Spectral shock waves and spectral viscosity in dynamic random matrix models

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Outline

- Dynamics in random matrix models
- Dysonian dynamics and turbulence in hermitian RMM (Blaizot, Warchoł, MAN, Grela [2011-2014])
  1. Complex Burgers-like equations in hermitian RMT
  2. Trivia on real Burgers equation
  3. Where are the shocks?
  4. What plays the role of spectral viscosity?
- General set-up for nonhermitian RMM (Janik, MAN, Papp, Zahed [1996], Jarosz, MAN [2004])
- Unraveling hidden dynamics in non-hermitian RMM (Burda, Grela, MAN, Tarnowski, Warchoł [2014])
  1. Coevolution of eigenvectors and eigenvalues
  2. Universality

Inviscid complex Burgers equation

After considerable and fruitless efforts to develop a Newtonian theory of ensembles, we discovered that the correct procedure is quite different and much simpler...... from F.J. Dyson, J. Math. Phys. 3 (1962) 1192

- $H_{ij} \to H_{ij} + \delta H_{ij}$ with $< \delta H_{ij} = 0 >$ and $< (\delta H_{ij})^2 >= (1 + \delta_{ij})\delta t$
- For GUE, $dHe^{-N\mathrm{Tr}H^2} \to \prod_i dx_i e^{-N\sum_i x_i^2} \prod_{i\neq j}(x_i - x_j)^2$
- For eigenvalues x_i, random walk undergoes in the "electric field" (Dyson) $< \delta x_i >\equiv E(x_i)\delta t = \sum_{i\neq j}\left(\frac{1}{x_j - x_i}\right)\delta t$ and $< (\delta x_i)^2 >= \delta t$
- Resulting SFP equation for the resolvent in the limit $N = \infty$ and $\tau = Nt$ reads $\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0$ where $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z - H(\tau)}\right\rangle$ is the resolvent
- Non-linear, inviscid complex Burgers equation (special case of more general Voiculescu equation)

Inviscid complex Burgers equation - details [Biane, Speicher 2001]

- SFP eq:
  $\partial_t P(\{x_j\}, t) = \frac{1}{2} \sum_i \partial^2_{ii} P(\{x_j\}, t) - \sum_i \partial_i (E(x_i) P(\{x_j\}, t))$
- Integrating, normalizing densities to 1 and rescaling the time $\tau = Nt$ we get
  $\partial_\tau \rho(x) + \partial_x \rho(x) P.V. \int dy \frac{\rho(y)}{x-y} =$
  $\frac{1}{2N} \partial^2_{xx} \rho(x) + P.V. \int dy \frac{\rho_c(x,y)}{x-y}$
- r.h.s. tends to zero in the large N limit
- $\frac{1}{x \pm i\epsilon} = P.V. \frac{1}{x} \mp i\pi\delta(x)$
- Taking Hilbert transform of the above equation and using above Sochocki formula converts pair of singular integral-differential equations onto complex inviscid Burgers equation.

Real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = \mu \partial_{xx} f(x,t)$
  $f(x,t)$ is the velocity field at time t and position x of the fluid with viscosity μ.
- One-dimensional toy model for turbulence [Burgers 1939]
- But, equation turned out to be exactly integrable [Hopf 1950],[Cole 1951]
  If $f(x,t) = -2\mu\partial_x \ln d(x,t)$, then
  $\partial_t d(x,t) = \mu\partial_{xx} d(x,t)$ (diffusion equation), so general solution comes from Cole-Hopf transformation where
  $d(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{4\mu t} - \frac{1}{2\mu}\int_0^{x'} f(x'',0)dx''} dx'$

Inviscid real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = 0$
  where $f(x,0) = f_0(x)$.
- Solution by the method of characteristics:
- If $x(t)$ is the solution of ODE $dx(t)/dt = f(x(t),t)$, then $F(t) \equiv f(x(t),t)$ is constant in time along characteristic curve on the (x,t) plane ($\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial f}$)
- Then $dx/dt = F$ and $dF/dt = 0$ lead to $x(t) = x(0) + tF(0)$ and $F(t) = F(0)$
- Defining $\xi \equiv x(0)$ we get
  $f(x,t) = f(\xi,0) = f_0(\xi) = f_0(x - tf(x,t))$, i.e. implicit relation determining the solution of the Burgers equation.
- When $d\xi/dx = \infty$, we get the shock wave.

Inviscid real Burgers equation

- In the case of inviscid Burgers equation, characteristics are straight lines, but with different slopes (velocity depends on the position)
- Characteristics method fails when lines cross (shock wave)
- Finite viscosity (or diffusive constant) smoothens the shock
- Inviscid limit of viscid Burgers equation is highly non-trivial

Complex inviscid Burgers Equation

- Complex Burgers equation $\partial_\tau G + G\partial_z G = 0$
- Complex characteristics, trivial initial conditions
  $G(z,\tau) = G_0(\xi[z,\tau)])$ $\qquad G_0(z) = G(\tau = 0, z) = \frac{1}{z}$
  $\xi = z - G_0(\xi)\tau \quad (\xi = x - vt)$, so solution reads
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Equivalently, $dz/d\xi = 0$, then $\xi_c = \pm\sqrt{\tau}$, so
  $z_c = \xi_c + G_0(\xi_c)\tau = \pm 2\sqrt{\tau}$
- Since explicit solution easily reads
  $G(z,\tau) = \frac{1}{2\pi\tau}(z - \sqrt{z^2 - 4\tau})$, i.e. $\rho(x,\tau) = \frac{1}{2\pi\tau}\sqrt{4\tau - x^2}$,
  we see that shock waves appear at the edges of the spectrum
  $(x = \pm 2\sqrt{\tau})$.

Where is the viscosity?

- Let us define $D_N(z,\tau) \equiv \langle \det(z - H(\tau)) \rangle$
- Opening the determinant with the help of auxilliary Grassmann variables and perfoming the averaging one gets easily (after rescaling the time)
  $D_N(z,\tau) = \int \exp\left(\sum_i \bar{\eta}_i z \eta_i - \frac{\tau}{N} \sum_{i<j} \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j \right) \prod_{l,r} d\bar{\eta}_l d\eta_r$
- Differentiating and using the properties of the Grassmann variables one gets that D_N obeys complex equation
  $\partial_\tau D_N(z,\tau) = -\frac{1}{2N} \partial_{zz} D_N(z,\tau)$.

Where is the viscosity? - cont.

- $\partial_\tau D_N(z,\tau) = -\frac{1}{2N}\partial_{zz} D_N(z,\tau)$.
- Then complex Cole Hopf transformation $f_N(z,\tau) = \frac{1}{N}\partial_z \ln D_N(z,\tau)$ leads to exact for any N, viscid complex Burgers equation $\partial_\tau f_N + f_N \partial_z f_N = -\mu \partial_{zz} f_N \qquad \mu = \frac{1}{2N}$
- Positive viscosity "smoothens" the shocks, negative is "roughening" them, triggering violent oscillations
- Note than $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{Tr} \frac{1}{z-H(\tau)} \right\rangle = \partial_z \left\langle \frac{1}{N} \mathrm{Tr} \ln(z - H(\tau)) \right\rangle = \partial_z \left\langle \frac{1}{N} \ln \det(z - H(\tau)) \right\rangle$ so f_N and G coincide only when $N = \infty$ (cumulant expansion).
- $\left\langle \frac{1}{N} \ln \det(z - H(\tau)) \right\rangle \overset{N=\infty}{=} \frac{1}{N} \ln \langle \det(z - H(\tau)) \rangle$,

Airy function as the herald of the shock

- Shock wave corresponds to square root singularities
- Number of eigenvalues in the narrow strip of width s around branch point scales like $n = N \int_{\rm strip} \lambda^{1/2} d\lambda = Ns^{3/2}$, so the spacing between the eigenvalues ($n = 1$) scales like $N^{-2/3} \sim \mu^{2/3}$
- Then $\pm x = 2\sqrt{\tau} + \mu^{2/3} s$ and $f_N(x, \tau) \sim \pm \frac{1}{\sqrt{\tau}} + \mu^{1/3} \xi_N(s, \tau)$
- Solving viscid Burgers equation with above parametrization yields, in the large N, limit Riccati equation, with solution $\xi_N \sim \partial_s \ln Ai(\frac{s}{2\sqrt{\tau}})$
- Herald of "soft edge" universality
- Non-trivial initial conditions lead to Pearcey universality.

Chiral GUE alias Wishart Ensemble

- Temporal dynamics of the matrix $W(\tau)$

$$W(\tau) = \begin{pmatrix} 0 & K^\dagger(\tau) \\ K(\tau) & 0 \end{pmatrix}$$

  where K is a $M \times N$ complex matrix ($M > N$), whose elements are undergoing complex Brownian walk. We define "zero modes number" $\nu = M - N$ and "rectangularity number" $r = N/M$.

- We define
  $D_N^\nu(z,\tau) = \langle \det(w - W(\tau)) \rangle = w^\nu \left\langle \det(w^2 - K^\dagger K) \right\rangle$
  $\equiv z^{\nu/2} R_N^\nu(z,\tau)$, where $w^2 = z$.
- Using Grassmannian tricks we derive exact for any finite M, N evolution equations.

Chiral GUE - "cylindrical" diffusion

- Test No 0: [Marcenko-Pastur 1968] - Burgers equation (!)
- Test No 1: For $M, N \to \infty$, N/M fixed, our equation for R_N^ν agrees with [Guionnet,Cabanal-Duvillard 2001] obtained in free martingale theory (complex Bru process, diffusing MP formula)
- Test No 2: For finite M, N, equations solved by $R_N^\nu = (-\tau)^N N! L_N^\nu(z/\tau)$ (time-dependent associated Laguerres)
- Main result reads $\partial_\tau D_N^\nu(w,\tau) = -\frac{1}{2\cdot 2M}\frac{1}{w}\partial_w(w\partial_w)D_N^\nu(w,\tau) + \frac{1}{2\cdot 2M}\frac{\nu^2}{w^2}D_N^\nu(w,\tau)$
- For $\nu = 0$, this is a complex analog of the cylindrical diffusion equation with viscosity equal to the inverse of the size of the matrix (i.e. $2M$).
- CH transformation $f_{N+M} = \frac{1}{M+N}\partial_w \ln D_N^\nu(w,\tau)$ generates corresponding complex "Burgers-like" equation

Chiral GUE - three types of shocks

- In large N, M limit ($r \to 1$), but ν fixed, we again recover inviscid Burgers equation for $g(w,\tau) = \lim_{M,N\to\infty} f_{M+N}(w,\tau)$, i.e. we get $\partial_\tau g(w,\tau) + g(w,\tau)\partial_w g(w,\tau) = 0$
- For general boundary conditions $g(w,0) = g_0(w) = \frac{1}{2}\left(\frac{1}{w-1} + \frac{1}{w+1}\right)$ we get again cubic equation with three types of shock waves:
- If we define $w - w^* \equiv p$ we get three types of scalings
  1. $p \to (N+M)^{-2/3} s$ for $\tau < 1$
  2. $p \to (N+M)^{-3/4} s$ for $\tau = 1$
  3. $p \to (N+M)^{-1} s$ for $\tau > 1$

Chiral GUE - Bessel and Bessoid "heralds" of shocks

Solutions in the vicinity of shocks $\xi_N = \partial_s \ln \phi$

1. For $\tau < 1$, Airy edge, where $\phi(s) = Ai(-\sqrt{2\dot{g}_0} s)$,
2. For $\tau > 1$, Bessel edge, where $\phi(s) = s^{-\nu/2} J_\nu(\pi\rho(0)\sqrt{s})$
3. For $\tau = 1$, generalized Bessoid
   $\phi(m, r) = \int_0^\infty y^{\nu+1} e^{-y^4/2 - y^2 r} J_\nu(2my) dy$
   where variables $m = -is, r$ scale with N like $N^{3/4}, N^{1/2}$, respectively.

Note that Bessoid cusp in chiral GUE superimposes the Pearcey cusp in GUE (CUE), where no additional symmetries are imposed on the ensemble.

Applications

- Complex systems evolve as a function of some external parameter (time, length of the wire/box, area, temperature...)
- Lookout for universality windows where simplified dynamics of RM is shared by non-trivial theories
- Ex. 1: Strong-weak coupling transition in large N_c Yang-Mills theory (Durhuus-Olesen transition) as the shock wave collision on unitary circle [Narayanan, Neuberger, Blaizot, MAN, Lohnmayer, Wettig 2006-2012], Pearcey's critical exponents confirmed by lattice simulations in 3 and 4 dimensions.
- Ex. 2: Chiral symmetry breakdown in Quantum Chromodynamics as the chiral shock wave collisions [Blaizot, MAN,Warchol, 2011-2012], Bessoid criticality [Janik,MAN,Papp,Zahed 1998],[Brezin,Hikami 1998], ..., [Bleher, Kuijlaars 2010], [Forrester 2012]

Non-hermitian case - electrostatic analogy (Dyson gas)

Analytic methods break down, since spectra are complex
$\rho(z,\bar{z}) = \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i)\right\rangle$.

- Potential $\Phi(z,\bar{z}) =$ $\lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\,\ln[(z\mathbf{1}_N - X)(\bar{z}\mathbf{1}_N - X^\dagger)+\epsilon^2\mathbf{1}_N]\right\rangle$
- Poisson law $\frac{\partial^2\Phi}{\partial z\partial\bar{z}} = \pi\rho(z,\bar{z})$, since $\delta^{(2)}(z) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{(|z|^2+\epsilon^2)^2}$
- Electric field $G(z,\bar{z}) = \frac{\partial\Phi}{\partial z}$
- Gauss law $\frac{1}{\pi}\partial_{\bar{z}}G(z,\bar{z}) = \rho(z,\bar{z})$ [Brown;1986],[Sommers,Crisanti,Sompolinsky,Stein;1988]
- Bad news: $G(z,\bar{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\,\frac{\bar{z}\mathbf{1}-X^\dagger}{(z\mathbf{1}_N-X)(\bar{z}\mathbf{1}_N-X^\dagger)+\epsilon^2\mathbf{1}}\right\rangle$
- No similarity to the hermitian case $G(z) = \left\langle \frac{1}{N}\mathrm{tr}\,\frac{1}{z\mathbf{1}_N - X}\right\rangle$
- Important in applications (dissipation, directed percolation, lagged correlations..), interesting in mathematics [Biane,Lehner;1999]

Non-hermitian case - Remedy: $\mathrm{tr}\ \ln A = \ln \det A$

$$\mathrm{tr}\ \ln[(z\mathbf{1}_N - X)(\bar{z}\mathbf{1}_N - X^\dagger) + \epsilon^2 \mathbf{1}_N] = \ln \det \begin{pmatrix} z\mathbf{1}_N - X & i\epsilon\mathbf{1}_N \\ i\epsilon\mathbf{1}_N & \bar{z}\mathbf{1}_N - X^\dagger \end{pmatrix}$$

- Duplication trick [Janik,MAN,Papp,Zahed;1996], note also "hermitization" trick [Girko;1990], [Feinberg,Zee;1997], [Fyodorov,Khoruzhenko,Sommers;1997]
  $\mathrm{tr}_{\,\mathrm{b}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{tr}\, A & \mathrm{tr}\, B \\ \mathrm{tr}\, C & \mathrm{tr}\, D \end{pmatrix}$
- $\mathcal{Z}_N = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \otimes \mathbf{1}_N \equiv \mathcal{Z} \otimes \mathbf{1}_N \quad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$
- 2 by 2 objects $\mathcal{G}(\mathcal{Z}) = \frac{1}{N} \left\langle \mathrm{tr}_{\,\mathrm{b}} \frac{1}{\mathcal{Z}_N - \mathcal{X}} \right\rangle = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix}$

Benefits of the duplication trick

- Upper-left corner of $\mathcal{G}$, i.e. $\mathcal{G}_{11} = G(z,\bar{z})$, so $\frac{1}{\pi}\partial_{\bar{z}}\mathcal{G}_{11} = \rho(\lambda)$
- Product of off-diagonal elements of $\mathcal{G}$, i.e. $C(z,\bar{z}) \equiv \mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1}$ gives the correlator between the left and right eigenvectors $O(z) \equiv \left\langle \frac{1}{N}\sum_i O_{ii}\delta^{(2)}(z-\lambda_i)\right\rangle = -\frac{N}{\pi}C(z,\bar{z})$ where $O_{ij} =< L_i|L_j >< R_j|R_i >$
  [Savin,Sokolov;1997],[Chalker,Mehlig;1998]
  [Janik,MAN,Papp,Zahed,Nörenberg;1998]

Hidden algebraic structure unveiled [Jarosz,MAN;2004]

- Each generic 2 by 2 matrix Q which has appeared before has the structure $Q = \begin{pmatrix} z & i\bar{v} \\ iv & \bar{z} \end{pmatrix} \equiv \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$
- Q is a quaternion
- $Q = q_0 \mathbf{1}_2 + i\sigma_i q_i = \begin{pmatrix} q_0 + iq_3 & i(q_1 - iq_2) \\ i(q_1 + iq_2) & q_0 - iq_3 \end{pmatrix}$
- One can exploit the whole space of Q, instead of staying infinitesimaly close (ϵ) in transverse directions $(1, 2)$
- Formal similarity to hermitian case, algebraic structure of quaternions simplifies considerably technical aspect of the calculations

Hidden Burgulence in Ginibre ensemble

- We study now analytic properties of additional variable w.
- We define the determinant $D_N(z,\bar{z},w,\bar{w}) = \langle \det(Q - X_D) \rangle$
- Naively, connection between the poles of the Green's function and zeroes of the characteristic determinant seems not to hold for complex matrices, see e.g. Ginibre ensemble (complex Gaussian), $< \det(z - X) >= z^N$ and $\rho = \frac{1}{\pi}\theta(1 - |z|)$.
- But since $\epsilon = v \equiv -i\bar{w}$ does not need to be small, we can now study analytic properties of D_N as a function of w.
- Note that $\det(Q - X_D) = \det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -\bar{w}L^\dagger L \\ wR^\dagger R & \bar{z}\mathbf{1}_N - \Lambda^\dagger \end{pmatrix}$

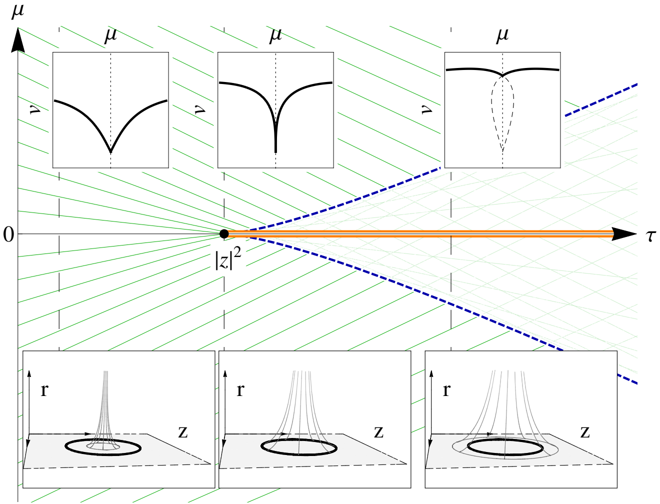
Ginibre random walk

- For random walk of complex elements of the nonhermitian matrix X, we arrive (using similar tricks alike in the hermitian case) at $\partial_\tau D_N(z,w,\tau) = +\frac{1}{N}\partial_{w\bar{w}} D_N$.
- Note that 2-dimensional Laplace operator has cylindrical symmetry and the sign is positive.
- We define $g(z,r,\tau) = \frac{1}{2N}\partial_z \ln D_N$ and $v(z,r,\tau) = \frac{1}{2N}\partial_r \ln D_N$, where r is the radial variable $w = re^{i\phi}$.
- Above functions fulfill exact equations (for any N) $\partial_\tau g = \frac{1}{N}\Delta_r g + v\partial_z v$, $\partial_\tau v = \frac{1}{N}(\Delta_r - \frac{1}{r^2})v + v\partial_r v$ where $\Delta_r = \frac{1}{4}(\partial_{rr} + \frac{1}{r}\partial_r)$

Solution in the $N = \infty$ case

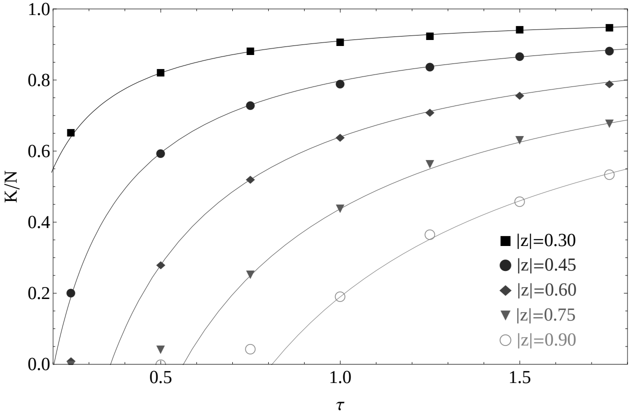
- Second equation becomes Euler equation $\partial_\tau v = v \partial_r v$ and is immediately solvable using radial characteristics. For $r \to 0$ solution reads $v^2 = (\tau - |z|^2)/\tau^2$ inside unit circle, reproducing known results for eigenvector correlations for Ginibre ensemble.
- First equation leads to spectral density $\rho(z, \tau) = \frac{1}{\pi\tau}\theta(\tau - |z|^2)$.
- For "frozen" time $\tau = 1$, above results reproduce known results for eigenvectors (Chalker, Mehlig [1998]) and for eigenvalues (Ginibre [1965])

The flow of singularities



The coevolution of eigenvalues and eigenvectors

Petermann factor $\frac{K}{N} \equiv \frac{v^2(|z|,\tau)}{\pi\rho(|z|,\tau)}$



Beyond the $N = \infty$ case

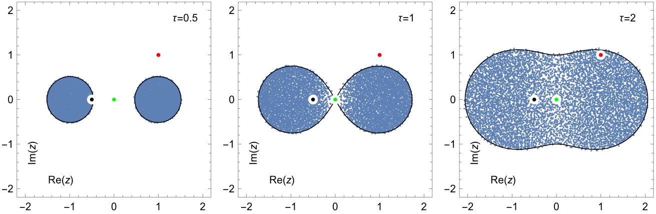
- Finite postive viscosity smoothens the sharp edge at $|z|^2 = \tau$
- Calculations using Burgers equations reproduce known scaling at the edge $|z| - \sqrt{\tau} = \eta N^{-1/2}$, yielding universal result $K(\eta) = \sqrt{\frac{N\pi}{2}}\tau^N e^{-N}\mathrm{Erfc}\left(\sqrt{\frac{2}{\tau}}\eta\right)$

Burgers vs KPZ

- Alternative Cole-Hopf-like transform $\phi_\tau(z,w) = \frac{1}{N} \ln D_\tau(z,w)$ yields noiseless KPZ
- Large N leads $\partial_\tau \phi_\tau = \frac{1}{4}(\partial_r \phi_\tau)^2$
- Solution by Hopf-Lax formula
  $\phi_\tau(z,r) = max_{y \geq 0} \left(\phi_{\tau=0}(z,y) - (r-y)^2/\tau\right)$

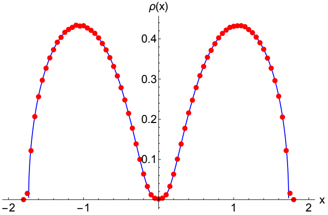
The spiric example

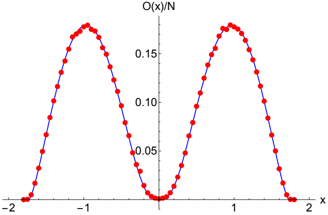
Initial condition $X_0 = \mathrm{diag}(a, ..a, -a, ... - a)$.



The spiric example cont.

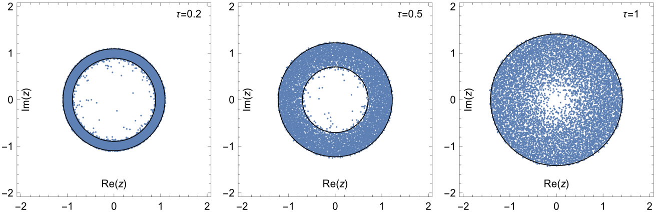
Spectral density (left) and eigenvector correlator (right) snapshots





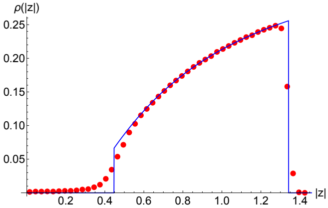
The non-normal example

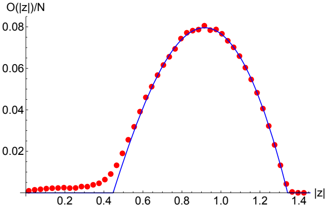
Initial condition: Non-normal $(X_0)_{ij} = \alpha\delta_{i,j-1}$.



The spiric example cont.

Spectral density (left) and eigenvector correlator (right) snapshots





Summary

- Formalism of Dysonian dynamics for non-hermitian RMM, involving **coevolution of eigenvalues and eigenvectors**
- Conjecture, that above presented scenario, based on Ginibre ensemble, is generic for all non-hermitian RMM - **paramount role of eigenvectors**
- Unexpected **similarity** between hermitian and non-hermitian RMM based on **"Burgulence"** concepts.
- Potential verification in various application of hermitian and non-hermitian random matrix models
- Unexplored mathematics

References

J.-P. Blaizot, MAN, Phys. Rev. Lett. 101, (2008) 102001
J.-P. Blaizot, MAN, Phys. Rev. E82 (2010) 051115
J.-P. Blaizot, MAN, P. Warchoł, Phys. Rev. E87 (2013) 052134
J.-P. Blaizot, MAN, P. Warchoł, Phys. Lett. B724 (2013) 170
J.-P. Blaizot, MAN, P. Warchoł, Phys. Rev. E89 (2014) 042130
Z. Burda, J. Grela, MAN, W. Tarnowski, P. Warchoł, Phys. Rev. Lett. 113 (2014) 104102
J-.P. Blaizot, J. Grela, MAN, P. Warchoł, arXiv 1405.5244