Ginibre evolutions and interacting particle systems

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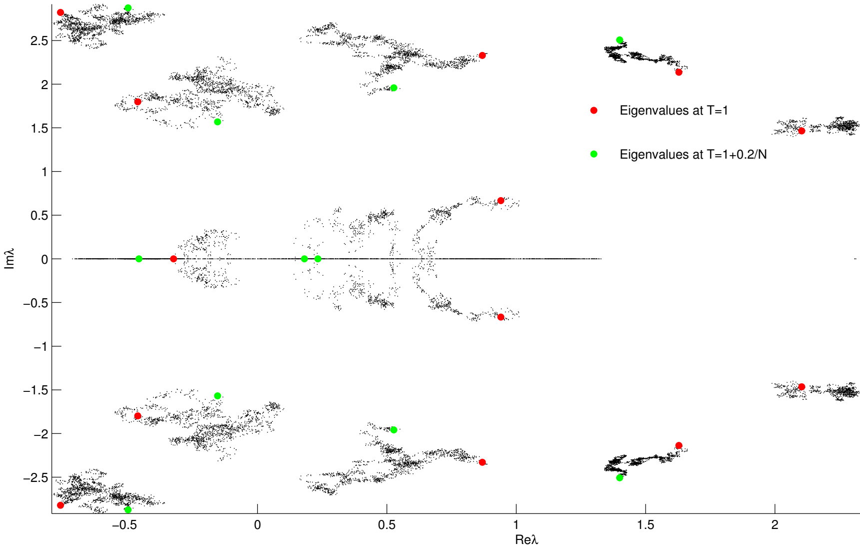
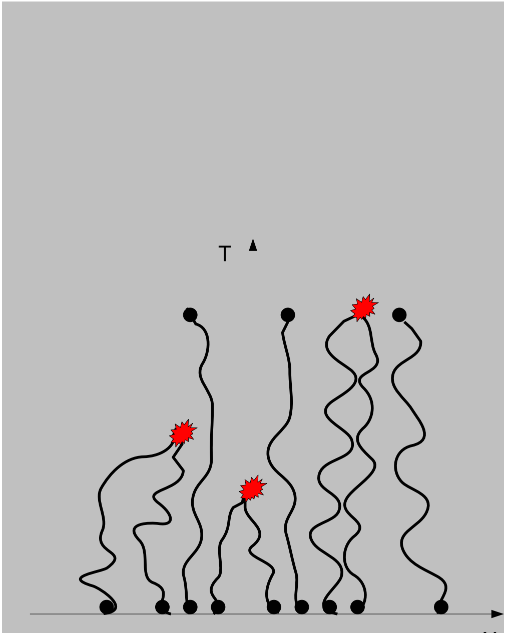
Eigenvalue evolution for Real Ginibre, $N = 100$



Annihilating Brownian motions



Pfaffian point process

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- Let A be an antisymmetric $2m \times 2m$ matrix. Then $\det(A) = \mathrm{Pf}(A)^2$ - example

  $$\mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

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- are solutions to

$$\frac{\partial u}{\partial t}(t,x,y) = L_x^X u(t,x,y) = L_y^Y u(t,x,y)$$

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- Product characteristic function identities (e.g. Desrosiers)

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- Consequence: X_t is a Pfaffian point process

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F depends on initial conditions

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- $\lim_{N\to\infty} K_t^{(N)} = K_t(x,y)$ - the kernel for annihilating BMs
- Natural maximal initial condition

$$F_t(x,y) = \mathrm{erfc}\left(\frac{y-x}{\sqrt{t}}\right)$$

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&= \quad C_K t^{-\frac{K}{2}} \Delta(yt^{-1/2}) e^{-\frac{1}{t} Tr(Y^2)} \int_{aU(K)} \mu(dW) e^{+\frac{1}{t} Tr(WYW^\dagger Y)}
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Introduce 'transform variables'

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Single-time statistics: 'The square root' of Itzykson-Zuber integral

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