Topology (de) trivialization transition in high-dimensional random fields and landscapes [1]

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[1] Based on: **YVF** & **B.A. Khoruzhenko**, under preparation.; **YVF** & **C. Nadal** *PRL* **109** (2012), 167203; **YVF**, **A. Lerario** & **E. Lundberg** *arXiv:1404.5349* ; **YVF** **arXiv:1307.2379**

May-Wigner Instabilty Scenario :

"Will a Large Complex System be Stable?"

This question was posed by **Robert May** (*NATURE* **238**, 413 (1972)) who introduced a toy **linear** model for (in)stability of a large system of many interacting species:

$$\dot{\mathbf{x}} = -\mu\mathbf{x} + B\mathbf{x}, \quad \mu > 0, \, \mathbf{x} \in \mathbb{R}^N$$

Without interactions the part $\dot{\mathbf{x}} = -\mu\mathbf{x}$ describes a **simple exponential** relaxation of N uncoupled degrees of freedom x_i with the same rate $\mu > 0$ towards the **stable equilibrium** $\mathbf{x} = 0$. A complicated interaction between dynamics of different degrees of freedom is mimicked by a general **real asymmetric** $N \times N$ **random matrix** B with mean zero and prescribed variance α^2 of all entries. As a typical eigenvalue of B with the largest real part grows as $\alpha\sqrt{N}$ the equilibrium $\mathbf{x} = 0$ becomes **unstable** as long as $\mu < \alpha\sqrt{N}$.

This scenario is known in the literature as the "**May-Wigner** instability" and despite its oversimplifying and schematic nature attracted very considerable attention in mathematical ecology and complex systems theory over the years.

A Nonlinear Analogue of May-Wigner model:

We suggest a natural **nonlinear extension** of the May's model to a system of N coupled **nonlinear autonomous** random ODE's:

$$\dot{x}_i = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N$$

where couplings $f_i(\mathbf{x})$ represent components of an $N-$dimensional vector field and are chosen as a sum of a "**gradient**" and "**solenoidal**" contributions:

$$f_i(\mathbf{x}) = -\frac{\partial V(\mathbf{x})}{\partial x_i} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\partial A_{ij}(\mathbf{x})}{\partial x_j}, \quad i = 1, \ldots, N$$

where we require the fields $A_{ij}(\mathbf{x})$ to be antisymmetric: $A_{ij} = -A_{ji}$. To make the model as simple as possible and amenable to a detailed mathematical analysis we choose the scalar potential $V(\mathbf{x})$ and the fields $A_{ij}(\mathbf{x})$ to be independent mean zero **Gaussian** random fields, with additional assumptions of **stationarity** and **isotropy** reflected in the covariance structure

$$\mathbb{E}\{V(\mathbf{x}_1)V(\mathbf{x}_2)\} = F\left(|\mathbf{x}_1 - \mathbf{x}_2|^2\right)$$

$$\mathbb{E}\{A_{ij}(\mathbf{x}_1)A_{nm}(\mathbf{x}_2)\} = (\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn})\,\Phi\left(|\mathbf{x}_1 - \mathbf{x}_2|^2\right)$$

Counting equilibria via Kac-Rice formulae:

A standard analysis of autonomous ODE's starts with finding **equilibrium points** and classifying them by **stability** properties.

We would like to know the **total number** $\mathcal{N}_{tot}(D)$ of all possible **equilibria** of our system of nonlinear ODEs, i.e. the number of simultaneous solutions of N equations $-\mu x_i + f_i(x_1, \ldots, x_N) = 0$, $i = 1, \ldots, N$ in a domain D of $\mathbb{R}^N$.

As is well-known this can be formally represented by a multidimensional **Kac-Rice** integral of the type $\mathcal{N}_{tot}(D) = \int_D \rho_{tot}(\mathbf{x})\, d\mathbf{x}$ where:

$$\rho_{tot}(\mathbf{x}) = \left|\det\left(-\mu\delta_{ij} + \partial_j f_i(\mathbf{x})\right)\right| \prod_{k=1}^{N} \delta\left(-\mu x_k + f_k(x_1, \ldots, x_N)\right)$$

Similarly the **number of stable equilibria** $\mathcal{N}_{st}(D)$ can be formally written as

$$\mathcal{N}_{st}(D) = \int_D \rho_{tot}(\mathbf{x}) \prod_{j=1}^{N} \theta\left[\mu - Re(z_j)\right]\, d\mathbf{x}$$

where z_j are (complex) eigenvalues of the corresponding $N \times N$ Jacobian matrix $J_{ij}(x) = \partial_j f_i(\mathbf{x})$ and $\theta(a) = 1$ for positive $a > 0$, and zero otherwise.

Ideally, we would like to have the full statistical characterization of $\mathcal{N}_{tot}(D)$ and $\mathcal{N}_{st}(D)$, but it looks currently as a **very challenging** problem.

Mean number of equilibria and the Elliptic Ensemble:

Using Kac-Rice approach we are able to count the **mean** total number $\mathbb{E}\{\mathcal{N}_{tot}\}$ of all possible **equilibria** in the system of nonlinear ODEs under consideration. This turns out to be given by (**YVF** $\&$ **Khoruzhenko**, *in progress*):

$$\mathbb{E}\{\mathcal{N}_{tot}\} = \frac{1}{m^N N^{(N-1)/2}} \int_{-\infty}^{\infty} \left\langle \left| \det \left((m + t\sqrt{\tau})\sqrt{N} - \mathbf{X} \right) \right| \right\rangle_X \frac{e^{-\frac{Nt^2}{2}} dt}{\sqrt{2\pi}}$$

where $m = \mu/\mu_c$ with some characteristic scale μ_c, and the random **real asymmetric** matrix $\mathbf{X}$ being taken from the **Gaussian Elliptic Ensemble**:

$$\mathcal{P}(\mathbf{X}) = C_N(\tau) e^{-\frac{1}{2(1-\tau^2)}\left[\mathrm{Tr}\mathbf{X}\mathbf{X}^T - \tau \mathrm{Tr}\mathbf{X}^2\right]}, \quad \tau \in [0,1]$$

The parameter τ depends on the ratio of variances of **gradient** and **solenoidal** components of the field such that the **real Ginibre ensemble** with $\tau = 0$ corresponds to **purely solenoidal**, and GOE with $\tau = 1$ to **purely gradient** flow.

Let us denote $\rho_N^{(\mathbf{r})}(\lambda)$ the mean density of **real** eigenvalues of $N \times N$ matrices $\mathbf{X}$ for the elliptic ensemble at λ. Then it turns out that (cf. **Edelman, Kostlan, Schub** '94.)

$$\langle |\det(\lambda - \mathbf{X})| \rangle_X = 2\sqrt{1+\tau} \frac{(N-1)!}{(N-2)!!} \rho_{N+1}^{(\mathbf{r})}(\lambda) \, e^{\frac{\lambda^2}{2(1+\tau)}}$$

A Nonlinear Analogue of May-Wigner Instability as Topology Detrivialization:

The mean density $\rho_N^{(\mathbf{r})}(\lambda)$ of real eigenvalues for the elliptic ensemble was computed explicitly by **Forrester** $\&$ **Nagao** '08 in terms of Hermite polynomials, and its large-N asymptotic behaviour was studied as well.

Asymptotic analysis of the counting problem for $N \gg 1$ reveals then a **topology detrivialization** transition, with the total number of equilibria **abruptly** changing from **a single equilibrium** for $\mu > \mu_c = \sqrt{N(F''(0) + \Phi''(0))}$ to **exponentially many** equilibria as long as $\mu < \mu_c$:

$$\mathbb{E}\{\mathcal{N}_{tot}\} \approx \sqrt{\frac{2(1+\tau)}{1-\tau}}\, e^{N\Sigma_{tot}(m)}, \quad \Sigma_{tot}(m) = \frac{m^2-1}{2} - \ln m > 0 \quad \text{for } m = \frac{\mu}{\mu_c} < 1$$

Similar transition was reported recently for a model of randomly coupled nonlinear ODE's describing neural networks (**G. Wainrib** and **J. Touboul** (2013)).

In the scaling vicinity of the transition for $|1-m| \sim 1/\sqrt{N}$ we further computed a (*presumably universal*) critical crossover function smoothly interpolating between $\mathbb{E}\{\mathcal{N}_{tot}\} = 1$ for $m > 1$ and $\mathbb{E}\{\mathcal{N}_{tot}\} \sim e^{N\Sigma_{tot}(m)}$ for $m < 1$.

Landscape topology (de)trivialization for gradient dynamics:

In the case of purely **gradient** dynamics $\dot{\mathbf{x}} = -\mu\mathbf{x} - \nabla V(x) = -\nabla\mathcal{L}(\mathbf{x})$ where :

$$\mathcal{L}(\mathbf{x}) = \frac{\mu}{2}\sum_{i=1}^{N-1} x_i^2 + V(x_1, \ldots, x_{N-1}), \quad \mu > 0, \, -\infty < x_i < \infty$$

is the Lyapunov function (or "energy functional"). Correspondingly the equilibria points are simply **stationary** points of the Lyapunov function whereas the stable equilibria are local **minima**.

Taking as before $V(\mathbf{x})$ to be **stationary isotropic** random Gaussian field with covariance structure $\mathbb{E}\{V(\mathbf{x})V(\mathbf{y})\} = F\left((\mathbf{x}-\mathbf{y})^2\right)$ we find that the Jacobian matrix $J_{ij} = \frac{\partial^2 V}{\partial x_i \partial x_j}$ becomes a real symmetric **Hessian** with Gaussian entries.

As a result, one is able to calculate not only $\mathbb{E}\{\mathcal{N}_{tot}\}$ but also $\mathbb{E}\{\mathcal{N}_{min}\}$ which is given in terms of the distribution $\mathcal{F}_N(t)$ of the **largest** eigenvalue of the standard GOE matrix (**YF** $\&$ **Nadal** '12):

$$\mathbb{E}\{\mathcal{N}_{min}\} = \frac{2^{(N+1)/2}}{N^{(N-3)/2}} \frac{e^{-\frac{N}{2}m^2}}{m^{N-1}} \Gamma\left(\frac{N}{2}\right) \int_{-\infty}^{\infty} e^{-\frac{N}{2}\left(t^2 - 2\sqrt{2}mt\right)} d\mathcal{F}_N(t)$$

where $m = \frac{\mu}{\sqrt{NF''(0)}}$ is the main dimensionless control parameter of the theory.

Landscape topology (de)trivialization for gradient dynamics:

The asymptotics $\mathcal{F}_{N\gg 1}(t)$ is well known (**Tracy** $\&$ **Widom** '94; **Borot et al** '11). Using it for a fixed $m\neq 1$ we find for the mean number of minima:

$$\mathbb{E}\left\{\mathcal{N}_{min}\right\}\approx\left\{\begin{array}{cc} 1, & m>1 \\ \sim e^{N\Sigma_{st}(m)}, & m<1 \end{array}\right.$$

Here the complexity of stable equilibria (minima) is given by

$$\Sigma(m)=-\frac{1}{2}(m^2-4m+3)-\ln m>0 \text{ for } m<1$$

so that $\Sigma(m)\propto(1-m)^3$ for $m\to 1$. This should be compared with the complexity of **all** equilibria: $\Sigma_{tot}(m)\propto(1-m)^2$ for $m\to 1$. We conclude that for gradient dynamics **stable equilibria** are relatively **rare**.

Moreover, after scaling $1-m=\delta/N^{1/3}$ and assuming the parameter δ to be of the order of unity when $N\to\infty$ one finds

$$\lim_{N\to\infty}\mathbb{E}\{\mathcal{N}_{min}\}=2e^{-\delta^3/3}\int_{-\infty}^{\infty}e^{\delta\zeta}F_1'(\zeta)d\zeta$$

in terms of the **Tracy-Widom** density $F_1'(\zeta)$. This describes the **critical crossover** accross the instability transition.

Part II: Topology of Random Algebraic Varieties :

Recently, the problem of computing the expectation of topological properties of random algebraic varieties has attracted a lot of interest (see e.g. the works by **Burgisser '07, Nazarov-Sodin '09, Gayet-Welshinger '11, Sarnak '11, Lerario-Lundberg '12, Sarnak-Wigman '13**) and others. An important class of problems addresses estimates for Betti numbers of "generic" (=random) real hypersurfaces given by **zero set** of real random homogenious polynomials of degree d in $n+1$ variables restricted to the unit sphere. E.g. for $d = 60$ and $n = 2$ a typical picture is:

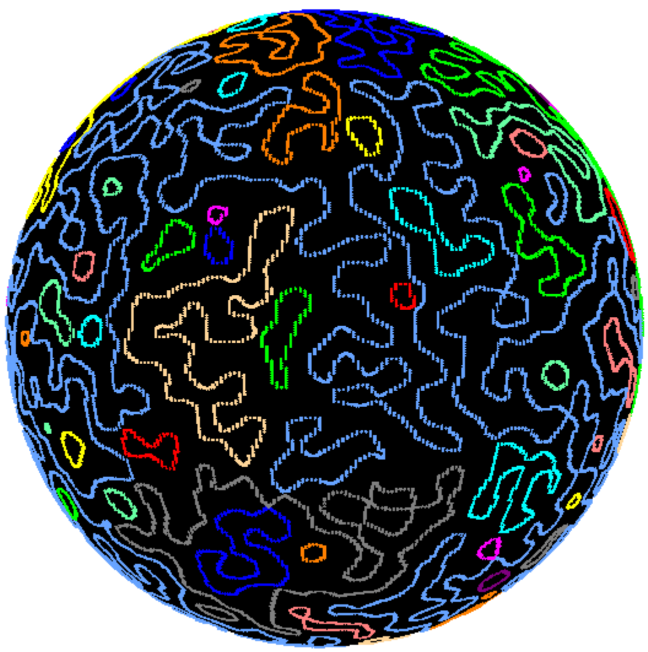

Figure 1: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Upper bound on b_0 by Random Matrix Theory:

It turns out that the methods and results just exposed allow one to provide a useful **upper bound** to the **expected number of connected components** $b_0(f)$. Indeed, every component of the zero locus of the polynomials restricted to the sphere bounds a region where the function attains at least a maximum or a minimum, and consequently $\mathbb{E}\{b_0(f)\} \leq \mathbb{E}\{N_{min} + N_{max}\}$, where $N_{min/max}$ are numbers of minima/maxima on the sphere. The problem then amounts to counting minima of a random function on a sphere.

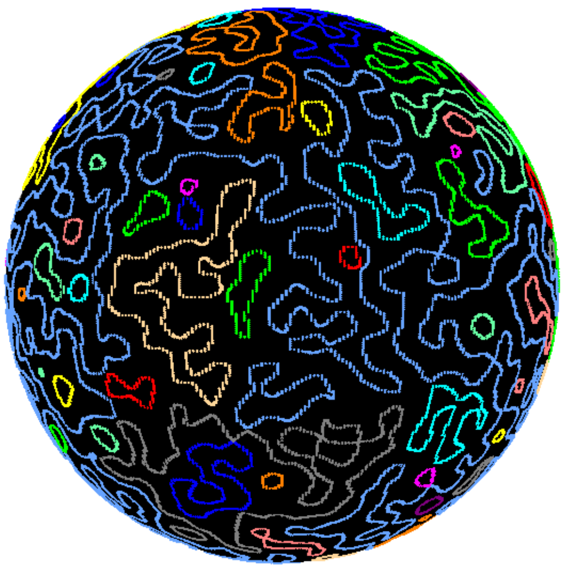

Figure 2: Zero locus of a random polynomial of degree $d = 60$ on the sphere (**M. Nastasescu**)

Counting Stationary points for Isotropic Gaussian Landscapes:

In recent years there was a steady progress in counting $\&$ classifying the **mean number** of **stationary points** of smooth isotropic Gaussian random fields $V(\mathbf{x})$ on the sphere $|\mathbf{x}| = R$ such that

$$\mathbb{E}\left\{V(\mathbf{x})V(\mathbf{x}')\right\} = F(\mathbf{x}\cdot\mathbf{x}')$$

Using the multidimensional **Kac-Rice** integrals it was shown, in particular, that $\mathbb{E}\left\{\mathcal{N}_{min}\right\}$ can be again directly related to the the distribution $\mathcal{F}_N(t)$ of the maximal eigenvalue of **random GOE** matrices H such that $\mathcal{P}(H) \propto \exp\left(-\frac{N}{4}\text{Tr}H^2\right)$. Namely

$$\mathbb{E}\left\{\mathcal{N}_{min}\right\} = 2\left(\frac{1+B}{1-B}\right)^{N/2}\sqrt{1-B}\int_{-\infty}^{\infty} e^{-NB\frac{t^2}{2}}\, d\mathcal{F}_N$$

where the only control parameter is

$$B = \frac{R^2\cdot F''(R^2) - F'(R^2)}{R^2\cdot F''(R^2) + F'(R^2)}$$

One can also give similar formulae for the mean number of stationary points of any **index**, and even specify their number to a given **height** of the landscape function, see **Auffinger, Ben Arous $\&$ Cerny**'11; **Auffinger $\&$ Ben Arous**'12; **Nicolaescu** '12; **YVF $\&$ Le Doussal** '13; **YVF** '13

Upper bound on b_0 for Gaussian rotationally invariant polynomials:

Endowing polynomials with a rotationally-invariant Gaussian distribution we can find $\mathbb{E}\{\mathcal{N}_{min}\}$ for any n and d from our formalism. We will mostly be interested in the limits $d \to \infty$ for a fixed n or $n \to \infty$ for a fixed d.

Let $\{Y_l^j\}$ denote the standard basis of spherical harmonics of degree l on sphere S^n, then a random invariant Gaussian polynomial of degree d in $n+1$ variables can be constructed as :

$$f(\mathbf{x}) = \sum_{d-l\in 2\mathbb{N}} p_d(l) \sum_j \xi_l^j |\mathbf{x}|^{d-l} Y_l^j\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad p_d(l) \geq 0 \quad \textbf{Kostlan}$$

where ξ_l^j are i.i.d. Gaussian coefficients, and nonnegative weights $p_d(d), p_d(d-2), \ldots,$ parametrize *all* invariant ensembles.

We assume that there exists such $0 < \lambda \leq 1$ that as $d \to \infty$ the polynomials assume the *scaling* form: $p_d(d^\lambda x)d^\lambda \to \psi(x) \neq 0$ pointwise. Further assuming that $\psi(x)$ is bounded by $c_1 e^{-c_2 x^2}$ we can prove the following Theorem (YVF, Lerario, Lundberg):

For any integrable $\psi(x) : (0,\infty) \to \mathbb{R}$ with subgaussian tails define the moments

$$\mu_k(\psi) = \int_0^\infty \psi^2(x)\, x^k\, dx, \quad k \in \mathbb{N}$$

Then there exists a constant $c > 0$ such that for any random hypersurface X

$$\lim_{d\to\infty} \mathit{sup} \frac{\mathbb{E}b_0(X)}{d^{\lambda n}} \leq \lim_{d\to\infty} \frac{2\mathbb{E}N_{min}}{d^{\lambda n}} \sim c\left(\frac{\mu_3(\psi)}{\mu_2(\psi)}\right)^{n/2} n^{-\frac{n}{2}-\frac{17}{36}} e^{-n+\frac{4\sqrt{2}}{3}\sqrt{n}}$$

Summary :

- As a generalization of the linear model by **R. May** we suggest to consider an autonomous system of N nonlinear differential equations driven by a random Gaussian field:

$$\dot{\mathbf{x}} = -\mu\mathbf{x} + f(\mathbf{x})$$

When magnitude of the random field increases the **single equilibrium** is replaced by **exponentially many** equilibria via a sharp "**topology (de)trivialization**" transition. The mean number of equilibria can be evaluated rigorously starting from the Kac-Rice formulae and mapping onto the problem of evaluating the **modulus of characteristic polynomial** for **real elliptic** Gaussian ensemble of asymmetric matrices. The latter can be related to the mean density of real eigenvalues of that ensemble.

- In the vicinity of the transition the mean number of equilibria is of order of unity and is given by a (presumably) **universal** crossover expression described in terms of the "edge density" of real eigenvalues.

- For the special case of **purely gradient** flows one can also find explicit expression for the number of **stable** equilibria. The latter are exponential in N but their fraction among all equilibria is negligible. The crossover expression in that case is given in terms of the **Tracy-Widom** density of the largest GOE eigenvalue.

- Similar techniques can be used to get a useful **upper bound** for the mean number of **connected components** (zero's Betti number) of real random algebraic varieties.

Challenges:

- Mean number of stable equilibria for a general non-gradient flow. Classification of equilibria.

- **Fluctuations** in the number of equilibria.

- Dynamics and global phase portrait. Periodic trajectories? Chaos?

- Further insights into topology of real random algebraic varieties.