^{1/2} e^{\eta/2}\psi \\ e^{-\eta/2}\lambda^{-1/2}(\partial_z + \partial_z\eta)\psi) \end{pmatrix} = \begin{pmatrix} e^{-\eta/2}\lambda^{1/2}(\partial_{\bar z} + \partial_{\bar z}\eta)\bar\psi) \\ \lambda^{-1/2} e^{\eta/2}\bar\psi \end{pmatrix}$$

Eliminating ψ_2 we find

$$\boxed{\left[-\partial_z^2 + (\partial_z\eta)^2 - \partial_z^2\eta + \lambda^2 p(z)\right]\psi(z,\bar z) = 0}$$

Similarly

$$\boxed{\left[-\partial_{\bar z}^2 + (\partial_{\bar z}\eta)^2 - \partial_{\bar z}^2\eta + \lambda^{-2} p(\bar z)\right]\bar\psi(z,\bar z) = 0}$$

Functional relations and Bethe ansatz equations

By studying the solutions of

$$\boxed{\left[\partial_z^2 + (\partial_z \eta)^2 - \partial_z^2 \eta + \lambda^2 p(z)\right] \psi(z, \bar{z}) = 0}$$

for complex z with $\bar{z}$ considered a parameter we find Bethe ansatz equations of the form

$$\prod_{j=0}^{\infty} \left(\frac{E_j(l) - \omega^2 E_k(l)}{E_j(l) - \omega^{-2} E_k(l)} \right) \left(\frac{E_j(-1-l) - \omega^{-2} s^{4M} / E_k(l)}{E_j(-1-l) - \omega^2 / E_k(l)} \right) = -\omega^{2l+1}$$

where the *eigenvalues* $E_j(l), E_j(-1-l)$ depend on the parameters s, M as

$$E(l) = s^{2m} \lambda^{2M/(1+M)}$$

(and implicitly on l)

The Bethe ansatz equations exactly match those of the massive quantum integrable sine-Gordon model

Massive to massless

The nonlinear wave equation

$$\boxed{[\partial_z^2 + (\partial_z \eta)^2 - \partial_z^2 \eta + \lambda^2 p(z)]\, \psi(z, \bar{z}) = 0}$$

reduces to the Schrödinger equation

$$\boxed{[\partial_x^2 + x^{2M} + \frac{l(l+1)}{x^2} - E]\, \psi(x) = 0}$$

in the *massless* limit

$$\bar{z} \to 0 \quad , \quad z \sim s \to 0 \quad , \quad \ln \lambda \to \infty$$

with

$$x = \lambda^{1/(1+M)} z \quad , \quad E = s^{2m} \lambda^{2M/(1+M)}$$

reproducing the massless IM/ODE correspondence

The massive IM/ODE for A_n-type model

Start with the A_n affine Toda field equations

$$2\,\partial_{\bar{w}}\partial_w \eta_i = e^{2\eta_i - 2eta_{i-1}} - e^{2\eta_{i+1}-2\eta_i} \quad i = 1, \ldots, n$$

with $\eta_{n+1}(x,t) = \eta_1(x,t)$ and $\sum_{i=1}^{n} \eta_i = 0$

Make the change of variables

$$\mathrm{d}w = p(z)^{\frac{1}{n}}\,\mathrm{d}z\,, \quad \mathrm{d}\bar{w} = p(\bar{z})^{\frac{1}{n}}\,\mathrm{d}\bar{z} \quad , \quad p(t) = t^{nM} - s^{nM}$$

and

$$\eta_i(z,\bar{z}) \to \eta_i(z,\bar{z}) + (n-(2i-1))\ln(p(z)p(\bar{z}))/4n$$

to obtain the modified affine Toda field equations

$$\begin{aligned} 2\,\partial_{\bar{z}}\partial_z \eta_1 &= p(z)p(\bar{z})\,e^{2\eta_1 - 2\eta_n} - e^{2\eta_2 - 2\eta_1}\,, \\ 2\,\partial_{\bar{z}}\partial_z \eta_i &= e^{2\eta_i - 2\eta_{i-1}} - e^{2\eta_{i+1} - 2\eta_i} \qquad i = 2, \ldots, n-1 \\ 2\,\partial_{\bar{z}}\partial_z \eta_n &= e^{2\eta_{n-1} - 2\eta_n} - p(z)p(\bar{z})\,e^{2\eta_1 - 2\eta_n}\,. \end{aligned}$$

Classical integrability

The modified affine Toda equations arrise from the zero-curvature condition $V_z - U_{\bar{z}} + [U, V] = 0$ of the linear problem

$$\big(\partial_z + U(z, \bar{z}, \lambda)\big)\boldsymbol{\Psi} = 0\,, \quad \big(\partial_{\bar{z}} + V(z, \bar{z}, \lambda)\big)\boldsymbol{\Psi} = 0$$

with Lax matrices

$$(U(z, \bar{z}, \lambda))_{ij} = \partial_z \eta_i \, \delta_{ij} + \lambda \, (C(z))_{ij}$$

and

$$(V(z, \bar{z}, \lambda))_{ij} = -\partial_{\bar{z}} \, \eta_i \, \delta_{ij} + \lambda^{-1} (C(\bar{z}))_{ji}$$

$$(C(z))_{ij} = \begin{cases} \exp(\eta_{j+1} - \eta_j) \, \delta_{i-1,j} & , \quad j = 1, \ldots, n-1 \\ p(z) \, \exp(\eta_{j+1} - \eta_j) \, \delta_{i-1,j} & , \quad j = n\,, \end{cases}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i \equiv j \mod n \\ 0 & \text{otherwise} \end{cases}$$

A specific solution of the modified ATFT

Require a unique, real solution of the modified ATFT that is finite everywhere except at $\rho = 0$ with

- Periodicity

$$\eta_i\big(\rho, \phi + 2\pi/nM\big) = \eta_i(\rho, \phi)$$

- Asymptotic behaviour as $\rho \to 0$:

$$\eta_i(\rho, \phi) = \big(n - i - g_{n-i}\big) \ln \rho + O(1)$$

- Asymptotic behaviour as $\rho \to \infty$:

$$\eta_i(\rho, \phi) = \tfrac{1}{2}(2i - 1 - n)M \ \ln \rho + o(1)$$

Associated linear problem

The linear problem

$$\big(\partial_z + U(z,\bar z,\lambda)\big)\boldsymbol{\Psi} = 0\,, \qquad \big(\partial_{\bar z} + V(z,\bar z,\lambda)\big)\boldsymbol{\Psi} = 0$$

has solutions

$$\begin{aligned}\Psi_i(z,\bar z,\lambda) &= \begin{cases} -\lambda^{-1} e^{\eta_i-\eta_{i+1}}\big(\partial_z\Psi_{i+1} + \partial_z\eta_{i+1}\Psi_{i+1}\big) \\ e^{\eta_n}\psi \end{cases} \\ &= \begin{cases} e^{-\eta_1}\bar\psi \\ -\lambda\, e^{\eta_{i-1}-\eta_i}\big(\partial_{\bar z}\Psi_{i-1} - \partial_{\bar z}\eta_{i-1}\Psi_{i-1}\big) \end{cases}\end{aligned}$$

Eliminating $\Psi_1,\ldots,\Psi_{n-1}$ or $\Psi_2,\ldots,\Psi_n$, we obtain

$$\begin{aligned}\Big((-1)^{n+1}D_n(\eta) + \lambda^n p(z)\Big)\psi &= 0\,, \\ \Big((-1)^{n+1}\bar D_n(\eta) + \lambda^{-n} p(\bar z)\Big)\bar\psi &= 0\end{aligned}$$

with n^{th}-order differential operators

$$\begin{aligned}D_n(\eta) &= (\partial_z + 2\,\partial_z\eta_1)(\partial_z + 2\,\partial_z\eta_2)\cdots(\partial_z + 2\,\partial_z\eta_n)\,, \\ \bar D_n(\eta) &= (\partial_{\bar z} - 2\,\partial_{\bar z}\eta_n)\cdots(\partial_{\bar z} - 2\,\partial_{\bar z}\eta_2)(\partial_{\bar z} - 2\,\partial_{\bar z}\eta_1)\end{aligned}$$

Massive to massless

In the *massless* limit

$$\bar{z} \to 0 \quad , \quad z \sim s \to 0 \quad , \quad \ln \lambda \to \infty$$

with

$$x = z\, e^{\frac{\theta}{M+1}}\,,\ \bar{x} = \bar{z}\, e^{-\frac{\theta}{M+1}} \quad , \quad E = s^{nM} e^{\frac{n\theta M}{M+1}}$$

yields

$$\boxed{\Big((-1)^{n+1} D_n(\mathbf{g}) + p(x,E)\Big)\psi(x,E) = 0\,, \quad p(x,E) = x^{nM} - E}$$

where

$$D_n(\mathbf{g}) = \left(\partial_z - \frac{g_{n-1} - (n-1)}{x}\right)\left(\partial_z - \frac{g_{n-2} - (n-2)}{x}\right)\cdots\left(\partial_z - \frac{g_0}{x}\right)$$

This is precisely the n^{th}-order ODE appearing in the massless A_{n-1} ODE/IM correspondence

Conclusion and outlook

The
eigenvalues
of linear systems of certain partial differential equations
are the Bethe roots of certain massive quantum integrable models
IM
ODE

Reference: Bethe ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories, P Adamopoulou and C Dunning 2014 J. Phys. A: Math. Theor. 47 205205

Next steps: Exploit and extend this link between classical and quantum integrable models