Universität Bielefeld

Unterstützt von / Supported by

Alexander von Humboldt
Stiftung / Foundation

Level statistics of the correlated and uncorrelated Wishart ensemble

Mario Kieburg

Fakultät für Physik
Universität Bielefeld (Germany)

X Brunel-Bielefeld Workshop on Random Matrix Theory
December 13th, 2014

My recent Collaborators

Gernot Akemann

Universität Bielefeld

Rene Wegner

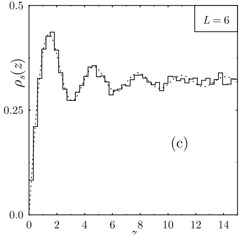
Thomas Guhr

UNIVERSITÄT DUISBURG ESSEN

Tim Wirtz

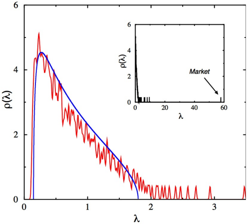
Wishart matrices in real life

Wishart matrix: $WW^\dagger$, $W \in \mathbb{R}^{N_1 \times N_2}, \mathbb{C}^{N_1 \times N_2}, \mathbb{H}^{N_1 \times N_2}$



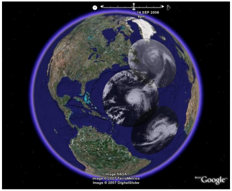
Lattice QCD

(image by Göckeler, Hehl, Rakow, Schäfer, Wettig (1998))



Financial Market

(image by Bouchaud, Potters, Laloux (2005))



Climate Research

Telecommunications

What do we know about uncorrelated Wishart matrices?

$$P(W) = \exp[-\mathrm{tr}\, WW^\dagger]$$

- **Determinantal ($\beta = 2$) or Pfaffian ($\beta = 1, 4$) point processes?**
  Yes: for averages of $\dfrac{\prod \det(WW^\dagger - \lambda_j \mathbb{1})}{\prod \det(WW^\dagger - \kappa_j \mathbb{1})}$
  and k-point correlation functions
- **Level density?**
  Yes: Marcenko-Pastur distribution
- **Local Statistics?**
  Yes: Sine-kernel (bulk),
  Airy-kernel (soft edge) $\rightarrow$ Tracy-Widom distribution,
  Bessel-kernel (hard edge)

What do we know about correlated Wishart matrices?

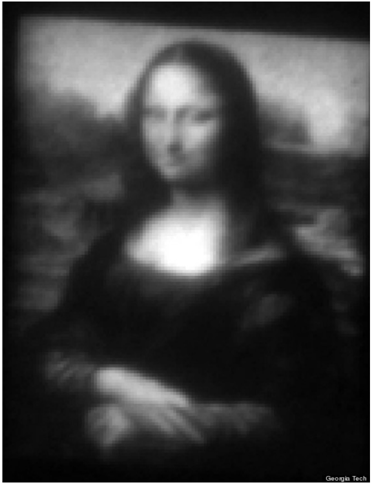
$$P(W) = \exp[-\mathrm{tr}\, C_2^{-1} W C_1^{-1} W^\dagger]$$

- **Determinantal ($\beta = 2$) or Pfaffian ($\beta = 1, 4$) point processes?**
  Yes: for the case $\beta = 2$, $C_2 = \mathbb{1}$
  otherwise we don't know
- **Level density?**
  Yes: for the case $\beta = 1$ and $C_2 = \mathbb{1}$ and
  for $\beta = 2$ and $C_1, C_2 \neq \mathbb{1}$
  otherwise we didn't know
- **Local Statistics?**
  Yes: for the case $\beta = 2$ and $C_2 = \mathbb{1}$
  otherwise we don't know (problems partially circumvented)

Outline of this talk

- Distribution of the smallest eigenvalue (uncorrelated)
- Correlated Wishart matrices
- Outlook: What is with non-Gaussian ensembles?

Distribution of the smallest eigenvalue (uncorrelated)



Smallest painting of the world, “Mona Lisa" on a hair by Georgia Institute of Technology (image from huffingtonpost.co.uk)

Distribution of the Smallest Eigenvalue

$$E_{\min}(s) = \partial_s P(0,s)$$

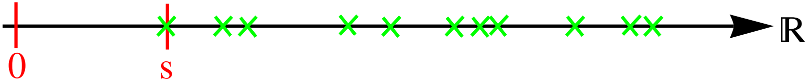
$$P(0,s) = \int d[W] P(W) \Theta[WW^\dagger - s\mathbb{1}]$$

Laguerre ensemble $P(W) \propto \det^\gamma WW^\dagger\, e^{-\mathrm{tr}\, WW^\dagger/(2\sigma^2)}$:

$$\beta = 1: \; E_{\min}(s) \propto s^{(\nu-1)/2} e^{-Ns} \times \left\langle \det^{(\nu-1)/2}(WW^\dagger + s\mathbb{1}) \right\rangle_{W:\,(N-1)\times(N+2)}$$

$$\beta = 2: \; E_{\min}(s) \propto s^{\nu} e^{-Ns} \left\langle \det^{\nu}(WW^\dagger + s\mathbb{1}) \right\rangle_{W:\,(N-1)\times(N+1)}$$

$$\beta = 4: \; E_{\min}(s) \propto s^{2\nu+1} e^{-Ns} \times \left\langle \det^{\nu+1/2}(WW^\dagger + s\mathbb{1}) \det^{3/2} WW^\dagger \right\rangle_{W:\,2(N-1)\times 2(N-1)}$$



Well known results for: $\beta = 2$ and ν arbitrary; $\beta = 1$ and ν odd

$E_{\min}$ for $\beta = 2$

$$E_{\min}(s) \;\;\propto\;\; s^{-\nu} e^{-Ns} \det\left[\mathcal{L}^{(b-a-2)}_{N+\nu+a-b}(-s)\right]$$

$E_{\min}$ for $\beta = 1$, odd ν (integer power of det)

$$E_{\min}(s) \;\;\propto\;\; s^{-(9\nu+1)/2} e^{-Ns} \mathrm{Pf}\left[(a-b)\mathcal{L}^{(2\nu-a-b-4)}_{N+a+b+1}(-2s)\right]$$

$\mathcal{L}^{(\alpha)}_n$ is the modified Laguerre polynomial in monic normalization

Nagao, Forrester[98], for $N \to \infty$ with ν fixed by Damgaard, Nishigaki, Wettig[98], Damgaard, Nishigaki[00]

$E_{\min}$ for $\beta = 1$, even ν

The square root is a problem! (?)

$$E_{\min}(s) \quad \propto \quad s^{(\nu-1)/2} e^{-Ns} \left\langle \frac{\det^{\nu/2}(WW^\dagger + s\mathbb{1})}{\sqrt{\det(WW^\dagger + s\mathbb{1})}} \right\rangle_{W:\,(N-1)\times(N+2)}$$

Edelman's Approach[91]

$$E_{\min}(s) = s^{(\nu-1)/2} e^{-Ns} \left[\mathcal{U}(a_\nu, b_\nu, s) Q_\nu(s) + \mathcal{U}(a'_\nu, b'_\nu, s) Q'_\nu(s)\right]$$

- $Q_\nu(s)$ and $Q'_\nu(s)$ polynomials given via a recursion
- $\mathcal{U}(a, b, x)$ is Tricomi's confluent hypergeometric function
- simple expressions for $\nu = 0, 2$

Main idea: Recursion in ν!

$E_{\min}$ for $\beta = 1$, even ν

The square root is a problem! (?)

$$E_{\min}(s) \propto s^{(\nu-1)/2} e^{-Ns} \left\langle \frac{\det^{\nu/2}(WW^\dagger + s\mathbb{1})}{\sqrt{\det(WW^\dagger + s\mathbb{1})}} \right\rangle_{W:\,(N-1)\times(N+2)}$$

$$\propto s^{(\nu+2)/2} e^{-Ns} \left\langle \frac{\det^{\nu/2}(WW^\dagger + s\mathbb{1})}{\sqrt{\det(W^\dagger W + s\mathbb{1})}} \right\rangle_{W}$$

$$\propto s^{(\nu+2)/2} e^{-Ns} \left\langle \frac{1}{\sqrt{\det(W^\dagger W + s\mathbb{1})}} \right\rangle_{W} \left\langle \det^{\nu/2}(WW^\dagger + s\mathbb{1}) \right\rangle_{\widehat{\mu}}$$

Combination of orthogonal polynomial theory and SUSY is the solution!

Redefinition of the two-point weight:

$$d\widehat{\mu}(x_1, x_2) = \mathrm{sign}(x_1 - x_2) \frac{x_1 x_2 e^{-x_1 - x_2}}{\sqrt{(x_1 + s)(x_2 + s)}} dx_1 dx_2$$

De Bruijn Integral

$$Z_{2k}^{(2N)}(\kappa) = \mathrm{Pf}\left[\int d\mu(x_a, x_b)\right] \Delta_{2N}(x) \prod_{l=1}^{2k} \det(x - \kappa_l \mathbb{1}_{2N})$$

$$\propto \frac{1}{\Delta_{2k}(\kappa)} \mathrm{Pf}\left[(\kappa_a - \kappa_b) Z_2^{(2(N+k-1))}(\kappa_a, \kappa_b)\right]_{1 \leq a,b \leq 2k}$$

with antisymmetric two point measure:

$$d\mu(x_1, x_2) = -d\mu(x_2, x_1)$$

Examples

- real Wishart ($\beta = 1$):
  $d\mu(x_1, x_2) = \mathrm{sign}(x_1 - x_2)(x_1 x_2)^{(\nu-1)/2} e^{-x_1 - x_2} dx_1 dx_2$
- quaternion Wishart ($\beta = 4$):
  $d\mu(x_1, x_2) = (x_1 x_2)^{2\nu+1} e^{-x_1-x_2}(\partial_{x_1} - \partial_{x_2})\delta(x_1 - x_2) dx_1 dx_2$
- complex Wishart ($\beta = 2$):
  $d\mu(x_1, x_2) = \dfrac{(x_1^N - x_2^N)^2}{x_1 - x_2}(x_1 x_2)^{\nu} e^{-x_1-x_2} dx_1 dx_2$

Also the complex case fits into this framework!

$$\Delta_{2N}(x) = \pm \det\left[x_a^{b-1}\right] = \pm \mathrm{Pf}\left[\frac{(x_a^N - x_b^N)^2}{x_a - x_b}\right]$$

- **And our case:**
  $d\widehat{\mu}(x_1, x_2) = \mathrm{sign}(x_1 - x_2)\dfrac{x_1 x_2 e^{-x_1-x_2}}{\sqrt{(x_1 + s)(x_2 + s)}} dx_1 dx_2$

$E_{\min}$ for $\beta = 1$, even ν

- normalization constant:

$$\left\langle \det{}^{-1/2}(W^\dagger W + s\mathbb{1}) \right\rangle_W \propto \int_0^\infty dx e^{-sx} \frac{x^{N/2}}{(1+x)^{(N-1)/2}} \propto \mathcal{U}\left(\frac{N+2}{2}, \frac{5}{2}, s\right)$$

- skew-orthogonal polynomials of even order $2j$:

$$\left\langle \det(y\mathbb{1} - WW^\dagger) \right\rangle_{\widehat{\mu}} = \mathcal{L}_{2j}^{(2)}(2y) - 2j \frac{\mathcal{U}\left(j+\frac{3}{2}, \frac{3}{2}, s\right)}{\mathcal{U}\left(j+\frac{3}{2}, \frac{5}{2}, s\right)} \mathcal{L}_{2j-1}^{(2)}(2y)$$

$\mathcal{U}(a, b, x)$ is Tricomi's confluent hypergeometric function

$E_{\min}$ for $\beta = 1$, $\nu \in 4\mathbb{N}$

Distribution of the smallest eigenvalue:

$$E_{\min}(s) \propto s^{(\nu+2)/2} e^{-Ns} \mathcal{U}\left(\frac{N+2}{2}, \frac{5}{2}, s\right)$$

$$\times \mathrm{Pf}\left[\partial_{s_1}^{a-1}\partial_{s_2}^{b-1}(s_1 - s_2)\left\langle \det(WW^\dagger + s_1 \mathbb{1})\det(WW^\dagger + s_2 \mathbb{1})\right\rangle_{\widehat{\mu}}\Big|_{s_1=s_2=s}\right]$$

Two point partition function:

- either as a sum of skew-orthogonal polynomials or as

$$\left\langle \det(WW^\dagger + s_1 \mathbb{1})\det(WW^\dagger + s_2 \mathbb{1})\right\rangle_{\widehat{\mu}}^{W:j\times(j+3)}$$

$$\propto \frac{1}{\mathcal{U}\left(\frac{j+3}{2}, \frac{5}{2}, s\right)} \int_0^\infty dx \int_{CSE(4)} d\mu_{\mathrm{Haar}}(U) e^{-sx + \mathrm{tr}\,\mathrm{diag}(s_1, s_1, s_2, s_2)U}$$

$$\times \frac{x^{(j+1)/2} \det^{-j/2} U \det^{j/2+1}(\mathbb{1} + U) \det^{1/2}((1+x)\mathbb{1} + U)}{(1+x)^{j/2+2}}$$

New result for finite ν and N!

(Akemann, Guhr, Kieburg, Wegner, Wirtz[14])

Smallest Eigenvalue Distr. ($\beta = 4$)

$$E_{\min}(s) \propto s^{2\nu+1} e^{-Ns} \left\langle \det{}^{\nu+1/2}(WW^\dagger + s\mathbb{1}) \det{}^{3/2} WW^\dagger \right\rangle_W$$

Microscopic universality with dynamical fermions

M.E. Berbenni-Bitsch[1], S. Meyer[1], and T. Wettig[2]

$$P(\lambda_{\min}) = \frac{2}{(\alpha+1)!(\alpha+3)!} \lambda_{\min}^{2\alpha+3} e^{-\frac{1}{2}\lambda_{\min}^2} T(\lambda_{\min}^2) \,, \quad (7)$$

where $T(x) = 1 + \sum_{d=1}^{\infty} a_d\, x^d$ with

$$a_d = \sum_{\substack{|\kappa|=d \\ l(\kappa)\le\alpha+1}} \prod_{(i,j)\in\kappa} \frac{\alpha+2j-i}{\alpha+2j-i+4} \times \frac{1}{[\kappa'_j-i+2(\kappa_i-j)+1][\kappa'_j-i+2(\kappa_i-j)+2]} \,. \quad (8)$$

Here, κ denotes a partition of the integer d, $l(\kappa)$ its length, $|\kappa|$ its weight, and κ' the conjugate partition. In Eq. (8), a partition κ is identified with its diagram, $\kappa = \{s = (i,j); 1 \le i \le l(\kappa), 1 \le j \le \kappa_i\}$. The Taylor series for $T(x)$ is rapidly convergent, and the curves for $N_f = 2$ and 4 can easily be computed to any desired accuracy.

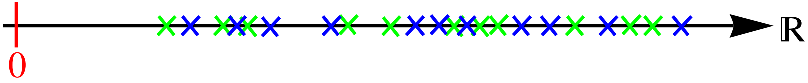
Correlated Wishart matrices

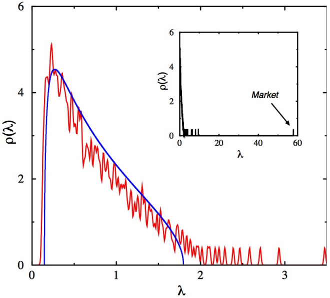
“The Three Wise Apes" (image from thomasbrasch-wordpress.com)

The RMT Model

$$P(W) \propto \exp[-\mathrm{tr}\, C^{-1} WW^\dagger]/\det^{\gamma(\beta)(N+\nu)} C$$

Empirical correlation matrix: $C = U\Lambda U^\dagger = U \mathrm{diag}(\Lambda_1, \ldots, \Lambda_N) U^\dagger$





(Bouchaud, Potters, Laloux (2005))

For $\beta = 2$

Harish-Chandra, Itzykson-Zuber integral:

$$p(x) \propto \det{}^{\nu} x \Delta_N^2(x) \int_{\mathrm{U}(N)} d\mu_{\mathrm{Haar}}(U) \exp[-\mathrm{tr}\,\Lambda^{-1} U x U^{\dagger}] / \det{}^{N+\nu} \Lambda$$

$$\propto \frac{\det{}^{\nu} x \Delta_N(x) \det\left[e^{-x_b/\Lambda_a}\right]}{\det{}^{N+\nu} \Lambda \Delta_N(\Lambda^{-1})}$$

The jpdf can be calculated! ⇒ Everything is calculated!?

- level density (Alfano, Tulino, Lozano, Verdu[04]):

$$\rho(y) \propto \det \begin{bmatrix} \Lambda_a{}^{b-2} & \Lambda_a{}^{-\nu-1} e^{-y/\Lambda_a} \\ y^{\nu+b-1}/\Gamma[\nu+b] & 0 \end{bmatrix} \Big/ \Delta_N(\Lambda)$$

- gap probability for smallest eigenvalue (Forrester[07]):

$$P(0,s) \propto e^{-\mathrm{tr}\, s/\Lambda} \det\left[\mathcal{L}_{\nu+b-1}^{(1-N-\nu-b)}\left(\frac{s}{\Lambda_a}\right)\right] \Big/ \Delta_N(s/\Lambda)$$

No group integral for $\beta = 1, 4$

Harish-Chandra integral (known):

$$\int_{\mathrm{G}} d\mu_{\mathrm{Haar}}(U) \exp[-\mathrm{tr}\, \Lambda^{-1} U x U^{\dagger}]$$

Λ^{-1} and x in the Lie algebra of the group G

Itzykson-Zuber integral (unknown for $\beta = 1, 4$):

$$\int_{\mathrm{G}} d\mu_{\mathrm{Haar}}(U) \exp[-\mathrm{tr}\, \Lambda^{-1} U x U^{\dagger}]$$

Λ^{-1} and x in a co-set corresponding to the group G

$\beta = 1$: Λ^{-1} and x real symmetric

$\beta = 4$: Λ^{-1} and x quaternion self-dual

“SUSY" circumvents group integrals!

For $\beta = 1$

- level density (Recher, Kieburg, Guhr, Zirnbauer[10])

  Finite sum of products of two decoupling one-fold integrals

- gap probability for the smallest eigenvalue (Wirtz, Guhr[14])

  Yields always a $(\nu - 1) \times (\nu - 1)$ or $\nu \times \nu$ Pfaffian!
  But: The kernel depends on all Λ_j

- level density: two-sided correlations (Waltner, Wirtz, Guhr[14])

  Finite sum of four-fold integrals

Expressions drastically simplify
when Λ is double degenerate!

For $\beta = 1$

Gap probability for the largest eigenvalue (Wirtz, Kieburg, Guhr[14]):

$$P(s,\infty) \propto \int_{\mathbb{R}^{N\times(N+\nu)}} d[W] e^{-s\,\mathrm{tr}\,\Lambda^{-1}WW^\dagger} \frac{\Theta(\mathbb{1} - W^\dagger W)}{\det^{(N+\nu)/2}(\Lambda/s)}$$

Ingham-Siegel integral = Heaviside Θ-function:

$$P(s,\infty) \propto \frac{1}{\det^{(N+\nu)/2}(\Lambda/s)} \int_{\mathrm{Sym}(N+\nu)} d[H] \frac{e^{\imath\,\mathrm{tr}\,H}}{\det^{(N+\nu-1)/2}(\imath H + \mathbb{1})}$$
$$\times \prod_{j=1}^{N} \frac{1}{\sqrt{\det(\imath H - (s/\Lambda_j + 1)\mathbb{1})}}$$

Yields always an $(N+\nu)\times(N+\nu)$ Pfaffian!
But: The kernel depends on all Λ_j

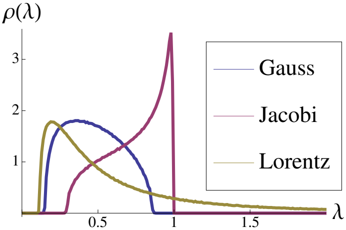
More details in the poster presentation by Tim Wirtz!

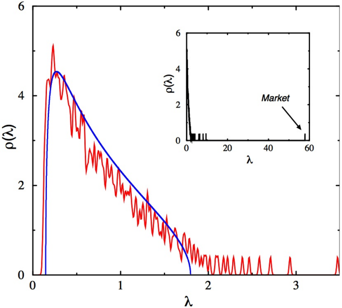
Outlook

“Outlook" by Peter Quidley (image from theoilpaintingsaless.com)

What is with non-Gaussian ensembles?

- Laguerre:
  - $P(W) = \exp[-\mathrm{tr}\, W^\dagger C^{-1} W]$
  - non-compact support
  - all moments exist
- Jacobi:
  - $P(W) = \det^\gamma(\mathbb{1} - W^\dagger C^{-1} W) \times \Theta(\mathbb{1} - W^\dagger C^{-1} W)$
  - compact support
  - all moments exist
- Lorentz (Cauchy):
  - $P(W) = \dfrac{1}{\det^\gamma(\mathbb{1} + W^\dagger C^{-1} W)}$
  - non-compact support
  - not all moments exist





(Bouchaud, Potters, Laloux (2005))

More Mathematical Challenges and Fun

Meijer G-ensemble = Product of rectangular matrices

Statistics of the Wishart matrix:

$$WW^\dagger = W_1 W_2 \cdots W_M C W_M^\dagger \cdots W_2^\dagger W_1^\dagger$$

- What is the level density with a fixed correlation matrix C?
- What are the distributions of the smallest eigenvalues at the hard edge (also for $C = \mathbb{1}$)?

New universality class! $\rightarrow$ Meijer G-kernel

- What happens when the W_j's are correlated with each other?

Thank you for your attention!

- C. Recher, M. Kieburg, T. Guhr: Phys. Rev. Lett. **105**, 244101 (2010) [arXiv:1006.0812]
- C. Recher, M. Kieburg, T. Guhr, M. R. Zirnbauer: J. Stat. Phys. **148**, 981 (2012) [arXiv:1012.1234]
- T. Wirtz, T. Guhr: Phys. Rev. Lett. **111**, 094101 (2013) [arXiv:1306.4790]
- T. Wirtz, T. Guhr: J. Phys. A **47**, 075004 (2014) [arXiv:1310.2467]
- G. Akemann, T. Guhr, M. Kieburg, R. Wegner, T. Wirtz: accepted for publication in Phys. Rev. Lett. [arXiv:1409.0360]
- T. Wirtz, M. Kieburg, T. Guhr: [arXiv:1410.4719]
- D. Waltner, T. Wirtz, T. Guhr: [arXiv:1412.3092]