University of BRISTOL

GSE statistics without spin

joint work with

Chris Joyner and Martin Sieber

Sebastian Müller

Spectral statistics

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Random matrix conjecture

Spectra of **chaotic** systems have universal statistics in agreement with **random matrix theory**.

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Ensemble depends on symmetries.

In absence of other symmetries it depends only on the behaviour under time reversal.

Time reversal invariance

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conventional time-reversal invariance:

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in general:

$\hat{H}$ must commute with anti-unitary operator $\mathcal{T}$

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together with $\mathcal{T}^2|\psi\rangle = c|\psi\rangle$ this implies $\mathcal{T}^2 = \pm 1$

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$$H_{nm} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = a_0 1 + a_1 \underbrace{i\sigma_1}_{=I} + a_2 \underbrace{i\sigma_2}_{=J} + a_3 \underbrace{i\sigma_3}_{=K}$$

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- example: a quantum graph
- background: discrete geometrical symmetries

Quantum graphs

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- networks of vertices connected by bonds (with lengths)

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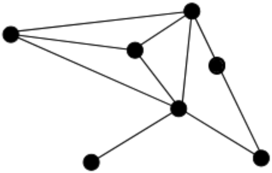

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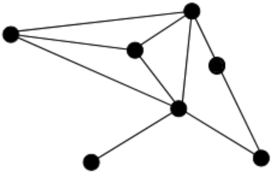

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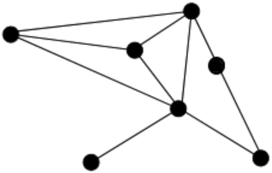

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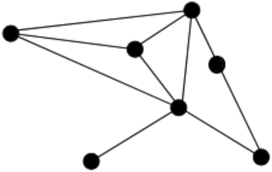

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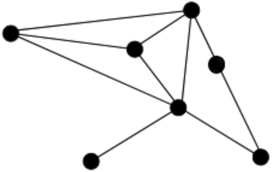

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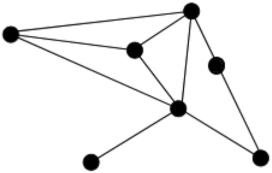

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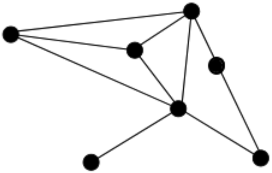

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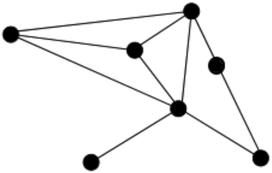

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- if Hamiltonian and vertex conditions symmetric w.r.t. complex conjugation: GOE

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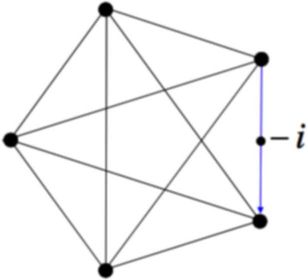
- here time-reversal invariance is broken by a complex phase factor: GUE



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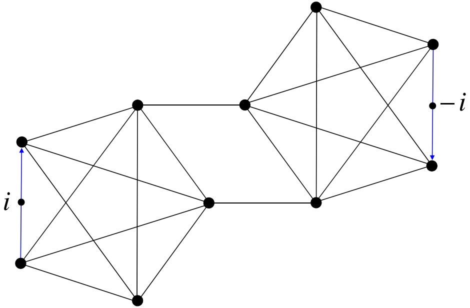

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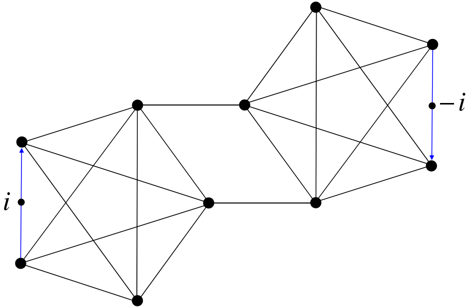

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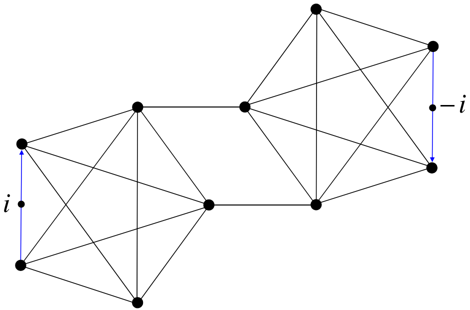

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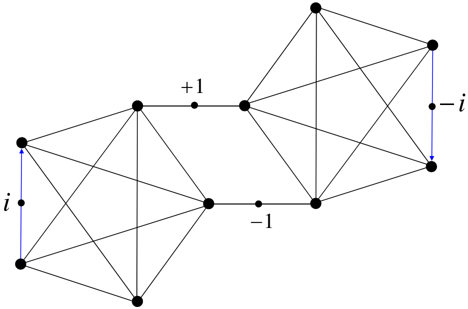

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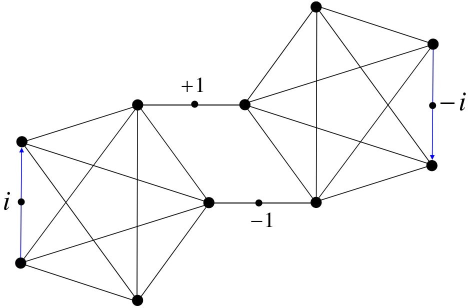

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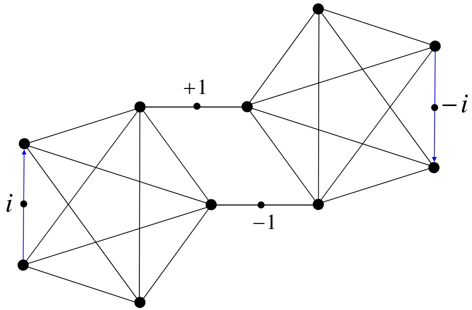

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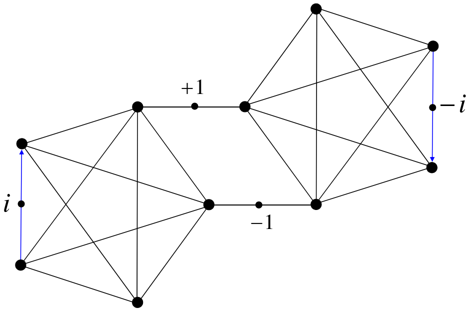

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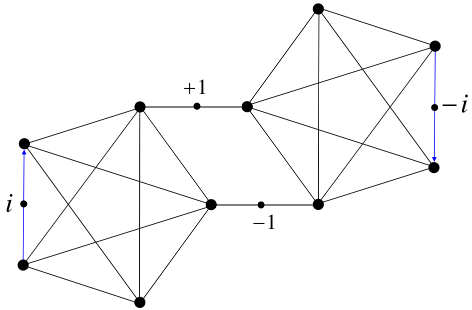

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General approach to symmetries

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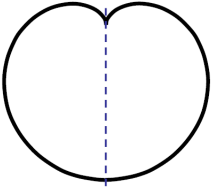

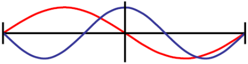

Spectral statistics in systems with (discrete) **geometric symmetries**?

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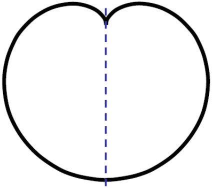

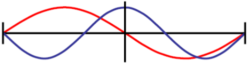

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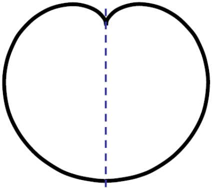

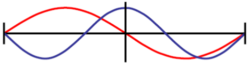

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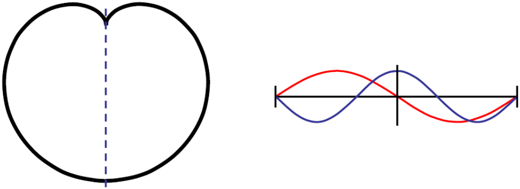

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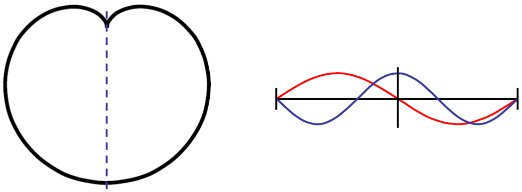

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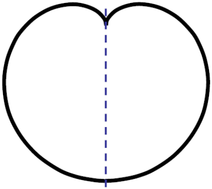

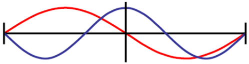

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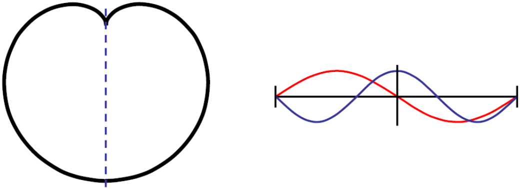

two subspectra:

- eigenfunctions even under reflection $\Rightarrow$ GOE
- eigenfunctions odd under reflection $\Rightarrow$ GOE
- subspectra uncorrelated

General discrete symmetries

- group of **classical** symmetry operations g

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- **quantum** symmetries

$$U(g)\psi(\mathbf{r}) = \psi(g^{-1}\mathbf{r})$$

commute with Hamiltonian,

they form a representation of the classical symmetry group, i.e.,

$$U(gg') = U(g)U(g')$$

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- blocks $M_\alpha(g)$ are (irreducible) matrix representations of the classical group, they satisfy

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Find a graph whose symmetry group has a pseudo-real representation.

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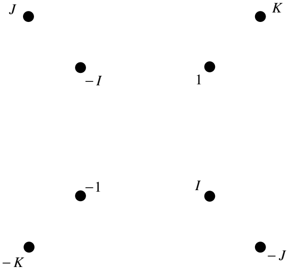

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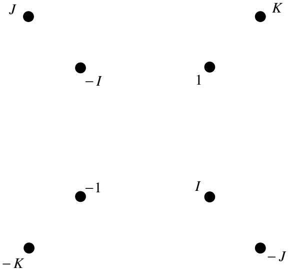

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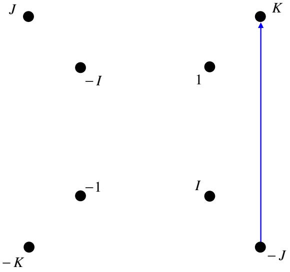

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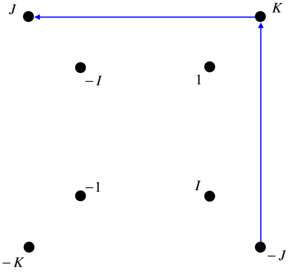

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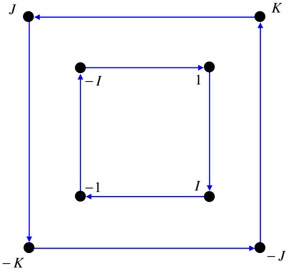

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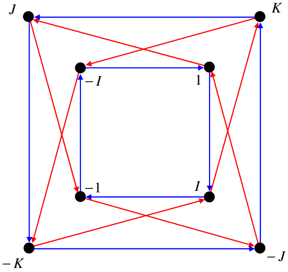

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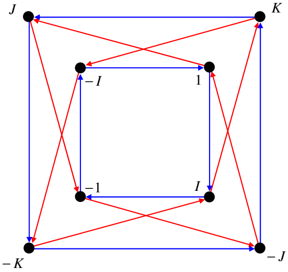

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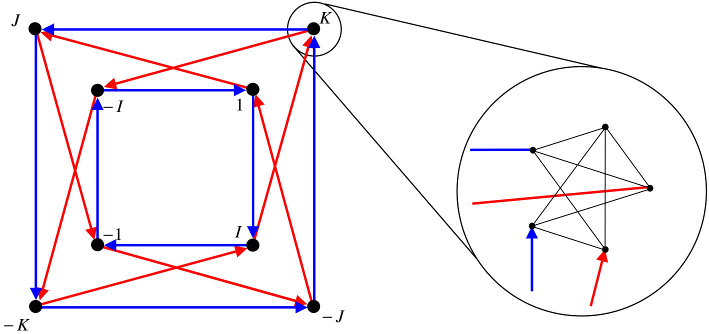

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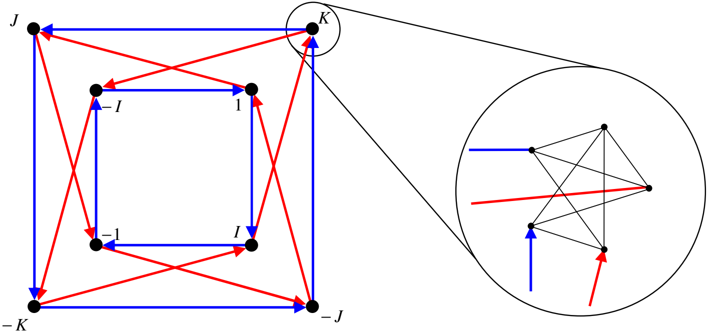

graph with GSE subspectrum

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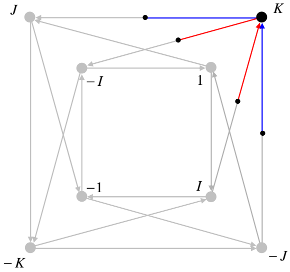

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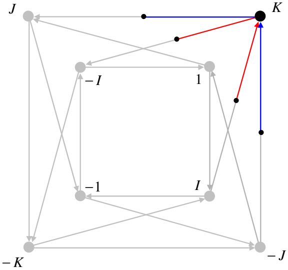

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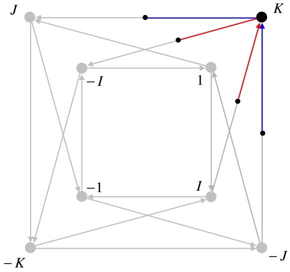

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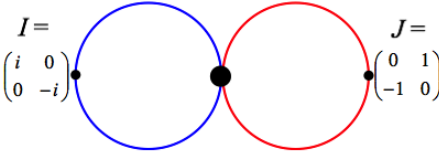

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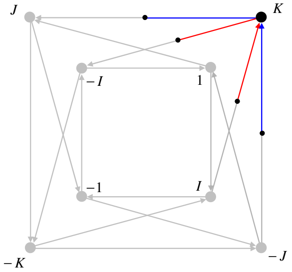

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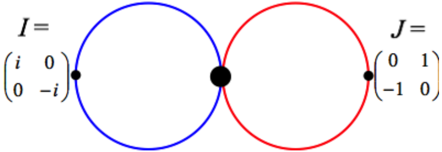

graph with pure GSE statistics

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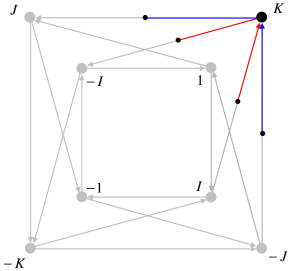

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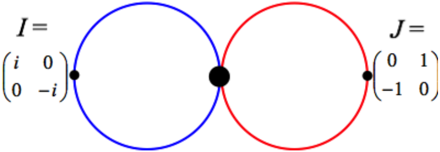

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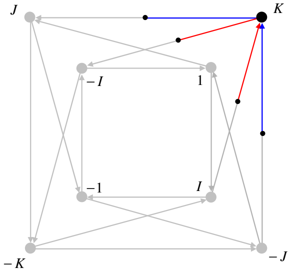

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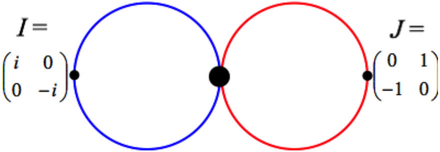

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... but boundary conditions mix pairs of degenerate eigenfunctions

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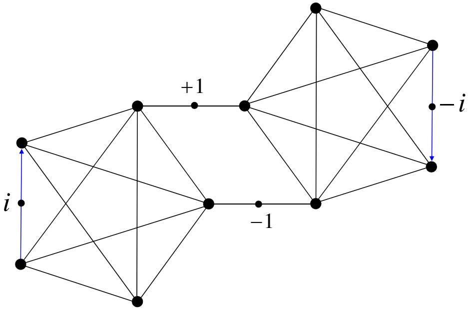

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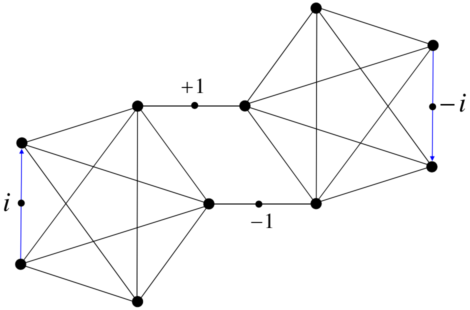

graph with a pure GSE spectrum and no resemblance of spin

Numerical Results

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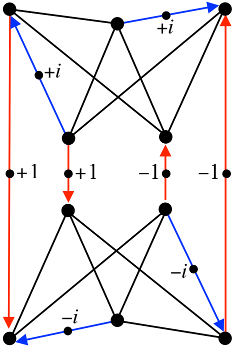

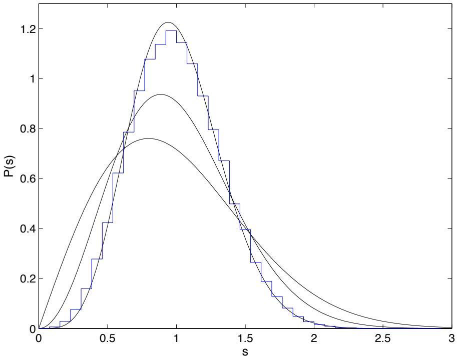

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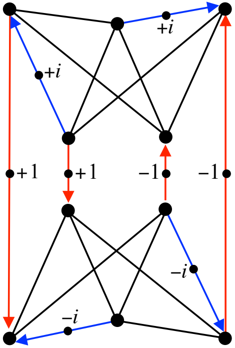

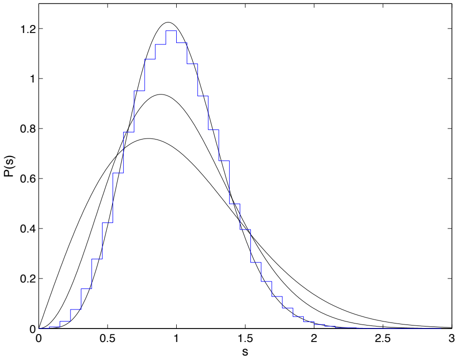

Agreement with GSE ☺

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- Generalisation to the ’tenfold way’?
- Experimental realisation?

Further example

Spectral statistics in systems with (discrete) **geometric symmetries**?

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Example: symmetry w.r.t. rotations by $\frac{2\pi}{3}$
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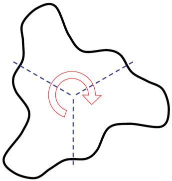

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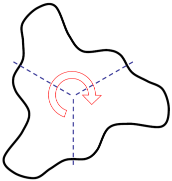

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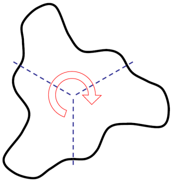

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Further example

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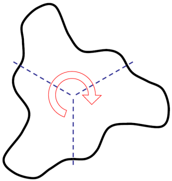

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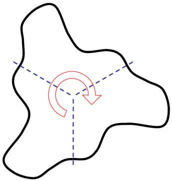

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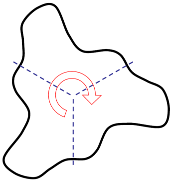

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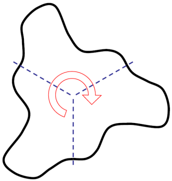

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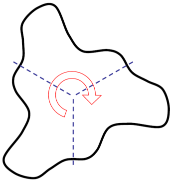

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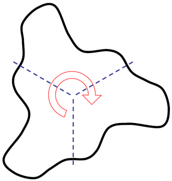

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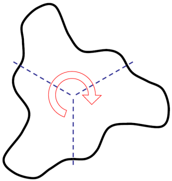

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