On the Optimum Asymptotic Multiuser Efficiency of Randomly Spread CDMA

Ralf R. Müller

Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
Lehrstuhl für Digitale Übertragung

13 December 2014

Linear Vector Channel

Consider a linear vector channel

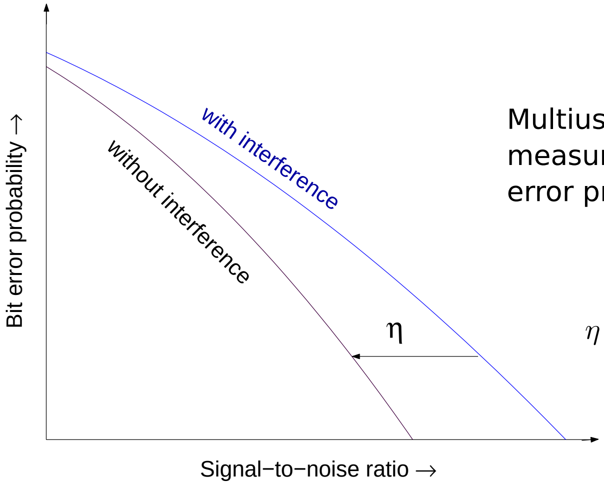
$$\boldsymbol{y} = \frac{1}{\sqrt{N}} \boldsymbol{H} \boldsymbol{b} + \boldsymbol{n}$$

with

$$\boldsymbol{b} \in \{\pm 1\}^K$$
$$\boldsymbol{n} \sim N(0, \sigma^2 \mathbf{I})$$

and $\boldsymbol{H}$ being an $N \times K$ random matrix with iid entries of unit variance and vanishing odd moments.

Multiuser Efficiency



Multiuser efficiency measures bitwise error probability.

$$\eta \in [0;1]$$

More on Multiuser Efficiency

For unit variance Gaussian data, the multiuser efficiency is trivially related to the Stieltjes transform $G(s)$ in the large matrix limit:

$$\eta(\sigma^2) = \lim_{s \to \sigma^2} sG(-s).$$

More on Multiuser Efficiency

For unit variance Gaussian data, the multiuser efficiency is trivially related to the Stieltjes transform $G(s)$ in the large matrix limit:

$$\eta(\sigma^2) = \lim_{s \to \sigma^2} sG(-s).$$

In general, the multiuser efficiency depends on both the eigenvalue spectrum of the random matrix $\boldsymbol{H}\boldsymbol{H}^\dagger$ and the distribution of the data.

Large System Result

Let $N, K \to \infty$, but $\beta = K/N$ fixed. Then, the multiuser efficiency with optimal detection is a solution to the fixed point equation

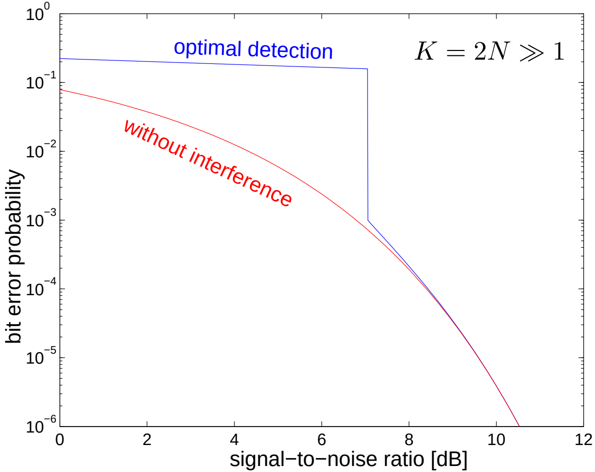
$$\frac{1}{\eta} = 1 + \frac{\beta}{\sigma^2}\left[1 - \sqrt{\frac{\eta}{2\pi\sigma^2}} \int_{\mathbb{R}} \tanh\left(\frac{\eta}{\sigma^2}x\right) \exp\left(-\frac{\eta(x-1)^2}{2\sigma^2}\right) \mathrm{d}x\right].$$

In case the fixed point equation has multiple solutions, the correct one is that solution for which the term

$$\frac{\eta}{\sigma^2} + \frac{\eta - \log\eta}{2\beta} - \sqrt{\frac{\eta}{2\pi\sigma^2}} \int_{\mathbb{R}} \log\left[\cosh\left(\frac{\eta}{\sigma^2}x\right)\right] \exp\left(-\frac{\eta(x-1)^2}{2\sigma^2}\right) \mathrm{d}x$$

is smallest (Tanaka 2002).

A Phase Transition in CDMA



For large SNR, i.e. $\sigma^2 \to 0$, we have $\eta \to 1$.

Asymptotic Multiuser Efficiency

The asymptotic multiuser efficiency is given by

$$\eta^* = \lim_{\sigma^2 \to 0} \eta = \min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \frac{\boldsymbol{x}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{x}}{N} \qquad \text{(Verdú 1986).}$$

Asymptotic Multiuser Efficiency

The asymptotic multiuser efficiency is given by

$$\eta^* = \lim_{\sigma^2 \to 0} \eta = \min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\mathbf{0}\}} \frac{\boldsymbol{x}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{x}}{N}$$ (Verdú 1986).

Theorem:
The optimum asymptotic multiuser efficiency converges to 1 almost surely as $N, K \to \infty$, but

$$\beta = \frac{K}{N} < \infty$$

fixed (Tse & Verdú 2000).

Asymptotic Multiuser Efficiency

The asymptotic multiuser efficiency is given by

$$\eta^* = \lim_{\sigma^2 \to 0} \eta = \min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \frac{\boldsymbol{x}^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{x}}{N} \qquad \text{(Verdú 1986).}$$

Theorem:
The optimum asymptotic multiuser efficiency converges to 1 almost surely as $N, K \to \infty$, but

$$\beta = \frac{K}{N} < \infty$$

fixed (Tse & Verdú 2000).

What about $\beta \to \infty$?

Uniquely Detecting Matrices

Definition

An $N \times K$ matrix $\boldsymbol{H}$ is called uniquely detecting for an alphabet $\mathcal{B}$, iff

$$\boldsymbol{H}\boldsymbol{b}_1 = \boldsymbol{H}\boldsymbol{b}_2 \Rightarrow \boldsymbol{b}_1 = \boldsymbol{b}_2$$

for all $\boldsymbol{b}_1 \in \mathcal{B}^K \ni \boldsymbol{b}_2$.

Logarithmically Infinite Overload

Theorem:
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed.
Then, there does not exist any $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$ such that it is uniquely detecting for binary data, iff

$$\zeta > \log_3 4 \approx 0.79$$

(Lindström 1966).

Logarithmically Infinite Overload

Theorem:
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed.
Then, there does not exist any $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$ such that the optimum asymptotic multiuser efficiency is greater than zero, if

$$\zeta > \log_3 4 \approx 0.79$$

(Lindström 1966).

An Existence Result

Theorem:
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$.
Then, the probability that $\boldsymbol{H}$ is uniquely detecting for binary data is 1, if

$$\zeta < \frac{1}{2}$$

(Erdős & Rényi 1963).

An Existence Result

Theorem:
Let $N, K \to \infty$, but

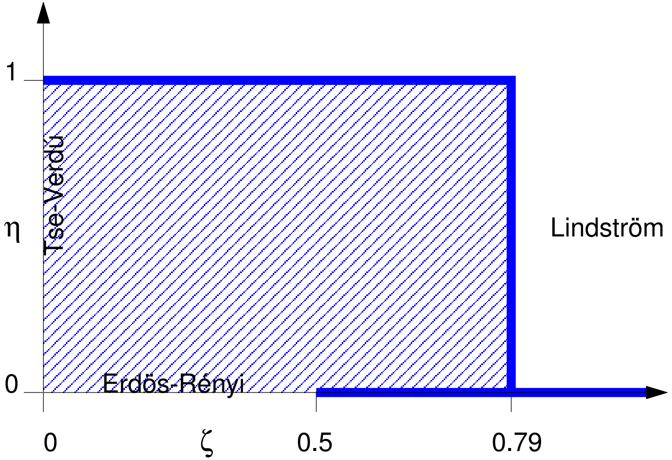
$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$.
Then, the optimum asymptotic multiuser efficiency converges to a nonzero value almost surely, if

$$\zeta < \frac{1}{2}$$

(Erdős & Rényi 1963).

Summary of Known Results



A New Binary Existence Result

Theorem:
Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$.
Then, the optimum asymptotic multiuser efficiency converges to 1 almost surely if

$$\zeta < \frac{3}{8}$$

(Sedaghat, RRM, Marvasti 2013).

Generalization of the Binary Existence Result

Theorem:
Let $N, K \to \infty$, but

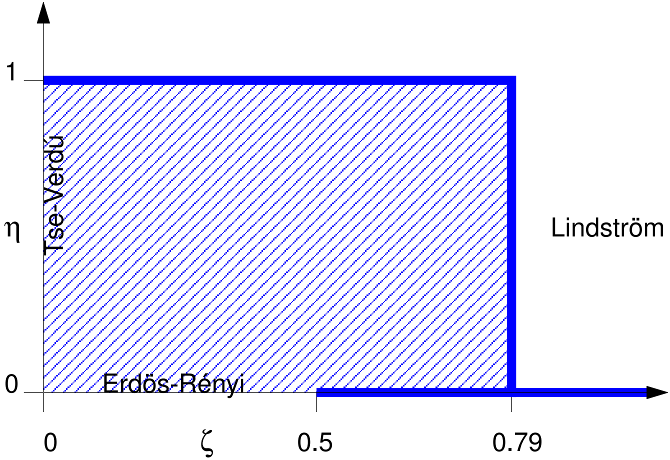
$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $\boldsymbol{H} \in \{\pm 1\}^{N \times K}$.
Then, the optimum asymptotic multiuser efficiency converges to value which is greater or equal to

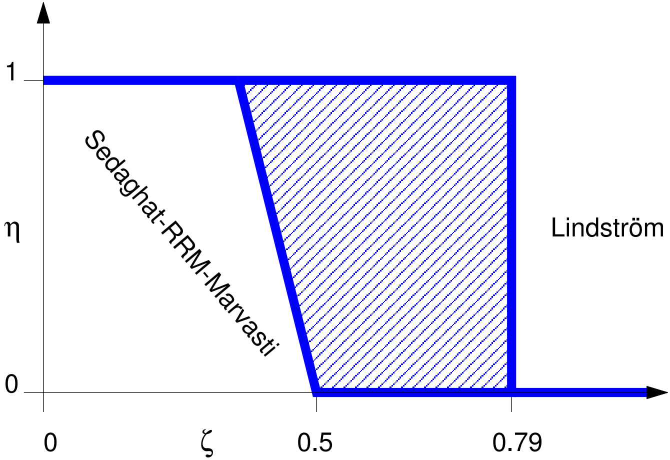
$$\min\{1, 4 - 8\zeta\}$$

(Sedaghat, RRM, Marvasti 2013).

Summary of Results



Summary of Results



Minimum Distance

- By Erdős and Rényi, the minimum distance is non-zero almost surely as long as $\zeta < \frac{1}{2}$ (despite logarithmically infinite overload).
- By the new result, the minimum distance is almost surely at least as big as in an hypercube downscaled by a factor of β as long as $\zeta < \frac{3}{8}$ (despite logarithmically infinite overload).

Minimum Distance

- By Erdős and Rényi, the minimum distance is non-zero almost surely as long as $\zeta < \frac{1}{2}$ (despite logarithmically infinite overload).
- By the new result, the minimum distance is almost surely at least as big as in an hypercube downscaled by a factor of β as long as $\zeta < \frac{3}{8}$ (despite logarithmically infinite overload).

The projection reduces the noise variance from K to N.

Minimum Distance

- By Erdős and Rényi, the minimum distance is non-zero almost surely as long as $\zeta < \frac{1}{2}$ (despite logarithmically infinite overload).
- By the new result, the minimum distance is almost surely at least as big as in an hypercube downscaled by a factor of β as long as $\zeta < \frac{3}{8}$ (despite logarithmically infinite overload).

The projection reduces the noise variance from K to N.

In a hypercube the kissing number (number of nearest neighbors is linear in K.
After the projection it could eventually be exponential in K degrading the asymptotic multiuser efficiency.

A Gaussian Existence Result

Theorem:

Let $N, K \to \infty$, but

$$\zeta = \frac{K}{N \log_3 K}$$

fixed. Let $\boldsymbol{H}$ have Gaussian entries.
Then, the optimum asymptotic multiuser efficiency converges to 1 almost surely if

$$\zeta < \frac{1}{2}$$

(Sedaghat, RRM, Marvasti 2013).

Sketch of Proof

We define $\boldsymbol{R} = \boldsymbol{H}^\dagger \boldsymbol{H}/N$ and start from Verdú 1986

$$\Pr(\eta^* < 1) = \Pr\left(\min_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right).$$

The union bound gives

$$\Pr(\eta^* < 1) \leq \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} \Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right)$$

We split the summation into shells of equal Hamming weight (zero norm):

$$\sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K \setminus \{\boldsymbol{0}\}} (\cdot) = \sum_{w=1}^{K} \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K : ||\boldsymbol{x}||_0 = w} (\cdot)$$

Lemma on Odd Weights

Lemma 1:
For any vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ with odd weight,

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} \geq 1.$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{K} H_{ij} x_j \right)^2}_{\geq 1}$$

The squared bracket is a positive integer, since the sum contains an odd number of binary antipodal $(+1, -1)$ terms.

Lemma on Odd Weights

Lemma 1:

For any vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ with odd weight,

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} \geq 1.$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{K} H_{ij} x_j \right)^2}_{\geq 1}$$

The squared bracket is a positive integer, since the sum contains an odd number of binary antipodal $(+1, -1)$ terms.

W. l. o. g., we can consider vectors with even weights.

Lemma on Even Weights

Lemma 2:

For any vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ with even weight and

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1,$$

we have

$$||\boldsymbol{H}\boldsymbol{x}||_0 < \frac{N}{4}$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{K} H_{ij} x_j \right)^2}_{\notin \{1,2,3\}}$$

Lemma on Even Weights

Lemma 2:

For any vector $\boldsymbol{x} \in \{\pm 1, 0\}^K$ with even weight and

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1,$$

we have

$$||\boldsymbol{H}\boldsymbol{x}||_0 < \frac{N}{4}$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{K} H_{ij} x_j \right)^2}_{\notin \{1,2,3\}}$$

If the empirical average is less then 1, the number of terms which are 4 or more must be less than $N/4$.

Subshell Uniformity

Lemma 3:

For any vectors $\boldsymbol{x}, \boldsymbol{y} \in \{\pm 1, 0\}^K$ with

$$||\boldsymbol{x}||_0 = ||\boldsymbol{y}||_0,$$

we have

$$\Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right) = \Pr\left(\boldsymbol{y}^\dagger \boldsymbol{R} \boldsymbol{y} < 1\right)$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \Bigg(\underbrace{\sum_{j=1}^{K} H_{ij} x_j}_{=: u_i} \Bigg)^2$$

Subshell Uniformity

Lemma 3:

For any vectors $\boldsymbol{x}, \boldsymbol{y} \in \{\pm 1, 0\}^K$ with

$$||\boldsymbol{x}||_0 = ||\boldsymbol{y}||_0,$$

we have

$$\Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right) = \Pr\left(\boldsymbol{y}^\dagger \boldsymbol{R} \boldsymbol{y} < 1\right)$$

Proof:

$$\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} = \frac{1}{N} \sum_{i=1}^{N} \Bigg(\underbrace{\sum_{j=1}^{K} H_{ij} x_j}_{=: u_i} \Bigg)^2$$

Characteristic function:

$$\phi_{u_i}(t) = \prod_{j=1}^{K} \phi_{H_{ij} x_j}(t) = \left[\phi_{H_{11}}(t)\right]^{||\boldsymbol{x}||_0}$$

Back to the Main Proof

After splitting into subshells, we had:

$$\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K : ||\boldsymbol{x}||_0 = w} \Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right)$$

Back to the Main Proof

After splitting into subshells, we had:

$$\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K : ||\boldsymbol{x}||_0 = w} \Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right)$$

With the three lemmas, we get

$$\Pr(\eta^* < 1) \leq \sum_{w=1}^{K/2} \left(\begin{array}{c} K \\ 2w \end{array} \right) 2^{2w} \Pr\left(||\boldsymbol{H}\boldsymbol{x}||_0 \Big|_{||\boldsymbol{x}||_0 = 2w} < \frac{N}{4} \right)$$

Back to the Main Proof

After splitting into subshells, we had:

$$\Pr(\eta^* < 1) \leq \sum_{w=1}^{K} \sum_{\boldsymbol{x} \in \{\pm 1, 0\}^K : ||\boldsymbol{x}||_0 = w} \Pr\left(\boldsymbol{x}^\dagger \boldsymbol{R} \boldsymbol{x} < 1\right)$$

With the three lemmas, we get

$$\Pr(\eta^* < 1) \leq \sum_{w=1}^{K/2} \binom{K}{2w} 2^{2w} \Pr\left(||\boldsymbol{H}\boldsymbol{x}||_0 \Big|_{||\boldsymbol{x}||_0 = 2w} < \frac{N}{4} \right)$$

$$= \sum_{w=1}^{K/2} \binom{K}{2w} 2^{2w} \sum_{i=0}^{N/4-1} \binom{N}{i} p_w^{N-i} (1 - p_w)^i$$

with

$$p_w = \Pr\left(\sum_{j=1}^{2w} H_{1j} = 0 \right) = \binom{2w}{w} 2^{-2w}.$$

A few more bounds lead to the desired result.

References

Erdős & Rényi On two problems of information theory, *Magyar Tud. Akad. Mat. Kutató Int. Közl*, vol. 8, pp. 229–243, 1963.

Lindström On a combinatorial detection problem II, *Studia Scientiarum Mathematicarum Hungarica*, vol. 1, pp. 353–361, 1966.

Tse & Verdú Optimum asymptotic multiuser efficiency of randomly spread CDMA, *IEEE Trans. Inform. Theory*, vol. 46, no. 7, 2000.

Tanaka A statistical mechanics approach to large-system analysis of CDMA multiuser detectors, *IEEE Trans. Inform. Theory*, vol. 48, no. 11, 2002.

Sedaghat, RRM, Marvasti On Optimum Asymptotic Multiuser Efficiency of Randomly Spread CDMA, arXiv:1410.8366v1.

On the Optimum Asymptotic Multiuser Efficiency of Randomly Spread CDMA

Ralf R. Müller

Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU)
Lehrstuhl für Digitale Übertragung

13 December 2014