Eigenvalue spectra of modular networks

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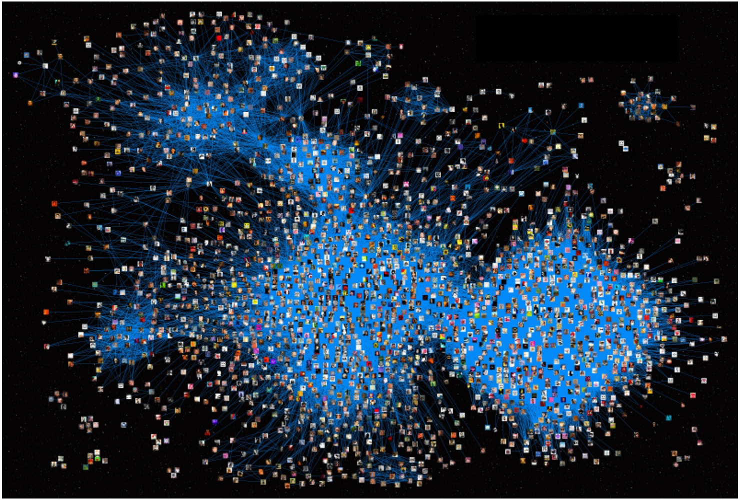
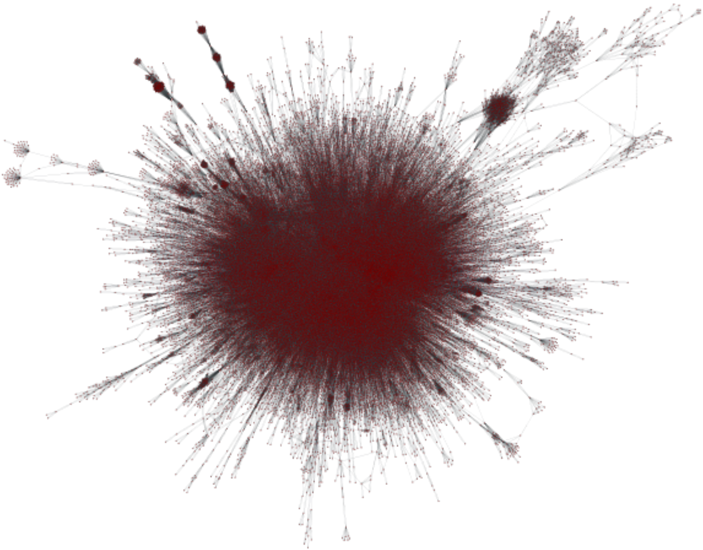
Global (or large-scale) structure

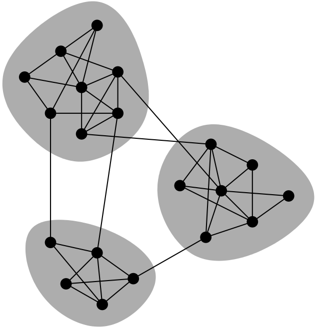
Modular structure



Problem: How to detect and characterize modular structure?



WHAT IS MODULAR STRUCTURE?



Example: "community structure"

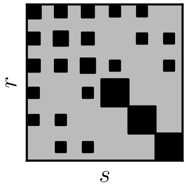
- How do we define what a module is?
- How do we find modules in networks?

A PRINCIPLED APPROACH: GENERATIVE MODELS

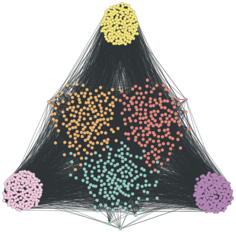
STATISTICAL INFERENCE OF **STOCHASTIC BLOCK MODELS**

N nodes divided into B blocks.

Parameters: $b_i \rightarrow$ block membership of node i
$e_{rs} \rightarrow$ number of edges from block r to s.



$\rightarrow$



P. W. Holland et al., Soc Networks 5, 109 (1983)

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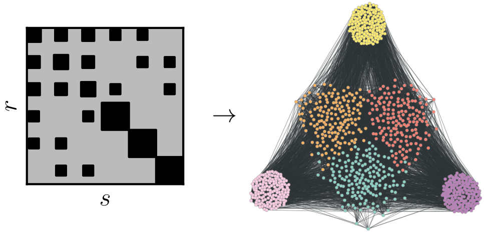


- Not restricted to assortative structures ("communities").
- Can be used to generalize the data and make predictions.
- Inference via maximum likelihood

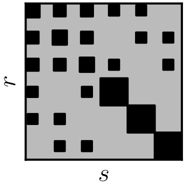
P. W. Holland et al., Soc Networks 5, 109 (1983)

INFERENCE VIA MAXIMUM LIKELIHOOD

Data likelihood: $\mathcal{P}(G|\theta)$

$G \rightarrow$ Observed network

$\theta \rightarrow$ Model parameters: $\{e_{rs}\}, \{b_i\}$



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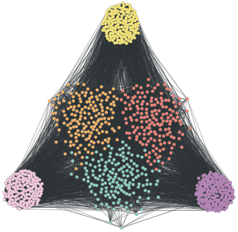
$\xrightarrow[P(G|\theta)]{}$



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Inference via maximum likelihood

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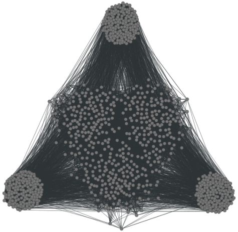

G

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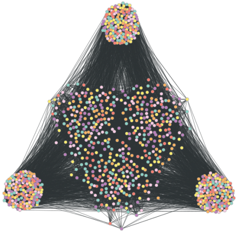

G

T. P. Peixoto, PRE 85, 056122 (2012); PRL, 148701 (2013); PRX 4, 011047 (2014)

INFERENCE VIA MAXIMUM LIKELIHOOD

Data likelihood: $\mathcal{P}(G|\theta)$

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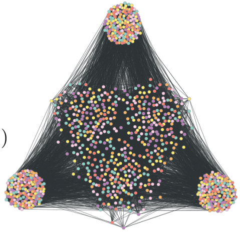
$\underset{\theta}{\operatorname{argmax}} P(G|\theta) \longleftarrow$



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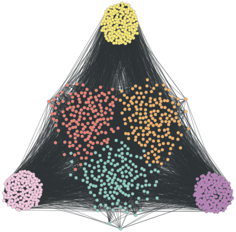
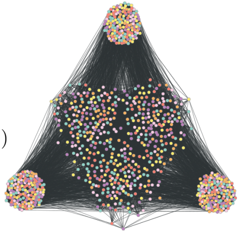


$\hat{\theta}$

$$\xleftarrow{\hspace{2em}}$$
$$\underset{\theta}{\operatorname{argmax}} P(G|\theta)$$



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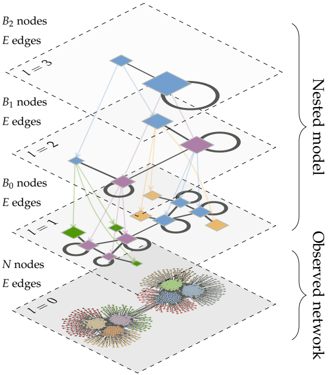
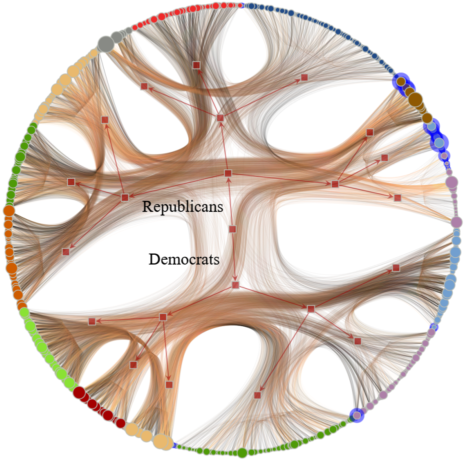
T. P. Peixoto, PRE 85, 056122 (2012); PRL, 148701 (2013); PRX 4, 011047 (2014)

POWERFUL AND VERSATILE



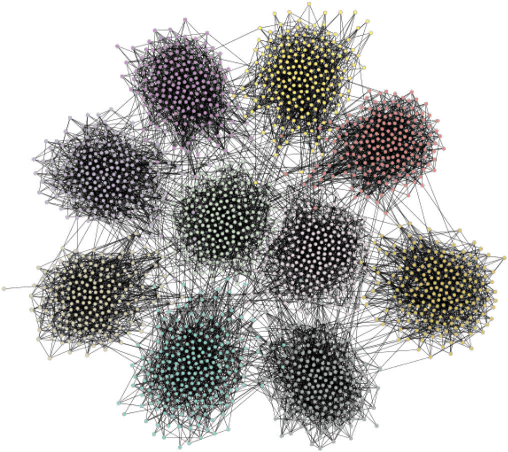
Political blogs



T. P. Peixoto, Phys. Rev. X 4, 011047 (2014)

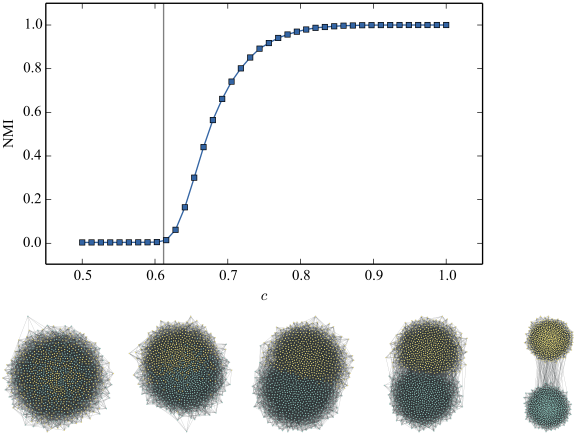
Planted partition model

- Groups have uniform size: $n_r = N/B$
- Community structure: $e_{rs} = 2E[\delta_{rs}c/B + (1-\delta_{rs})(1-c)/B(B-1)]$
- $c \in [0,1] \rightarrow$ mixing parameter



PP MODEL: DETECTABILITY TRANSITION

Phase transition: Undetectable vs. detectable

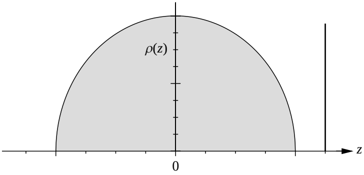


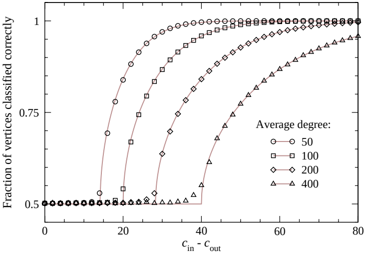
Model parameters can be inferred for up to $c^* = \frac{1}{B} \pm \frac{B-1}{B} \frac{1}{\sqrt{\langle k \rangle}}$.

Eigenvalue spectrum of the adjacency matrix

$$\text{Adjacency matrix } A_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ exists,} \\ 0 & \text{otherwise} \end{cases}$$

Planted partition model: Semicircle bulk + isolated eigenvalue





- Bulk: Random eigenvectors
- Isolated eingenvalue: Correlated with group structure

Eigenvalue merges with bulk: Transition at the exact same point!

$$c^* = \frac{1}{B} \pm \frac{B-1}{B}\frac{1}{\sqrt{\langle k \rangle}}$$

R. R. Nadakuditi, M. E. J. Newman, PRL 108, 188701 (2012)

BLOCKMODEL INFERENCE

Planted partition model with $B = 2$ groups

Log-likelihood:

$$\boxed{\log P = \sum_{ij} (A_{ij} - \nu k_i k_j) s_i s_j}$$

$s_i \in \{-1, +1\} \to$ group membership

Relaxation: $s_i \to \mathbb{R}, \quad \sum_i k_i s_i^2 = 2E$

Relaxation + Constrained maximization $\to$ Eigenvalue problem:

$$\boxed{D^{-1/2} A D^{-1/2} = \lambda u}$$

M. E. J. Newman, PRE 88, 042822 (2013)

Modularity

Planted partition model with $B = 2$ groups

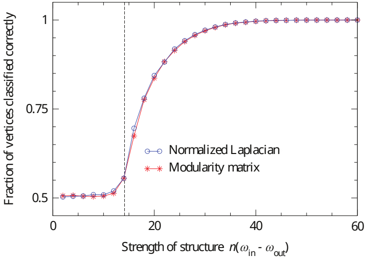
Modularity maximization:

$$\boxed{Q = \frac{1}{4m} \sum_{ij} \left[A_{ij} - \frac{k_i k_j}{2m} \right] (s_i s_j + 1)}$$

$s_i \in \{-1, +1\} \rightarrow$ group membership

Relaxation: $s_i \rightarrow \mathbb{R}, \quad \sum_i k_i s_i^2 = 2E$

Relaxation + Maximization $\rightarrow$ eigenvalue problem: $D^{-1/2} A D^{-1/2} = \lambda u$



M. E. J. Newman, PRE 88, 042822 (2013)

Hypothesis: Universal transition?

How far goes the correspondence between the different operators?

General framework

- Adjacency matrix $A_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ exists,} \\ 0 & \text{otherwise} \end{cases}$
  (percolation, ...)
- Laplacian matrix: $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$
  (unconserved flows, synchronization, ...)
- Normalized Laplacian: $\boldsymbol{\mathcal{L}} = \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$
  (conserved flows, random walks, ...)

General formulation

$$\boxed{\boldsymbol{W} = \boldsymbol{C} + \boldsymbol{M}}$$

$$\boldsymbol{C} \to \text{Random diagonal}, \boldsymbol{M} \to \text{Random symmetric}$$

$$\boldsymbol{A} \to \{\boldsymbol{C} = 0, \boldsymbol{M} = \boldsymbol{A}\}, \boldsymbol{L} \to \{\boldsymbol{C} = D, \boldsymbol{M} = -\boldsymbol{A}\},$$
$$\boldsymbol{\mathcal{L}} \to \{\boldsymbol{C} = \boldsymbol{I}, \boldsymbol{M} = -\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}\}$$

T. P. Peixoto, PRL 111, 098701 (2013)

General framework

$$\boldsymbol{W} = \boldsymbol{C} + \underbrace{\boldsymbol{\mathcal{M}}}_{\text{fluctuations}} + \underbrace{\langle \boldsymbol{M} \rangle}_{\text{mean}} = \underbrace{\boldsymbol{\mathcal{X}}}_{\text{random}} + \langle \boldsymbol{M} \rangle$$

$\boldsymbol{\mathcal{X}} = \boldsymbol{C} + \boldsymbol{M} - \langle \boldsymbol{M} \rangle \rightarrow$ zero-mean off-diagonals

Average resolvent and Stieltjes transform:

$$\rho(z) = -\frac{1}{N\pi} \operatorname{Im} \operatorname{Tr} \left\langle (z\mathbf{I} - \boldsymbol{\mathcal{X}})^{-1} \right\rangle$$

General framework

$$\mathbf{Y} = \begin{pmatrix} \boxed{\mathbf{Y}_n} & \boxed{\vec{a}} \\ \boxed{\vec{a}^T} & y_{nn} \end{pmatrix}$$

$$\left[\mathbf{Y}^{-1}\right]_{nn} = \frac{1}{y_{nn} - \vec{a}^T \mathbf{Y}_n^{-1} \vec{a}}$$

$$\left\langle \left[\mathbf{Y}^{-1}\right]_{nn} \right\rangle \simeq \frac{1}{\langle y_{nn} \rangle - \left\langle \vec{a}^T \mathbf{Y}_n^{-1} \vec{a} \right\rangle}$$

$$\left\langle [\boldsymbol{Y}^{-1}]_{ii} \right\rangle \simeq \sum_{Y_{ii}} \frac{P_i(Y_{ii})}{Y_{ii} - \sum_j \left\langle [\boldsymbol{Y}^{-1}]_{jj} \right\rangle \langle a_j^2 \rangle}$$

$$\left\langle [\boldsymbol{Y}^{-1}]_{ij} \right\rangle \simeq 0$$

$$Y \to zI - \mathcal{X}$$

T. P. Peixoto, PRL 111, 098701 (2013)

General framework

$$t_r(z) \equiv \left\langle [(z\boldsymbol{I} - \boldsymbol{\mathcal{X}})^{-1}]_{ii} \right\rangle \text{ for } i \in r$$

$$\boxed{t_r(z) = \sum_c \frac{p_c^r}{z - c - \sum_s \sigma_{rs}^2 n_s t_s(z)}}$$

$p_c^r \to$ diagonal distribution; $\sigma_{rs}^2 \to$ variance of $\boldsymbol{\mathcal{M}}$

The spectrum of $\boldsymbol{\mathcal{X}}$:

$$\boxed{\rho(z) = -\frac{1}{N\pi} \sum_r n_r \operatorname{Im} t_r(z)}$$

Spectrum of $\boldsymbol{W} = \boldsymbol{\mathcal{X}} + \langle \boldsymbol{M} \rangle$

$$\det(z\boldsymbol{I} - (\boldsymbol{\mathcal{X}} + \langle \boldsymbol{M} \rangle)) = 0$$

$$\det(z\boldsymbol{I} - \boldsymbol{\mathcal{X}}) \det(\boldsymbol{I} - (z\boldsymbol{I} - \boldsymbol{\mathcal{X}})^{-1} \langle \boldsymbol{M} \rangle) = 0$$

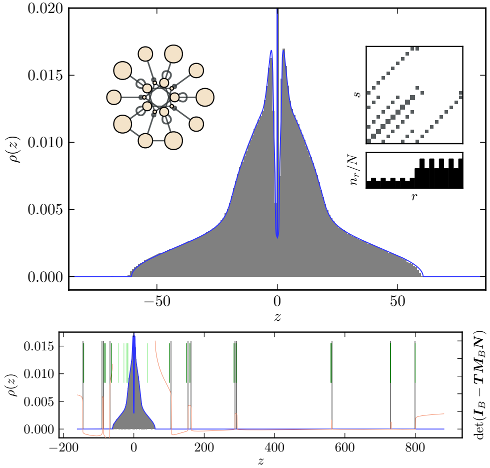
Additional eigenvalues: $\det(\boldsymbol{I} - \left\langle (z\boldsymbol{I} - \boldsymbol{\mathcal{X}})^{-1} \right\rangle \langle \boldsymbol{M} \rangle) = 0$

$$\boxed{\det(\boldsymbol{I}_B - \boldsymbol{T}(z) \boldsymbol{M}_B \boldsymbol{N}) = 0}$$

T. P. Peixoto, PRL 111, 098701 (2013)

EXAMPLES

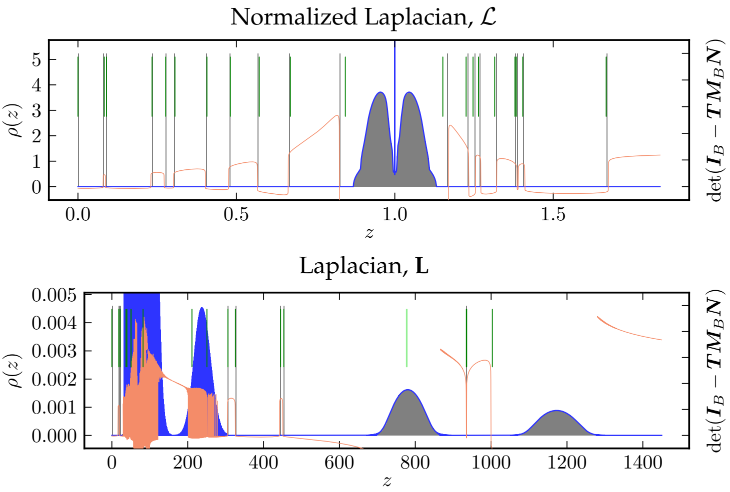
Adjacency matrix, **A**



- In general: No semicircle
- Sufficiently detached eigenvalues:
  $t_r \approx 1/(z - c_r)$
  $\det(z\boldsymbol{I}_B - (\boldsymbol{C}_B + \boldsymbol{M}_B\boldsymbol{N})) = 0$

T. P. Peixoto, PRL 111, 098701 (2013)

EXAMPLES

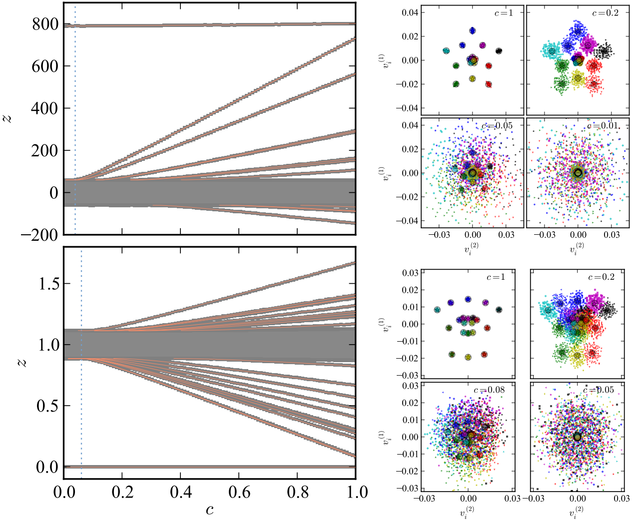


T. P. Peixoto, PRL 111, 098701 (2013)

Detachment transition

Interpolation with purely a random graph:

$$e_{rs} = ce^0_{rs} + (1-c)e^0_r e^0_s/2E$$

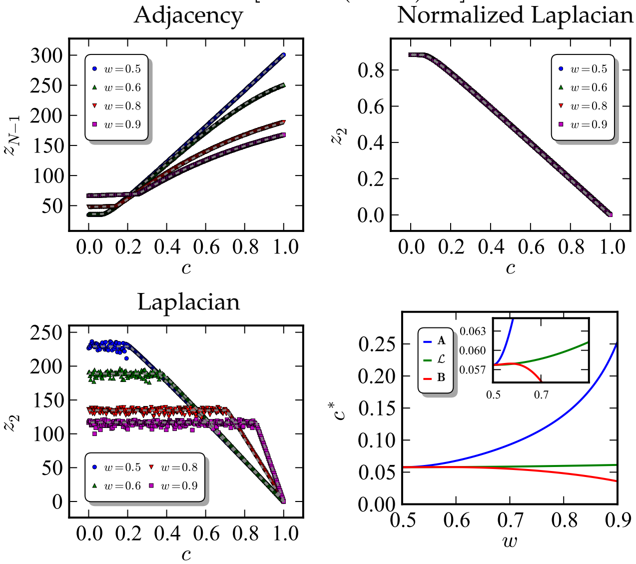


T. P. Peixoto, PRL 111, 098701 (2013)

Transition points

Simple two-block structure: $n_1/N = w$, $n_2/N = 1 - w$ and

$$e_{rs} = E[c\delta_{rs} + (1-c)/2]$$



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Transition points: Symmetric structures

$$\boxed{t_r(z) = \sum_c \frac{p_c^r}{z - c - \sum_s \sigma_{rs}^2 n_s t_s(z)}}$$

For $p_c^r = \delta_{d,c}$:

$$t(z) = (z - d \pm \sqrt{(d-z)^2 - 4a})/2a$$

with $a = a_r = N \sum_s \sigma_{rs}^2 / B$

$$\rho(z) = \sqrt{4a - (z-d)^2}/2a\pi$$

Detached eigenvalues: $z = d + at_i + 1/t_i$, where $t_i = B/N\lambda_i$

$\lambda_i \to$ eigenvalue of $\boldsymbol{M}_B$

General transition point:

$$\boxed{\lambda_m^2 = \frac{1}{\langle k \rangle B^2}}$$

$\lambda_m \to$ eigenvalue of $e_{rs}/N\langle k \rangle$.

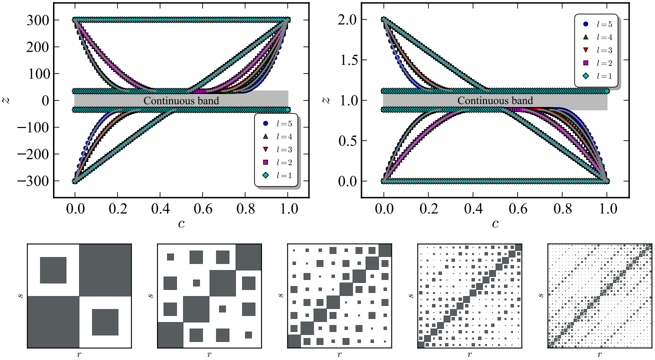
T. P. Peixoto, PRL 111, 098701 (2013)

Transition points: Symmetric structures

Hierarchical structure:

$$[\boldsymbol{m}_1]_{rs} = \delta_{rs} c / B_1 + (1 - \delta_{rs})(1 - c)/B_1(B_1 - 1)$$

$$\boldsymbol{m}_l = \boldsymbol{m}_{l-1} \otimes \boldsymbol{m}_{l-1} \otimes \rightarrow \text{the Kronecker product.}$$



$l = 1$ $l = 2$ $l = 3$ $l = 4$ $l = 5$

T. P. Peixoto, PRL 111, 098701 (2013)

CONCLUSION

- Detachment transition $\rightarrow$ detectability of modules
- General theory for spectra of modular networks
- In general, detachment transition is **dependent** on the actual operator
- Transitions for many operators collapse on the same point for symmetric structures

Open: What is the “best” operator? (Is there one?)