Semiclassical approach to quantum chaos as a matrix model

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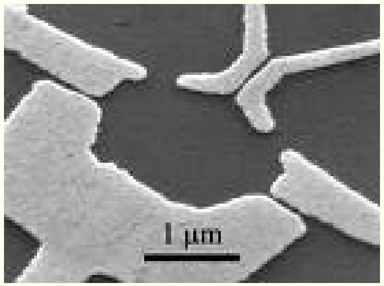
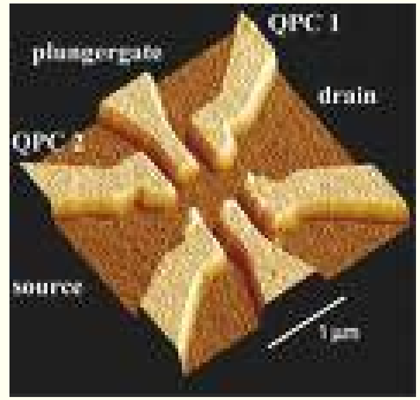
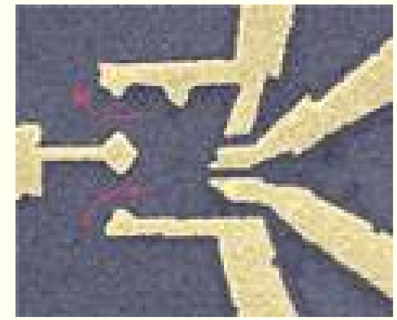
Transport through quantum dots







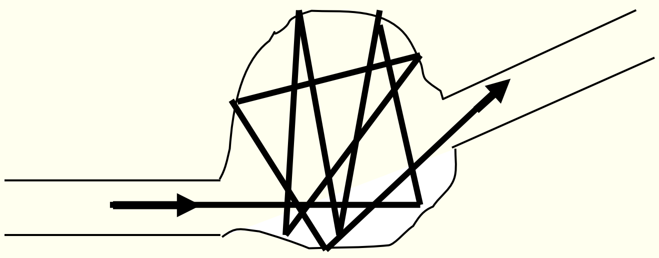
Mesoscopic systems: interference effects are important

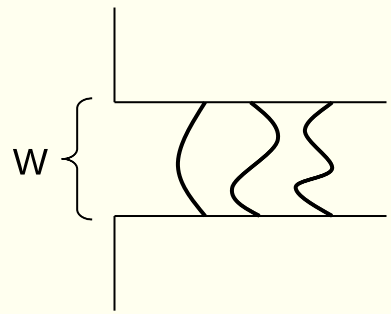
Chaotic dynamics: universality, well described by Random Matrix Theory (RMT)

Semiclassical description is effective

Central question: can semiclassics account for observed RMT universality?

Quantum dots and quantum chaos





Transverse modes: $E_n = \frac{\hbar^2 n^2 \pi^2}{2mW^2}$

Open channels: (propagating) $E_n < E_F$

Closed channels: (evanescent) $E_n > E_F$

M_1 Open incoming channels

M_2 Open outgoing channels

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

S matrix is unitary, $SS^\dagger = 1$ of dimension $M = M_1 + M_2$

(No time-reversal symmetry)

Random Matrix Universality

We assume $\tau_D \gg \tau_E$ τ_D : Dwell time τ_E : Ehrenfest time

In this regime RMT is effective:

S is a random element of the CUE= $\mathcal{U}(M)$

T is a random element of a Jacobi ensemble

Universality of chaotic systems

$$\langle |S_{11}|^2 |S_{12}|^2 \rangle = \frac{1}{M(M+1)} \qquad \langle |S_{11}|^2 |S_{23}|^2 \rangle = \frac{1}{(M^2-1)}$$

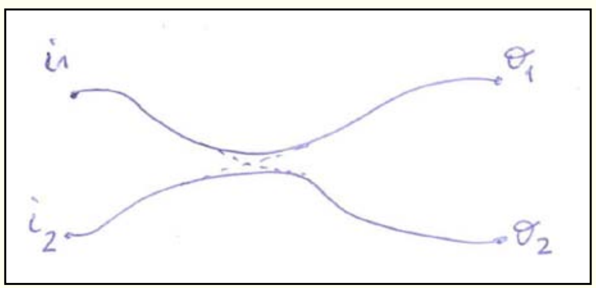
In general, $\left\langle \prod_{k=1}^{n} S_{a_k b_k} S^{\dagger}_{c_k d_k} \right\rangle$ (Weingarten functions)

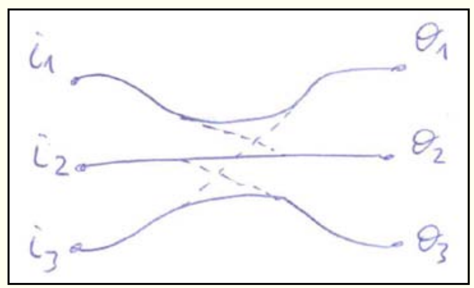
Semiclassical counting statistics

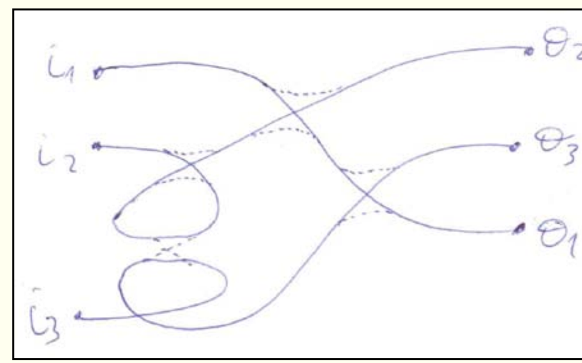
Semiclassical Approximation: $S_{oi} = \sum_{\gamma:i\to o} A_\gamma e^{i\mathcal{S}_\gamma/\hbar}$

$\left\langle \prod_{k=1}^{n} S_{a_k b_k} S^\dagger_{c_k d_k} \right\rangle_E$ will require 2n trajectories

Energy average in the semiclassical limit: Action correlations







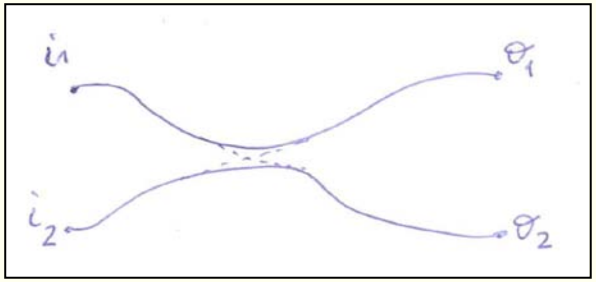
Encounters

Correlated sets of trajectories are represented by diagrams

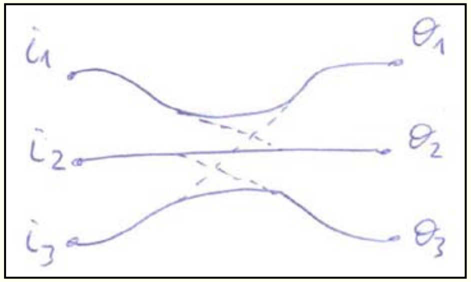
Encounters become vertices; Pieces of trajectories connecting them become edges

Diagrammatic rules: Each edge gives $\frac{1}{M}$, each vertex gives $-M$

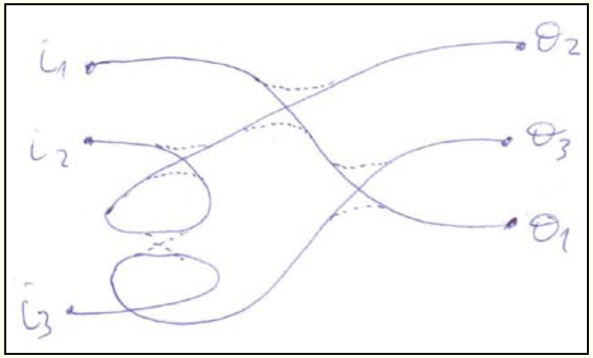
Calculation becomes naturally perturbative



$$\frac{(-M)}{M^4}$$



$$\frac{(-M)}{M^6}$$



$$\frac{(-M)^4}{M^{11}}$$

Some credits

Richter & Sieber (2002): First correction to $\mathcal{M}_1$

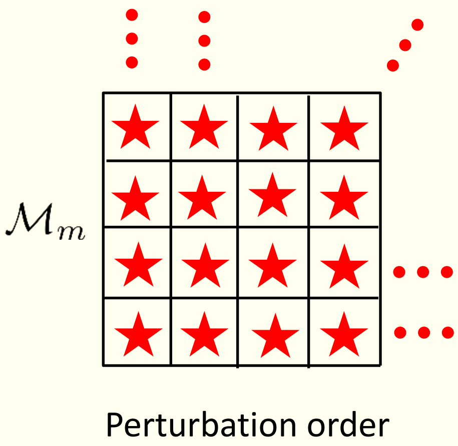
Schanz, Puhlmann & Geisel (2003): Leading order of $\mathcal{M}_2$

Heusler, Muller, Braun & Haake (2006): $\mathcal{M}_1$ and $\mathcal{M}_2$ to all orders

Berkolaiko, Harrison & Novaes (2008): Leading order to all $\mathcal{M}_m$

Berkolaiko & Kuipers (2011): Second order to all $\mathcal{M}_m$

Berkolaiko & Kuipers (2013):
Novaes (2013):
All orders to all $\mathcal{M}_m$



These works depend on heavy explicit manipulations involving either diagrams or permutations

(Not mentioning closed systems nor Ehrenfest time corrections)

New approach: Matrix Model

Complex $N_1 \times N_2$ matrices Z

Gaussian measure $\langle f \rangle_Z = \int e^{-M\mathrm{Tr}(ZZ^\dagger)} f dZ$

Covariance of matrix elements $\langle Z_{mj} Z^\dagger_{qr} \rangle_Z = \frac{\delta_{mr}\delta_{jq}}{M}$

Integration of matrix elements = Wick's rule

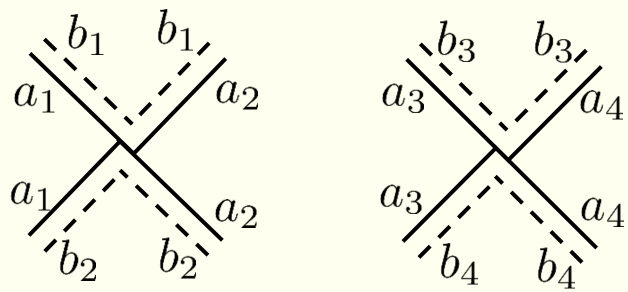
$$\left\langle \prod_{k=1}^{n} Z_{m_k j_k} Z^\dagger_{q_k r_k} \right\rangle_Z = \sum_{\sigma \in S_n} \prod_{k=1}^{n} \langle Z_{m_k j_k} Z^\dagger_{q_{\sigma(k)} r_{\sigma(k)}} \rangle_Z$$

All possible combinations of Z's and $Z^\dagger$'s, in pairs

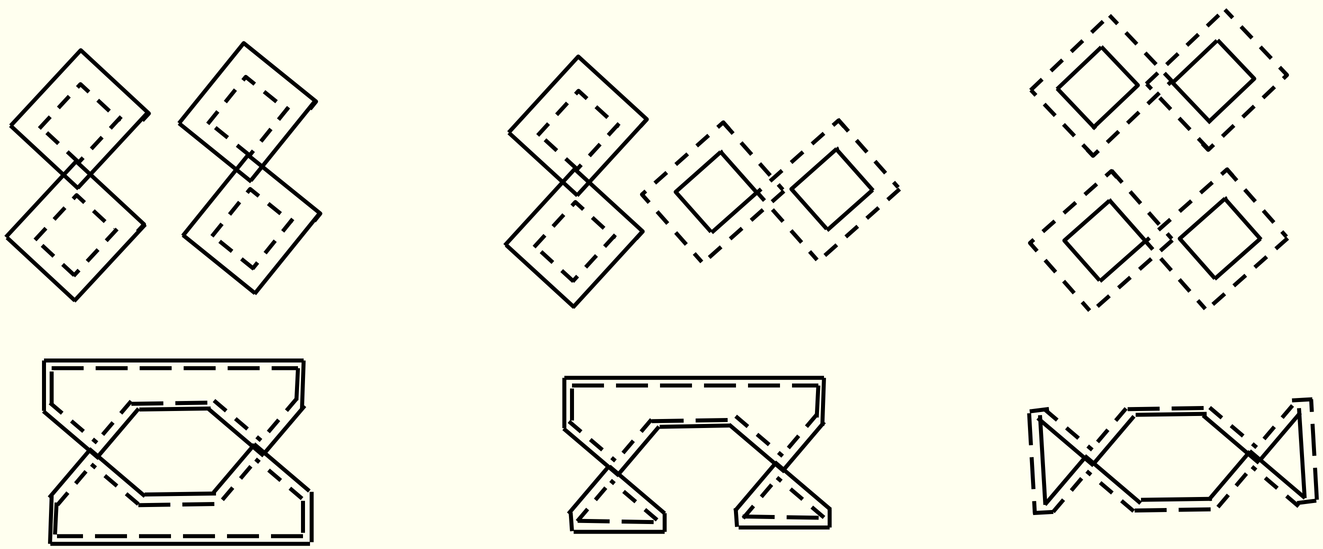
Diagrammatics

For example, consider the average value of

$$\mathrm{Tr}(ZZ^\dagger)^2\mathrm{Tr}(ZZ^\dagger)^2 = \sum_{a_1,a_2,a_3,a_4=1}^{N_1}\ \sum_{b_1,b_2,b_3,b_4=1}^{N_2} Z_{a_1b_1}Z^*_{a_2b_1}Z_{a_2b_2}Z^*_{a_1b_2}Z_{a_3b_3}Z^*_{a_4b_3}Z_{a_4b_4}Z^*_{a_3b_4}$$



Solid lines are a's and dashed lines are b's.

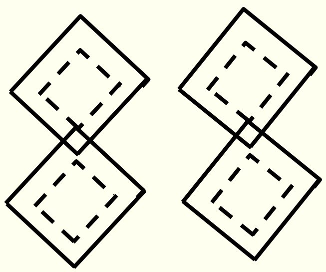


(Some examples out of 24 possibilities)

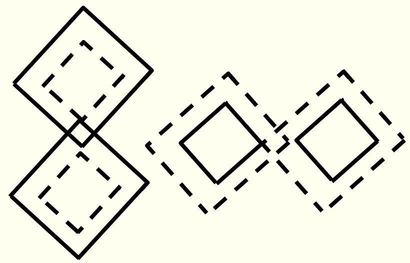
Diagrammatics

The covariance $\langle Z_{mj} Z^{\dagger}_{qr} \rangle_Z = \dfrac{\delta_{mr}\delta_{jq}}{M}$ implies that lines have a single index

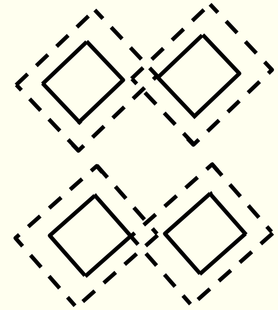
Summing over free indices, we get contributions of diagrams



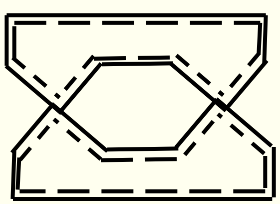
$$\frac{N_1^2 N_2^4}{M^4}$$



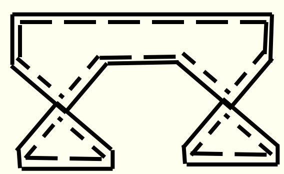
$$\frac{N_1^3 N_2^3}{M^4}$$



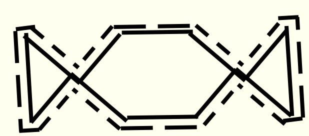
$$\frac{N_1^4 N_2^2}{M^4}$$



$$\frac{N_1^2 N_2^2}{M^4}$$



$$\frac{N_1^1 N_2^3}{M^4}$$

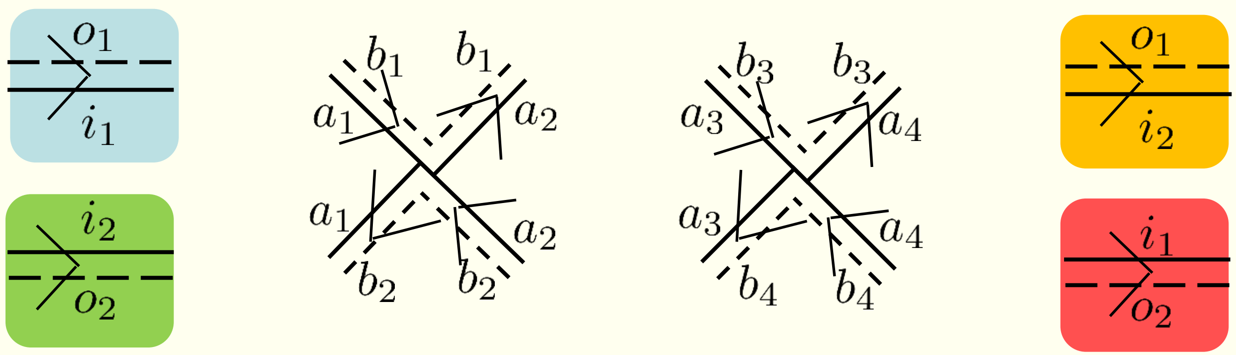


$$\frac{N_1^3 N_2^1}{M^4}$$

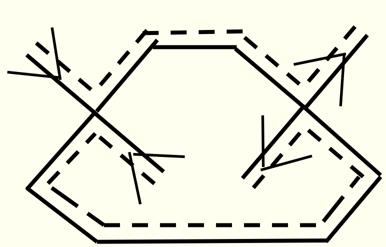
Transport Diagrammatics

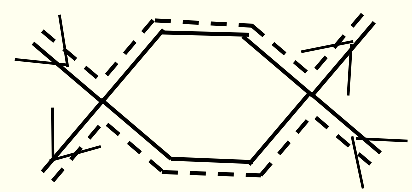
Transport diagrams must have channels

For example $\left\langle \mathrm{Tr}(ZZ^\dagger)^2 \mathrm{Tr}(ZZ^\dagger)^2 Z_{i_1 o_1} Z^\dagger_{o_1 i_2} Z_{i_2 o_2} Z^\dagger_{o_2 i_1} \right\rangle_Z$



Wick's rule will lead to





Acceptable diagrams

Non-acceptable diagrams

(contains a periodic orbit)

Getting rid of periodic orbits

Periodic orbits are free indices, summed over

If there are t_1 solid line orbits and t_2 dashed line orbits, then $N_1^{t_1} N_2^{t_2}$

Diagrams without p.o.'s have contribution independent of N_1, N_2

We may get rid of periodic orbits by letting $N_1, N_2 \to 0$

Therefore, the quantity

$$\lim_{N_1, N_2 \to 0} \left\langle \mathrm{Tr}(ZZ^\dagger)^2 \mathrm{Tr}(ZZ^\dagger)^2 Z_{a_1 b_1} Z^\dagger_{c_1 d_1} Z_{a_2 b_2} Z^\dagger_{c_2 d_2} \right\rangle_Z$$

Recovers the semiclassics for $\left\langle S_{a_1 a_1} S^\dagger_{c_1 d_1} S_{a_2 b_2} S^\dagger_{c_2 d_2} \right\rangle_E$ with two simple encounters

Producing all possible encounters

In order to produce all possible encounters, we need

$$e^{-M\sum_{q\geq 2}\frac{1}{q}\mathrm{Tr}(ZZ^{\dagger})^{q}}$$

When this is expanded in powers of M, every encounter gets multiplied by -M

Combined with Gaussian measure, this becomes

$$e^{-M\sum_{q\geq 1}\frac{1}{q}\mathrm{Tr}(ZZ^{\dagger})^{q}} = e^{M\log(1-ZZ^{\dagger})} = \det(1-ZZ^{\dagger})^{M}$$

Therefore, we get

$$\left\langle S_{a_1 a_1} S^{\dagger}_{c_1 d_1} S_{a_2 b_2} S^{\dagger}_{c_2 d_2} \right\rangle_E = \lim_{N_1, N_2 \to 0} \left\langle \det(1-ZZ^{\dagger})^{M} Z_{a_1 b_1} Z^{\dagger}_{c_1 d_1} Z_{a_2 b_2} Z^{\dagger}_{c_2 d_2} \right\rangle_Z$$

Equivalence with usual RMT

It remains to compute $\left\langle \prod_{k=1}^{n} S_{a_k b_k} S^{\dagger}_{c_k d_k} \right\rangle_E$

Introduce singular value decomposition $Z = UDV^{\dagger}$

Integrate over U and V: Weingarten functions $\int_{\mathcal{U}(N_1)} dU \prod_{k=1}^{n} U_{a_k b_k} U^{\dagger}_{c_k d_k}$

Integrate over D: Selberg-like integrals $\int_0^1 dX |\Delta(X)|^2 \det \left[(1-X)^M X^{N_2 - N_1} \right] s_{\lambda}(X)$

Take the $N_1, N_2 \to 0$ limit

Agreement is found in general

Remarks

If we forget about the physics, we have the curious fact:

$$\lim_{N_1\to 0}\ \lim_{N_2\to 0}\ \int_{ZZ^\dagger<1} dZ\ \det(1-ZZ^\dagger)^M \prod_{k=1}^{n} Z_{a_k b_k} Z^\dagger_{c_k d_k} = \int_{\mathcal{U}(M)} dU \prod_{k=1}^{n} U_{a_k b_k} U^\dagger_{c_k d_k}$$

Generalization to the orthogonal group:

$$\lim_{N_1\to 0}\ \lim_{N_2\to 0}\ \int_{ZZ^T<1} dZ\ \det(1-ZZ^T)^{M/2} \prod_{k=1}^{2n} Z_{a_k b_k} = \int_{\mathcal{O}(M+1)} dU \prod_{k=1}^{2n} U_{a_k b_k}$$

And to the symplectic group:

$$\lim_{N_1\to 0}\ \lim_{N_2\to 0}\ \int_{ZZ^D<1} dZ\ \det(1-ZZ^D)^{2M} \prod_{k=1}^{2n} Z_{a_k b_k} = \int_{Sp(M-1/2)} dU \prod_{k=1}^{2n} U_{a_k b_k}$$

Remarks

Limits can be made rigorous by analytic continuation based on Barnes G function

$$M^{N_1 N_2} \prod_{j=1}^{N_1} \frac{\Gamma(1+M+N_1-j)}{\Gamma(\lambda_j+1+M+N_1+N_2-j)} \qquad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$$ (assuming $N_2 \geq N_1$)

$$\xrightarrow{N_2 \to 0} \prod_{j=1}^{N_1} \frac{\Gamma(1+M+N_1-j)}{\Gamma(\lambda_j+1+M+N_1-j)} = \prod_{j=1}^{\ell} \frac{\Gamma(1+M+N_1-j)}{\Gamma(\lambda_j+1+M+N_1-j)}$$ (assuming $N_1 \geq \ell$)

$$\xrightarrow{N_1 \to 0} \prod_{j=1}^{\ell} \frac{\Gamma(1+M-j)}{\Gamma(\lambda_j+1+M-j)}$$