Painlevé equations, Cherednik algebras and q-Askey scheme

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Painlevé equations are non linear second order ODE

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \qquad z \in \overline{\mathbb{C}},$$

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1. Painlevé–Kowalevski property: $w(z; c_1, c_2)$ have no *critical points* that depend on c_1, c_2.
2. For generic c_1, c_2, $w(z; c_1, c_2)$ are new functions, Painlevé Transcendents.

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z, \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\gamma w}{z} + \\ + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\delta w(w+1)}{w-1}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right) w_z + \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

Painlevé transcendents in Random Matrix Theory:

Example: (S_N, P_N) group of permutations π of $1, 2, \ldots, N$ with uniform probability distribution $P_N(\pi) = \frac{1}{N!}$.

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$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{l_N(\pi) - 2\sqrt{N}}{N^{1/6}} \leq t\right) = \exp\left\{-\int_t^{\infty} (t-z) w^2(z)\, \mathrm{d}z\right\}$$

$w(z)$ satisfies PII. (Baik, Deift, Johansson 1999).

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Only special solutions appear in this way

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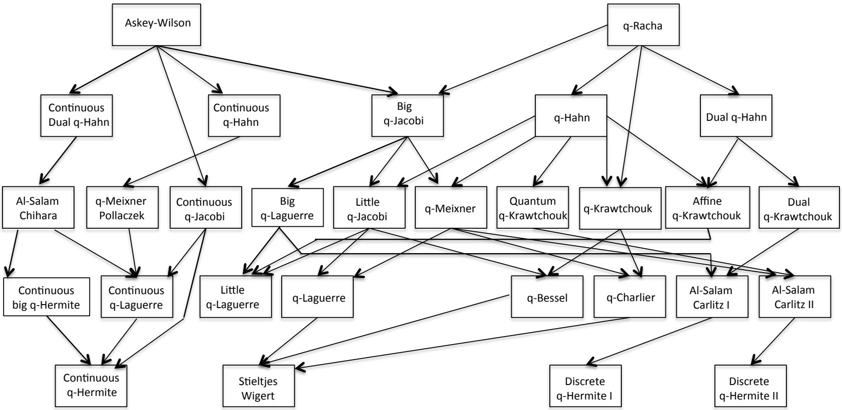
- To each Painlevé equation we associate a cubic surface called *monodromy manifold*.
- The monodromy manifold admits a Poisson bracket and can be quantised.
- These quantum algebras admit representations on the space of Laurent polynomials.
- The eigenspaces in this representation are spanned by q-orthogonal polynomials (q-Askey scheme).

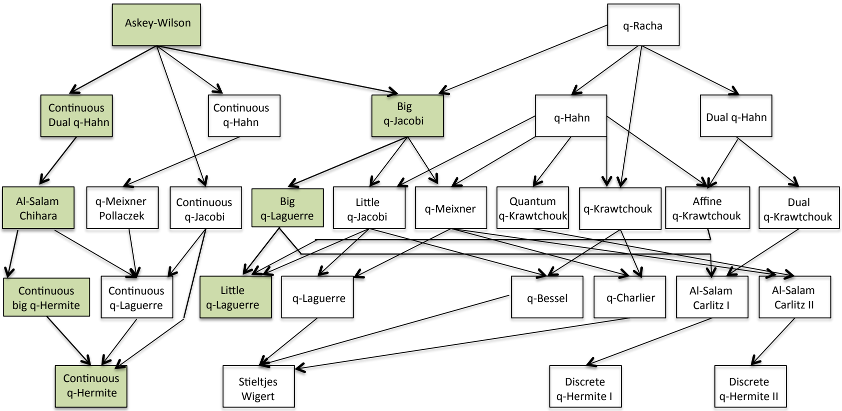
q-Askey scheme

Koekoek, Lesky, Swarttouw 2010

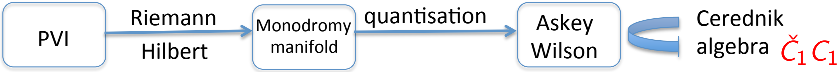


Painlevé equations and q-Askey polynomials

M.M. arXiv:1307.6140



Other results



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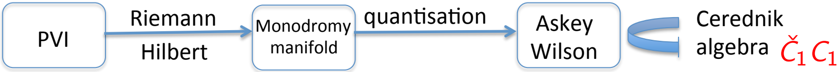

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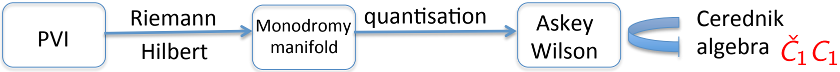

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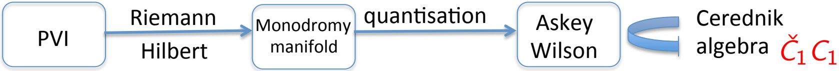

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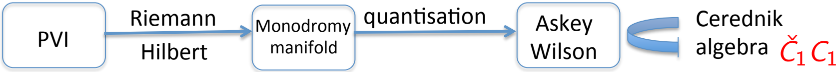

- Confluence Painlevé equations,
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- Confluence Askey Wilson algebra,
- Confluence Cherednik algebra $\Rightarrow$ seven new algebras. M.M. arXiv:1307.6140

Askey Wilson polynomials

q-Pochhammer symbol: $(a;q)_k := \Pi_{j=0}^{k-1}(1 - a\, q^j)$

$(a_1, \ldots, a_r; q)_k := (a_1; q)_k (a_2; q)_k \cdots (a_r; q)_k$

q-hypergeometric function:

$${}_r\varphi_{r-1}\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{array} ; q, x \right) := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, \ldots, b_{r-1}; q)_k} x^k.$$

Askey Wilson polynomials

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$(a_1,\dots,a_r;q)_k := (a_1;q)_k(a_2;q)_k\cdots(a_r;q)_k$

q-hypergeometric function:

$$ {}_r\varphi_{r-1}\left(\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,\dots,b_{r-1}\end{matrix};q,x\right) := \sum_{k=0}^{\infty}\frac{(a_1,a_2,\dots,a_r;q)_k}{(b_1,\dots,b_{r-1};q)_k}x^k. $$

Askey-Wilson polynomials:

$$ \begin{aligned} p_n(z;a,b,c,d\mid q) &:= {}_4\varphi_3\left(\begin{matrix} q^{-n},q^{n-1}abcd,az,az^{-1} \\ ab,ac,ad\end{matrix};q,q\right) \\ &= \sum_{k=0}^{n+1}\frac{(q^{-n},q^{n-1}abcd,az,az^{-1};q)_k}{(ab,ac,ad;q)_k}q^k. \end{aligned} $$

Facts about Askey Wilson polynomials

- They are special cases of Macdonald polynomials.
- Their algebraic structure is controlled by the Cherednik algebra of type $\check{C}_1 C_1$.
- They contain all other basic hypergeometric polinomyals as degenerations.

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- They are special cases of Macdonald polynomials.
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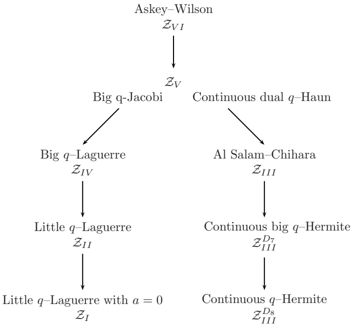
- They contain all other basic hypergeometric polinomyals as degenerations.

Open question: what algebras control all other elements of the q-Askey scheme?

Quantum algebra results

- For each Painlevé equation, a new quantum algebra and its representation theory is introduced.
- These algebras control some elements of the q-Askey scheme.



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Cherednik algebra of type $\check{C}_1 C_1$ Cherednik '92, Sahi '99

Algebra generated by $V_0, V_1, \check{V}_0, \check{V}_1$:

$$
\begin{aligned}
(V_0 - k_0)(V_0 + k_0^{-1}) &= 0\\
(V_1 - k_1)(V_1 + k_1^{-1}) &= 0\\
(\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) &= 0\\
(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) &= 0\\
\check{V}_1 V_1 V_0 \check{V}_0 &= q^{-1/2},
\end{aligned}
$$

k_0, k_1, u_0, u_1 scalars.

Representation theory:

Cherednik algebra on Laurent polynomials

$$V_1[f] = \frac{z\left[u_1 z^2 - \frac{1}{u_1} + \left(k_1 - \frac{1}{k_1}\right) z\right]}{z^2-1} f\left(\frac{1}{z}\right) - \frac{z\left(u_1 - \frac{1}{u_1}\right) + k_1 - \frac{1}{k_1}}{z^2-1} f(z)$$

$$V_0[f] = \frac{q k_0 - \frac{z^2}{k_0} + \sqrt{q}\left(u_0 - \frac{1}{u_0}\right) z}{z^2-1} f\left(\frac{q}{z}\right) - \frac{z\left[\left(k_0 - \frac{1}{k_0}\right) + \sqrt{q}\left(u_0 - \frac{1}{u_0}\right)\right]}{z^2-1} f(z)$$

The operator $\frac{1+\check{V}_1}{1+u_1^2}$ acts as symmetrizer $\Rightarrow$ representation of the spherical sub–algebra on symmetric Laurent polynomials.

Spherical sub-algebra:

generated by: $e = \frac{1+\check{V}_1}{1+u_1^2}$,

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$X_2 = e\left(\check{V}_1 V_0 + (\check{V}_1 V_0)^{-1}\right)$,

$X_3 = e\left(q^{1/2} V_1 V_0 + q^{-1/2} (V_1 V_0)^{-1}\right)$

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$$q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = (q^{-1}-q)X_3 + (q^{-1/2}-q^{1/2})\omega_3 e$$

$$q^{-1/2}X_2X_3 - q^{1/2}X_3X_2 = (q^{-1}-q)X_1 + (q^{-1/2}-q^{1/2})\omega_1 e$$

$$q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 = (q^{-1}-q)X_2 + (q^{-1/2}-q^{1/2})\omega_2 e$$

$$q^{\frac{1}{2}}X_2X_1X_3 - qX_2^2 - q^{-1}X_1^2 - qX_3^2 + q^{\frac{1}{2}}\omega_2 X_2 + q^{-\frac{1}{2}}\omega_1 X_1 + q^{\frac{1}{2}}\omega_3 X_3 = \omega_4 e.$$

Zhedanov '91, Oblomkov '04

Spherical sub–algebra and Askey Wilson polynomials

$$X_1\left(p_n(z,a,b,c,d|q)\right)=(z+\tfrac{1}{z})p_n(z,a,b,c,d|q),$$

$$X_2\left(p_n(z,a,b,c,d|q)\right)=\left(q^{-n}+abcdq^{n-1}\right)\left(p_n(z,a,b,c,d|q)\right).$$

Koornwinder '07

Recap:

- Cherednik algebra of type $\check{C}_1 C_1$:

$$\begin{array}{c}(V_0 - k_0)(V_0 + k_0^{-1}) = 0\\ (V_1 - k_1)(V_1 + k_1^{-1}) = 0\\ (\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) = 0\\ (\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0\\ \check{V}_1 V_1 V_0 \check{V}_0 = q^{-1/2}.\end{array}$$

Represented on Laurent polynomials Non-symmetric Askey-Wilson

- Spherical subalgebra (Zhedanov):

$$\begin{array}{c}q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 = (q^{-1}-q)X_3 + (q^{-1/2}-q^{1/2})\omega_3 e,\\ q^{-1/2}X_2X_3 - q^{1/2}X_1X_3 = (q^{-1}-q)X_1 + (q^{-1/2}-q^{1/2})\omega_1 e,\\ q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 = (q^{-1}-q)X_2 + (q^{-1/2}-q^{1/2})\omega_2 e.\end{array}$$

Representation spanned by Askey-Wilson polynomials

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa 1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]$$

$A = A(\lambda; z, w, w_z),\ B = B(\lambda; z, w, w_z) \in \mathfrak{sl}_2.$

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$$\frac{\mathrm{d}B}{\mathrm{d}\lambda}-\frac{\mathrm{d}A}{\mathrm{d}z}=[A,B]$$

$A=A(\lambda;z,w,w_z),\ B=B(\lambda;z,w,w_z)\in\mathfrak{sl}_2.$
This means that **the monodromy data** of the linear system

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda}=A(\lambda;z,w,w_z)Y$$

are **locally constant along solutions of the Painlevé equation**.
The monodromy data play the role of initial conditions

PVI monodromy data

$M_1, M_2, M_3, M_\infty \in SL(2,\mathbb{C})$ s.t.

$$\mathrm{eigenv}(M_i) = \exp(\pm i\, p_i), \quad i = 1, 2, 3, \infty,$$

$$M_\infty M_1 M_2 M_3 = 1.$$

Jimbo, Miwa '81

Cherednik algebra as quantisation of the monodromy group

$$
\begin{aligned}
(M_3 - e^{i\,p_3})(M_3 - e^{-i\,p_3}) &= 0,\\
(M_2 - e^{i\,p_2})(M_2 - e^{-i\,p_2}) &= 0,\\
(M_1 - e^{i\,p_1})(M_1 - e^{-i\,p_1}) &= 0,\\
(M_\infty - e^{i\,p_\infty})(M_\infty - e^{-i\,p_\infty}) &= 0\\
M_\infty M_3 M_2 M_1 &= 1.
\end{aligned}
$$

Cherednik algebra as quantisation of the monodromy group

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(M_3 - e^{i p_3})(M_3 - e^{-i p_3}) &= 0, & (V_0 - k_0)(V_0 + k_0^{-1}) &= 0,\\
(M_2 - e^{i p_2})(M_2 - e^{-i p_2}) &= 0, & (V_1 - k_1)(V_1 + k_1^{-1}) &= 0,\\
(M_1 - e^{i p_1})(M_1 - e^{-i p_1}) &= 0, & (\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) &= 0,\\
(M_\infty - e^{i p_\infty})(M_\infty - e^{-i p_\infty}) &= 0 & (\check{V}_0 - u_0)(\check{V}_0 + u_0^{-1}) &= 0,\\
M_\infty M_3 M_2 M_1 &= 1. & \check{V}_1 V_1 V_0 \check{V}_0 &= q^{-1/2}.
\end{aligned}
$$

$$
M_1 \to i\check{V}_1, \quad M_2 \to iV_1, \quad M_3 \to iV_0, \quad M_\infty \to i\check{V}_0.
$$

$$
u_1 = -ie^{-\frac{p_1}{2}}, \quad k_0 = -ie^{-\frac{p_3}{2}}, \quad k_1 = -ie^{-\frac{p_2}{2}}, \quad u_0 = -ie^{-S_1 - S_2 - S_3}.
$$

Monodromy manifold:

$$x_1 = \mathrm{Tr}(M_2 M_3), \quad x_2 = \mathrm{Tr}(M_2 M_3), \quad x_3 = \mathrm{Tr}(M_3 M_1),$$

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

Poisson bracket:

$$\{x_1, x_2\} = x_1 x_2 + 2x_3 + \omega_3, \qquad \{x_2, x_3\} = x_2 x_3 + 2x_1 + \omega_1,$$
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$$q^{-1/2} X_3 X_1 - q^{1/2} X_1 X_3 = (q^{-1} - q) X_2 + (q^{-1/2} - q^{1/2}) \omega_2 e$$

$$q^{\frac{1}{2}} X_2 X_1 X_3 - q X_2^2 - q^{-1} X_1^2 - q X_3^2 + q^{\frac{1}{2}} \omega_2 X_2 + q^{-\frac{1}{2}} \omega_1 X_1 + q^{\frac{1}{2}} \omega_3 X_3 = \omega_4 e.$$

Monodromy manifolds for the Painlevé equations

$$PVI \quad x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

$$PV \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

$$PIV \quad x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$$

$$PIII \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$$

$$PII \quad x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$$

$$PI \quad x_1x_2x_3 + x_1 + x_2 + 1 = 0$$

Saito and van der Put

Poisson structure on the monodromy manifolds

Cubic surface $M_\varphi := \mathbb{C}[x_1, x_2, x_3]/_{\langle \varphi(x_1,x_2,x_3)=0\rangle}$
Poisson bracket:

$$\{x_1, x_2\} = \frac{\partial\varphi}{\partial x_3}, \qquad \{x_2, x_3\} = \frac{\partial\varphi}{\partial x_1}, \qquad \{x_3, x_1\} = \frac{\partial\varphi}{\partial x_2}.$$

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Example PV: $\varphi := x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 - \omega_4,$

$$\{x_1, x_2\} = x_1 x_2 + \omega_3, \qquad \{x_2, x_3\} = x_2 x_3 + 2x_1 + \omega_1,$$
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$$\{x_1, x_2\} = \frac{\partial \varphi}{\partial x_3}, \qquad \{x_2, x_3\} = \frac{\partial \varphi}{\partial x_1}, \qquad \{x_3, x_1\} = \frac{\partial \varphi}{\partial x_2}.$$

Example PV: $\varphi := x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 - \omega_4,$

$$\{x_1, x_2\} = x_1 x_2 + \omega_3, \qquad \{x_2, x_3\} = x_2 x_3 + 2x_1 + \omega_1,$$
$$\{x_3, x_1\} = x_1 x_3 + 2x_2 + \omega_2.$$

Quantisation (Teichmüller theory and cluster algebra)

(L. Chekhov and M.M. J.Phys A 2010).

For $d \to 0$ $(k_0 \to 0,\ u_0 \to \epsilon u_0)$:

- First confluent Cherednik algebra:

$$
\begin{gathered}
V_0^2 + V_0 = 0, \\
(V_1 - k_1)(V_1 + k_1^{-1}) = 0 \\
\check{V}_0{}^2 + u_0^{-1}\check{V}_0 = 0, \\
(\check{V}_1 - u_1)(\check{V}_1 + u_1^{-1}) = 0 \\
q^{1/2}\check{V}_1 V_1 V_0 = \check{V}_0 + u_0^{-1}, \\
q^{1/2}\check{V}_0 \check{V}_1 V_1 = V_0 + 1.
\end{gathered}
$$

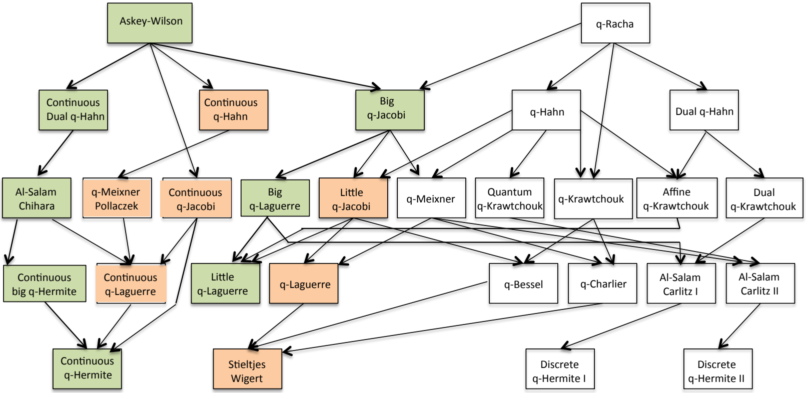
Represented on Laurent polynomials Non-symmetric Continuous dual q-Hahn polynomials (M.M. arXiv:1409.4287)

- Spherical subalgebra (confluent Zhedanov):

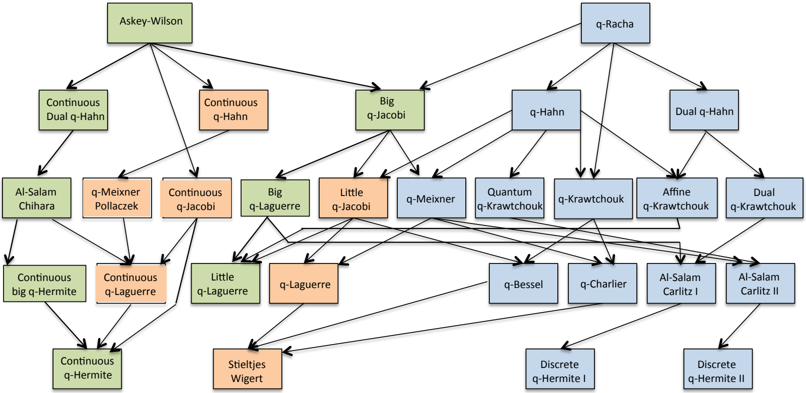
$$
\begin{aligned}
q^{-1/2}X_1X_2 - q^{1/2}X_2X_1 &= (q^{-1/2} - q^{1/2})\omega_3 e, \\
q^{-1/2}X_2X_3 - q^{1/2}X_1X_3 &= (q^{-1} - q)X_1 + (q^{-1/2} - q^{1/2})\omega_1 e, \\
q^{-1/2}X_3X_1 - q^{1/2}X_1X_3 &= (q^{-1} - q)X_2 + (q^{-1/2} - q^{1/2})\omega_2 e.
\end{aligned}
$$

Representation spanned by Continouos dual q-Hahn polynomials

Outlook


Askey-Wilson
q-Racha
Continuous Dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
Al-Salam Chihara
q-Meixner Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam Carlitz I
Al-Salam Carlitz II
Continuous q-Hermite
Stieltjes Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Outlook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- Additive discrete Painlevé equations$\rightarrow$ cover the whole q-Askey scheme.

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- q-difference and elliptic Painlevé equations may correspond to elliptic hypergeometric bi-orthogonal polynomials.
- Multivariable high order analogues of the Painlevé equations $\rightarrow$ *confluence scheme for Macdonald polynomials*