Modification of the Porter-Thomas distribution by rank-one interaction

Eugene Bogomolny

University Paris-Sud, CNRS
Laboratoire de Physique Théorique et Modèles Statistiques, Orsay
France

XII Brunel-Bielefeld Workshop on Random Matrix Theory, Uxbridge, 10.12.2016

Outlook

Porter–Thomas distribution = everything is Gaussian

C. E. Porter and R. G. Thomas Phys. Rev. **104** 483 (1956)

Fluctuations of nuclear reaction widths

- Time-invariant systems

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right)$$

- Nuclear physics : $x =$ reduced resonance width. Random matrices : $x = N|\Psi|^2$
- Time-non-invariant systems

$$P_2(x) = \frac{1}{I} \exp\left(-\frac{x}{I}\right)$$

- $I =$ mean value of x. Standard normalisation : $\langle x \rangle = 1 \longrightarrow I = 1$

Porter–Thomas distribution = everything is Gaussian

C. E. Porter and R. G. Thomas Phys. Rev. **104** 483 (1956)

Fluctuations of nuclear reaction widths

- Time-invariant systems

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right)$$

- Nuclear physics : $x =$ reduced resonance width. Random matrices : $x = N|\Psi|^2$
- Time-non-invariant systems

$$P_2(x) = \frac{1}{I} \exp\left(-\frac{x}{I}\right)$$

- $I =$ mean value of x. Standard normalisation : $\langle x \rangle = 1 \longrightarrow I = 1$

Old experiments agreed well with the PT distribution

Porter–Thomas distribution = everything is Gaussian

C. E. Porter and R. G. Thomas Phys. Rev. **104** 483 (1956)

Fluctuations of nuclear reaction widths

- Time-invariant systems

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right)$$

- Nuclear physics : $x =$ reduced resonance width. Random matrices : $x = N|\Psi|^2$
- Time-non-invariant systems

$$P_2(x) = \frac{1}{I} \exp\left(-\frac{x}{I}\right)$$

- $I =$ mean value of x. Standard normalisation : $\langle x \rangle = 1 \longrightarrow I = 1$

Old experiments agreed well with the PT distribution

Recent experiments P. E. Koehler *et al*

Reduced neutron widths in the nuclear data ensemble : experiment and theory do not agree
Phys. Rev. C **84**, 034312 (2011)
Neutron resonance data exclude random matrix theory
Fortschritte Phys. **61**, 80 (2013)

Modification of the model

- Within standard (invariant) random matrix ensembles : PT distribution = **theorem**
- Modifications required

Modification of the model

- Within standard (invariant) random matrix ensembles : PT distribution = **theorem**
- Modifications required

A. Volya, H. A. Weidenmüller, and V. Zelevinsky PRL **115**, 052501 (2015)
Neutron resonance widths and the Porter-Thomas distribution

Realistic model of nuclear *s*-wave resonances : $M_{ij} = G_{ij}^{(\beta)} + Z\,\delta_{i1}\delta_{j1}$

$G_{ij}^{(\beta)}$ = standard random matrix, $\beta = 1 \to$ GOE, $\beta = 2 \to$ GUE
$Z\,\delta_{i1}\delta_{j1}$ = interaction which couples resonances to decay channels

Modification of the model

- Within standard (invariant) random matrix ensembles : PT distribution = **theorem**
- Modifications required

A. Volya, H. A. Weidenmüller, and V. Zelevinsky — PRL **115**, 052501 (2015)
Neutron resonance widths and the Porter-Thomas distribution

Realistic model of nuclear s-wave resonances : $M_{ij} = G_{ij}^{(\beta)} + Z\,\delta_{i1}\delta_{j1}$

$G_{ij}^{(\beta)}$ = standard random matrix, $\beta = 1 \to$ GOE, $\beta = 2 \to$ GUE
$Z\,\delta_{i1}\delta_{j1}$ = interaction which couples resonances to decay channels

- Distribution (GOE or GUE) : $P(G_{ij}) \sim \exp\left(-\frac{\beta}{4\sigma^2}\mathrm{Tr}\,(G\,G^{\dagger})\right) \quad \beta = 1, 2$
- Density = Wigner semicircle law : $\rho_W(E) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 N - E^2}$
- Z may be complex. ReZ is due to coupling to neutron channel and is "immanent to the theory". ImZ is related with hypothetical non-statistical gamma decays
- Here only Hermitian matrix M is considered, Z = real. To get a nontrivial limit

$$\kappa = \frac{Z}{\sigma\sqrt{N}} \quad \text{remains constant when} \quad N \to \infty$$

Numerically : distribution of $x_\alpha = N|\Psi_1(\alpha)|^2$ does deviate from the PT law

Known results

Real $Z \longrightarrow$ local spectral statistics is independent on Z (after unfolding)

- For **GUE** there is a direct proof using the Itzykson-Zuber integral
  E. Brezin and S. Hikami, Nucl. Phys. B **479** 697 (1996)
- For **GOE** ?

Imaginary $Z = \frac{i\gamma}{\sigma\sqrt{N}} \longrightarrow$ there exists an exact solution (using non-linear sigma model) for the distribution of imaginary parts of eigenvalues, $E = e + \frac{i}{2}\Gamma$, $y = \pi\Gamma/\rho_W(e)$,

- **GUE**, Y. V. Fyodorov and H.-J. Sommers, JETP Lett. **63**, 1036 (1996)

$$P(y) = -\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{-yg}\frac{\sinh y}{y}\right), \qquad g = \frac{1}{2}(\gamma + \gamma^{-1})$$

- **GOE**, H.-J. Sommers, Y. V. Fyodorov, and M. Titov, J. Phys. A **32**, L77 (1999)

$$P(y) = \frac{1}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\int_{-1}^{1}(1-\lambda^2)\mathrm{e}^{2\lambda y}(g-\lambda)F(\lambda, y)\mathrm{d}\lambda$$

$$F(\lambda, y) = \int_g^\infty \frac{\mathrm{e}^{-yp_1}\,\mathrm{d}p_1}{(\lambda - p_1)^2\sqrt{(p_1^2-1)(p_1-g)}}\int_1^g \frac{\mathrm{e}^{-yp_2}\,\mathrm{d}p_2}{(\lambda-p_2)^2\sqrt{(p_2^2-1)(g-p_2)}}$$

Rank-one formalism $\quad M_{ij} = G_{ij}^{(\beta)} + Z\,\delta_{i1}\delta_{j1}$

- Two $N \times N$ Hermitian matrices G and M differ by a rank-one interaction

$$M_{ij} = G_{ij} + v_i^* v_j, \qquad v_j = \sqrt{Z}(1, 0, \dots, 0)$$

- Eigenvalues and eigenfunctions for Hermitian matrices assumed to be orthogonal

$$\sum_{j=1}^{N} G_{ij}\Phi_j(\alpha) = e_\alpha \Phi_i(\alpha), \qquad \sum_{j=1}^{N} M_{ij}\Psi_j(\alpha) = E_\alpha \Psi_i(\alpha)$$

- Expansions : $\quad \Psi_j(\alpha) = \sum_{\beta=1}^{N} C_{\alpha\beta}\Phi_j(\beta), \qquad \Phi_j(\alpha) = \sum_{\beta=1}^{N} C_{\alpha\beta}^{-1}\Psi_j(\beta)$

Rank-one formalism $M_{ij} = G_{ij}^{(\beta)} + Z\,\delta_{i1}\delta_{j1}$

- Two $N \times N$ Hermitian matrices G and M differ by a rank-one interaction

$$M_{ij} = G_{ij} + v_i^* v_j, \qquad v_j = \sqrt{Z}(1, 0, \dots, 0)$$

- Eigenvalues and eigenfunctions for Hermitian matrices assumed to be orthogonal

$$\sum_{j=1}^{N} G_{ij}\Phi_j(\alpha) = e_\alpha \Phi_i(\alpha), \qquad \sum_{j=1}^{N} M_{ij}\Psi_j(\alpha) = E_\alpha \Psi_i(\alpha)$$

- Expansions : $\Psi_j(\alpha) = \sum_{\beta=1}^{N} C_{\alpha\beta}\Phi_j(\beta), \qquad \Phi_j(\alpha) = \sum_{\beta=1}^{N} C_{\alpha\beta}^{-1}\Psi_j(\beta)$
- Consequences

$$C_{\alpha\,\beta} = \frac{a_\alpha b_\beta^*}{E_\alpha - e_\beta}, \qquad b_\beta = \sum_{j=1}^{N} v_j \Phi_j(\beta), \quad a_\alpha = \sum_\beta C_{\alpha\beta} b_\beta$$

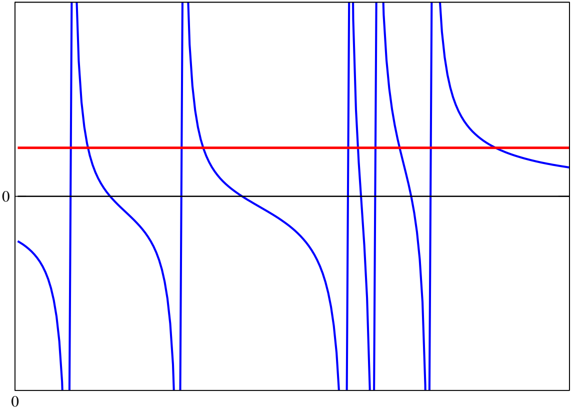
- Quantisation condition

$$\sum_\beta \frac{|b_\beta|^2}{E_\alpha - e_\beta} = 1, \quad E_\alpha, e_\alpha$$

- Complementary relations

$$C_{\alpha\,\beta}^{-1} = \frac{b_\alpha a_\alpha^*}{E_\beta - e_\alpha} \qquad \sum_\beta \frac{|a_\beta|^2}{e_\alpha - E_\beta} = -1$$

Schematic picture



Quantisation conditions : $\displaystyle\sum_\beta \frac{|b_\beta|^2}{E_\alpha - e_\beta} = 1, \qquad \sum_\beta \frac{|a_\beta|^2}{e_\alpha - E_\beta} = -1$

New and old eigenvalues are interlacing

$e_1 \leq e_2 \leq e_N,\ e_\alpha \leq E_\alpha \leq e_{\alpha+1},\ e_N \leq E_N$ (for $Z > 0$)

Main relations

- Solve for numerators

$$|b_\alpha|^2 = \frac{\prod_\gamma (E_\gamma - e_\alpha)}{\prod_{\gamma\neq\alpha}(e_\gamma - e_\alpha)}, \qquad |a_\alpha|^2 = -\frac{\prod_\gamma (e_\gamma - E_\alpha)}{\prod_{\gamma\neq\alpha}(E_\gamma - E_\alpha)}$$

Theorem (x_m, y_m are known)

$$\sum_{m=1}^{N} \frac{b_m}{x_m - y_n} = 1, \qquad \longrightarrow \qquad b_m = \frac{\prod_n (x_m - y_n)}{\prod_{s\neq m}(x_m - x_s)}$$

Proof (one among many others)

- Consider the function

$$f_n(x) = \frac{\prod_{r\neq n}(x - y_r)}{\prod_s (x - x_s)} = \frac{\prod_r (x - y_r)}{(x - y_n)\prod_s (x - x_s)}$$

- Asymptotically $f_n(x) \to 1/x \longrightarrow$ integral over a large contour encircling all poles equals 1
- Rewriting this integral as the sum over all finite poles gives

$$1 = \sum_m \frac{\prod_r (x_m - y_r)}{(x_m - y_n)\prod_{s\neq m}(x_m - x_s)} \qquad \text{Q.E.D.}$$

- Many different relations. E.g. $\sum_\beta |b_\beta|^2 = \sum_\beta |a_\beta|^2 = \sum_\beta (E_\beta - e_\beta)$

Initial probability distribution

- Eigenvalues e_α and eigenfunctions $\Phi_1(\alpha)$ of matrix G^β are distributed as in standard random matrix ensembles

$$P(\{e_\alpha\},\,\{r_\alpha\}) \sim \prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta\left(\sum_\alpha r_\alpha - 1\right) \exp(-V(\{e_\alpha\}))$$

- $r_\alpha = |\Phi_1(\alpha)|^2$ (the same is valid for other components as well)
- $V(\{e_\alpha\})$ = confinement term. For standard Gaussian ensembles

$$V(\{e_\alpha\}) = \frac{\beta}{4\sigma^2}\sum_\alpha e_\alpha^2$$

- Mean density of matrix eigenvalues = Wigner semicircle law

$$\rho_W(E) = \frac{1}{2\pi\sigma^2}\sqrt{4N\sigma^2 - E^2}$$

Initial probability distribution

- Eigenvalues e_α and eigenfunctions $\Phi_1(\alpha)$ of matrix G^β are distributed as in standard random matrix ensembles

$$P(\{e_\alpha\},\{r_\alpha\}) \sim \prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta\left(\sum_\alpha r_\alpha - 1\right) \exp(-V(\{e_\alpha\}))$$

- $r_\alpha = |\Phi_1(\alpha)|^2$ (the same is valid for other components as well)
- $V(\{e_\alpha\})$ = confinement term. For standard Gaussian ensembles

$$V(\{e_\alpha\}) = \frac{\beta}{4\sigma^2} \sum_\alpha e_\alpha^2$$

- Mean density of matrix eigenvalues = Wigner semicircle law

$$\rho_W(E) = \frac{1}{2\pi\sigma^2}\sqrt{4N\sigma^2 - E^2}$$

Main question

How new eigenvectors $\Psi_1(\alpha)$ and new eigenvalues E_α are distributed ?

Two steps of changing variables : $(e_\alpha, \Phi_1(\alpha)) \longrightarrow (e_\alpha, E_\alpha) \longrightarrow (\Psi_1(\alpha), E_\alpha)$

Joint distribution of old and new eigenvalues

$$P(\{e_\alpha\},\,\{r_\alpha\})\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha \sim \prod_{\alpha<\gamma}|e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1}\delta(\sum_\alpha r_\alpha - 1)\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha$$

- $b_\alpha = \sum_{j=1}^N v_j \Phi_j(\beta),\ v_j = \sqrt{Z}(1,0,\ldots,0)$ $\longrightarrow$ $\boxed{|b_\alpha|^2 = Z|\Phi_1(\alpha)|^2}$
- Expression of r_α through old and new eigenvalues,

$$r_\alpha \equiv |\Phi_1(\alpha)|^2 = \frac{1}{Z}|b_\alpha|^2 = \frac{1}{Z}\frac{\prod_\gamma (E_\gamma - e_\alpha)}{\prod_{\gamma\neq\alpha}(e_\gamma - e_\alpha)}$$

Joint distribution of old and new eigenvalues

$$P(\{e_\alpha\},\{r_\alpha\})\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha \sim \prod_{\alpha<\gamma}|e_\gamma-e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1}\delta(\sum_\alpha r_\alpha -1)\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha$$

- $b_\alpha = \sum_{j=1}^N v_j \Phi_j(\beta)$, $v_j=\sqrt{Z}(1,0,\dots,0)$ $\longrightarrow$ $\boxed{|b_\alpha|^2 = Z|\Phi_1(\alpha)|^2}$
- Expression of r_α through old and new eigenvalues,

$$r_\alpha \equiv |\Phi_1(\alpha)|^2 = \frac{1}{Z}|b_\alpha|^2 = \frac{1}{Z}\frac{\prod_\gamma(E_\gamma-e_\alpha)}{\prod_{\gamma\neq\alpha}(e_\gamma-e_\alpha)}$$

- **Change of variables :** N quantities $r_\alpha \longrightarrow N$ quantities E_α
- Derivatives : $\partial r_\alpha/\partial E_\gamma = r_\alpha/(E_\gamma-e_\alpha)$
- Cauchy determinant : $\det(\frac{1}{x_m-y_n}) = \frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_j-y_i)}{\prod_{i,j}(x_i-y_j)}$
- Jacobian

$$\det\left(\frac{\partial r_\alpha}{\partial E_\gamma}\right) = (\prod_\alpha r_\alpha)\det\left(\frac{1}{E_\gamma-e_\alpha}\right) = (\prod_\alpha r_\alpha)\frac{\prod_{\alpha<\beta}(E_\alpha-E_\beta)(e_\beta-e_\alpha)}{\prod_{\alpha,\beta}(E_\alpha-e_\beta)}$$

$$= \frac{1}{Z^N}\prod_{\alpha<\gamma}\frac{(E_\gamma-E_\alpha)}{(e_\alpha-e_\gamma)}, \qquad \sum_\alpha r_\alpha = \frac{1}{Z}\sum_\alpha(E_\alpha-e_\alpha)$$

Change from old eigenvalues to new eigenfunctions

I. L. Aleiner and K. A. Matveev *Shifts of random energy levels by a local perturbation*, PRL **80**, 814 (1998)

$$\tilde{P}(\{e_\alpha\},\{E_\alpha\}) \sim \frac{\prod_{\gamma>\alpha}(e_\gamma - e_\alpha)(E_\gamma - E_\alpha)}{\prod_{\gamma,\alpha}|e_\gamma - E_\alpha|^{\beta/2-1}}\delta\left(\sum_\alpha (E_\alpha - e_\alpha) - Z\right)\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}E_\alpha$$

- $\boxed{\Psi_1(\alpha) = \sum_\beta C_{\alpha\beta}\Phi_1(\beta) = \frac{1}{\sqrt{Z}}\sum_\beta C_{\alpha\beta}b_\beta = \frac{a_\alpha}{\sqrt{Z}}\sum_\beta \frac{|b_\beta|^2}{E_\alpha - e_\beta} = \frac{a_\alpha}{\sqrt{Z}}}$
- Expression of r_α through old and new eigenvalues

$$z_\alpha = |\Psi_1(\alpha)|^2 = \frac{1}{Z}|a_\alpha|^2 = \frac{1}{Z}\frac{\prod_\gamma (E_\alpha - e_\gamma)}{\prod_{\gamma\neq\alpha}(E_\alpha - E_\gamma)}$$

Change from old eigenvalues to new eigenfunctions

I. L. Aleiner and K. A. Matveev *Shifts of random energy levels by a local perturbation*, PRL **80**, 814 (1998)

$$\tilde{P}(\{e_\alpha\},\{E_\alpha\}) \sim \frac{\prod_{\gamma>\alpha}(e_\gamma - e_\alpha)(E_\gamma - E_\alpha)}{\prod_{\gamma,\alpha}|e_\gamma - E_\alpha|^{\beta/2-1}}\delta\left(\sum_\alpha (E_\alpha - e_\alpha) - Z\right)\prod_\alpha \mathrm{d}e_\alpha \mathrm{d}E_\alpha$$

- $$\boxed{\Psi_1(\alpha) = \sum_\beta C_{\alpha\beta}\Phi_1(\beta) = \frac{1}{\sqrt{Z}}\sum_\beta C_{\alpha\beta}b_\beta = \frac{a_\alpha}{\sqrt{Z}}\sum_\beta \frac{|b_\beta|^2}{E_\alpha - e_\beta} = \frac{a_\alpha}{\sqrt{Z}}}$$
- Expression of r_α through old and new eigenvalues

$$z_\alpha = |\Psi_1(\alpha)|^2 = \frac{1}{Z}|a_\alpha|^2 = \frac{1}{Z}\frac{\prod_\gamma (E_\alpha - e_\gamma)}{\prod_{\gamma\neq\alpha}(E_\alpha - E_\gamma)}$$

- N quantities $r_\alpha \longrightarrow N$ quantities e_α. Derivatives : $\frac{\partial z_\alpha}{\partial e_\gamma} = -\frac{z_\alpha}{E_\alpha - e_\gamma}$
- Jacobian

$$\det\left(\frac{\partial z_\alpha}{\partial e_\beta}\right) = \frac{1}{Z^N}\prod_{\alpha<\gamma}\frac{(e_\gamma - e_\alpha)}{(E_\alpha - E_\gamma)}$$

- Identities

$$\prod_{\alpha,\gamma}(E_\alpha - e_\gamma) = \prod_\alpha z_\alpha \Big(\prod_{\alpha<\gamma}(E_\alpha - E_\gamma)\Big)^2, \qquad \sum_\alpha z_\alpha = \frac{1}{Z}\sum_\alpha (E_\alpha - e_\alpha)$$

Distribution of new eigenvalues and new eigenfunctions

New joint distribution = the old one

$$\prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta(\sum_\alpha r_\alpha - 1) \prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha =$$
$$= \prod_{\alpha<\gamma} |E_\gamma - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1) \prod_\alpha \mathrm{d}E_\alpha \mathrm{d}z_\alpha$$

Distribution of new eigenvalues and new eigenfunctions

New joint distribution = the old one

$$\prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta(\sum_\alpha r_\alpha - 1) \prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha =$$

$$= \prod_{\alpha<\gamma} |E_\gamma - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1) \prod_\alpha \mathrm{d}E_\alpha \mathrm{d}z_\alpha$$

Symmetry : $G_{ij} = M_{ij} - v_i^* v_j$ $e_\alpha \leftrightarrow -E_\alpha$ $\tilde{P}(\{e_\alpha\}, \{E_\alpha\})$ is symmetric

Distribution of new eigenvalues and new eigenfunctions

New joint distribution = the old one

$$\prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta(\sum_\alpha r_\alpha - 1) \prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha =$$

$$= \prod_{\alpha<\gamma} |E_\gamma - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1) \prod_\alpha \mathrm{d}E_\alpha \mathrm{d}z_\alpha$$

Symmetry : $G_{ij} = M_{ij} - v_i^* v_j$ $\quad e_\alpha \leftrightarrow -E_\alpha \quad$ $\tilde{P}(\{e_\alpha\}, \{E_\alpha\})$ is symmetric

- Symmetry is valid only **without** the confinement term : $\exp(-\beta \mathrm{Tr}\, GG^\dagger/4\sigma^2)$
- $\mathrm{Tr}\, GG^\dagger = \mathrm{Tr}\,(M_{ij} - Z\delta_{i1}\delta_{j1})^2$, $M_{ij} = \sum_\alpha E_\alpha \Psi_i(\alpha)\Psi_j^*(\alpha)$, $M_{11} = \sum_\alpha E_\alpha |\Psi_1(\alpha)|^2$
- $\sum_\alpha e_\alpha^2 = \sum_\alpha E_\alpha^2 - 2Z\sum_\alpha E_\alpha z_\alpha + Z^2$

Full joint distribution of new eigenvalues E_α and new eigenvectors, $z_\alpha \equiv |\Psi_1(\alpha)|^2$

$$P(\{E_\alpha\}, \{z_\alpha\}) \sim \prod_{\alpha<\beta} |E_\beta - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1)$$

$$\times \exp\left[-\frac{\beta}{4\sigma^2}\Big(\sum_\alpha E_\alpha^2 - 2Z\sum_\alpha E_\alpha z_\alpha\Big)\right]$$

Mean level density

L. A. Pastur, *On the spectrum of random matrices*, TMP, 10, 67 (1972)

- Mean Green function

$$\bar{G}(E) = \frac{1}{N}\langle \mathrm{Tr}\,(E-M)^{-1}\rangle$$

- The Pastur equation

$$\bar{G}(E) = \frac{1}{N}\frac{1}{E-Z-\sigma^2 N\bar{G}(E)} + \frac{N-1}{N}\frac{1}{E-\sigma^2 N\bar{G}(E)}$$

- Solution in 2 lowest orders in N^{-1}

$$\bar{G}(E) = \bar{G}_0(E)\left(1+\frac{Z\bar{G}_0(E)}{N(1-Z\bar{G}_0(E))}\right)$$

- $\bar{G}_0(E) =$ the mean Green function for standard RM ensembles

$$\bar{G}_0(E) = \frac{1}{E-\sigma^2 N\bar{G}_0(E)}, \qquad \bar{G}_0(E) = \frac{E-\sqrt{E^2-4\sigma^2 N}}{2\sigma^2 N}$$

Formation of a collective state (= outlier)

- Mean level density

$$\rho(E) = \frac{1}{\pi}\mathrm{Tr}\, G(E - \mathrm{i}0),\ \ G_0(E) = \frac{E - \sqrt{E^2 - 4\sigma^2 N}}{2\sigma^2 N} \longrightarrow \rho_0(E) = \frac{1}{2\pi\sigma^2 N}\sqrt{4N\sigma^2 - E^2}$$

- For perturbed problem $E = 2\sigma\sqrt{N}\cos\phi$

$$\rho(\phi) = \Big(\frac{N}{2\pi} + \frac{2\kappa(2\cos\phi - \kappa)}{\pi(\kappa^2 - 2\kappa\cos\phi + 1)}\Big)\sin^2\phi$$

Formation of a collective state (= outlier)

- Mean level density

$$\rho(E)=\frac{1}{\pi}\mathrm{Tr}\, G(E-\mathrm{i}0),\ \ G_0(E)=\frac{E-\sqrt{E^2-4\sigma^2 N}}{2\sigma^2 N}\longrightarrow \rho_0(E)=\frac{1}{2\pi\sigma^2 N}\sqrt{4N\sigma^2-E^2}$$

- For perturbed problem $E=2\sigma\sqrt{N}\cos\phi$

$$\rho(\phi)=\Big(\frac{N}{2\pi}+\frac{2\kappa(2\cos\phi-\kappa)}{\pi(\kappa^2-2\kappa\cos\phi+1)}\Big)\sin^2\phi$$

- Total level number : $\int_0^{\pi}\rho(\phi)\mathrm{d}\phi=\begin{cases} N & \kappa^2<1\\ N-1+1/\kappa^2 & \kappa^2>1\end{cases}$

Formation of a collective state (= outlier)

- Mean level density

$$\rho(E)=\frac{1}{\pi}\mathrm{Tr}\,G(E-\mathrm{i}0),\;\; G_0(E)=\frac{E-\sqrt{E^2-4\sigma^2 N}}{2\sigma^2 N}\longrightarrow \rho_0(E)=\frac{1}{2\pi\sigma^2 N}\sqrt{4N\sigma^2-E^2}$$

- For perturbed problem $E=2\sigma\sqrt{N}\cos\phi$

$$\rho(\phi)=\Big(\frac{N}{2\pi}+\frac{2\kappa(2\cos\phi-\kappa)}{\pi(\kappa^2-2\kappa\cos\phi+1)}\Big)\sin^2\phi$$

- Total level number : $\int_0^\pi \rho(\phi)\mathrm{d}\phi=\begin{cases} N & \kappa^2<1 \\ N-1+1/\kappa^2 & \kappa^2>1\end{cases}$
- **Collective state** (outlier) appears at $\kappa^2>1$

$$E_c-Z-\sigma^2 N\bar{G}_0(E_c)=0\longrightarrow \boxed{E_c=\sigma\sqrt{N}(\kappa+\frac{1}{\kappa})}$$

- Close to E_c $\bar{G}(E)\approx\bar{G}_0(E)+\delta G(E)$ $(a=2\sigma(1-\kappa^{-2}))$

$$\delta G(E)=\frac{1}{N}[\frac{\kappa^2}{\kappa^2-1}(E-E_c)-\sigma^2 N\delta G(E)]^{-1}\longrightarrow \delta\rho(E)=\frac{1}{\pi\sigma a}\sqrt{a^2-(E-E_c)^2}$$

- $\int\delta\rho(E)\mathrm{d}E=1-\kappa^{-2}$. Total : $\int\rho(E)\mathrm{d}E=N$

Formation of a collective state (= outlier)

- Mean level density

$$\rho(E) = \frac{1}{\pi}\mathrm{Tr}\, G(E-\mathrm{i}0),\;\; G_0(E) = \frac{E-\sqrt{E^2-4\sigma^2 N}}{2\sigma^2 N} \longrightarrow \rho_0(E) = \frac{1}{2\pi\sigma^2 N}\sqrt{4N\sigma^2 - E^2}$$

- For perturbed problem $E = 2\sigma\sqrt{N}\cos\phi$

$$\rho(\phi) = \Big(\frac{N}{2\pi} + \frac{2\kappa(2\cos\phi - \kappa)}{\pi(\kappa^2 - 2\kappa\cos\phi + 1)}\Big)\sin^2\phi$$

- Total level number : $\int_0^{\pi}\rho(\phi)\mathrm{d}\phi = \begin{cases} N & \kappa^2 < 1 \\ N - 1 + 1/\kappa^2 & \kappa^2 > 1 \end{cases}$
- **Collective state** (outlier) appears at $\kappa^2 > 1$

$$E_c - Z - \sigma^2 N\bar{G}_0(E_c) = 0 \longrightarrow \boxed{E_c = \sigma\sqrt{N}(\kappa + \frac{1}{\kappa})}$$

- Close to E_c $\bar{G}(E) \approx \bar{G}_0(E) + \delta G(E)$ $(a = 2\sigma(1-\kappa^{-2}))$

$$\delta G(E) = \frac{1}{N}\Big[\frac{\kappa^2}{\kappa^2-1}(E-E_c) - \sigma^2 N\delta G(E)\Big]^{-1} \longrightarrow \delta\rho(E) = \frac{1}{\pi\sigma a}\sqrt{a^2 - (E-E_c)^2}$$

- $\int\delta\rho(E)\mathrm{d}E = 1 - \kappa^{-2}$. Total : $\int\rho(E)\mathrm{d}E = N$
- **Exact result :** $E - E_c$ = Gaussian variable with mean= 0, variance= $\frac{2(\kappa^2-1)\sigma^2}{\beta\kappa^2}$

Case $\kappa^2 < 1$

- $z_\alpha = |\Psi_1(\alpha)|^2$ for all energies E_α are $\mathcal{O}(N^{-1})$
- Condition $\sum_\alpha z_\alpha = 1$ can be taken into account by the Lagrange multiplier

$$\delta(\sum_\alpha z_\alpha - 1) \longrightarrow \exp(-\mu(\sum_\alpha z_\alpha - 1))$$

- $z_\alpha =$ independent, each $z_\alpha \equiv z(E_\alpha)$ has PDF

$$P(z,E) = \left(\mu - \frac{Z}{2\sigma^2}E\right)^{\beta/2} (\pi z)^{\beta/2-1} \exp\left(-\left(\mu - \frac{\beta Z}{2\sigma^2}E\right)z\right)$$

- μ has to be calculated from normalisation : $\sum_\alpha z_\alpha = 1$

$$\sum_\alpha \int_0^\infty z\, P(z, E_\alpha)\, \mathrm{d}z = 1, \quad \sum_\alpha \frac{1}{\mathcal{E} - E_\alpha} = \frac{Z}{\sigma^2}, \quad \mathcal{E} = \frac{2\mu\sigma^2}{\beta Z}$$

- The sum is mean unperturbed Green function, $\bar{G}_0(E)$

$$\bar{G}_0(E) = \frac{1}{N}\langle \mathrm{Tr}\,(E-G)^{-1}\rangle = \frac{E - \sqrt{E^2 - 4\sigma^2 N}}{2\sigma^2 N} \longrightarrow \boxed{\mu = \beta N \frac{\kappa^2+1}{2}}$$

- Distribution of perturbed problem = **local PT distribution**

$$P_\beta(x) = \frac{1}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right), \quad \boxed{I(E) = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}}$$

Case $\kappa^2 > 1$

- When $\kappa^2 > 1$ there exists one collective state.

$z_c = |\Psi_1(E_c)|^2 = \mathcal{O}(1)$

- Other z_α are $\mathcal{O}(N^{-1})$ and their distribution is the local PT distribution

$$P(z,E) = \left(\mu - \frac{Z}{2\sigma^2}E\right)^{\beta/2}(\pi z)^{\beta/2-1}\exp\left(-\left(\mu - \frac{\beta Z}{2\sigma^2}E\right)z\right)$$

- The only difference is that their normalisation is $\sum_{\alpha=1}^{N-1} z_\alpha = 1 - z_c \longrightarrow$

$$\mu(z_c) = \frac{\beta N}{2}\left(\kappa^2(1-z_c) + \frac{1}{1-z_c}\right)$$

Case $\kappa^2 > 1$

- When $\kappa^2 > 1$ there exists one collective state. $z_c = |\Psi_1(E_c)|^2 = \mathcal{O}(1)$
- Other z_α are $\mathcal{O}(N^{-1})$ and their distribution is the local PT distribution

$$P(z,E) = \left(\mu - \frac{Z}{2\sigma^2}E\right)^{\beta/2} (\pi z)^{\beta/2-1} \exp\left(-\left(\mu - \frac{\beta Z}{2\sigma^2}E\right)z\right)$$

- The only difference is that their normalisation is $\sum_{\alpha=1}^{N-1} z_\alpha = 1 - z_c \longrightarrow$

$$\mu(z_c) = \frac{\beta N}{2}\left(\kappa^2(1-z_c) + \frac{1}{1-z_c}\right)$$

Calculation of E_c and z_c

Joint distribution of $E \equiv E_c$ and $r \equiv z_c$

$$P(E,r) \sim \prod_\alpha |E - E_\alpha|^\beta (\mu(r) - \frac{\beta\sqrt{N}\kappa}{2\sigma}E_\alpha)^{-\beta/2}$$

$$\times \quad \exp\left[-\frac{\beta}{4\sigma^2}E^2 + \frac{\beta\sqrt{N}\kappa}{2\sigma}Er - (1-r)\mu(r)\right] \sim \mathrm{e}^{-N\beta\, F(E,r)}$$

$$F(E,r) = -\frac{1}{N}\sum_\alpha \ln(E - E_\alpha) + \frac{1}{4\sigma^2 N}E^2 + \frac{1}{2N}\sum_\alpha \ln\left(\nu(r) - \frac{\kappa}{2\sigma\sqrt{N}}E_\alpha\right)$$

$$- \frac{\kappa}{2\sigma\sqrt{N}}Er - (1-r)\nu(r), \qquad \nu(r) = \frac{1}{2}\left(\kappa^2(1-r) + \frac{1}{1-r}\right)$$

Final answer

- Saddle-point equations $\frac{\partial F(E,r)}{\partial E} = 0$ $\frac{\partial F(E,r)}{\partial r} = 0$ plus mean Green function $\longrightarrow$

$$\boxed{E_c = \sigma\sqrt{N}\left(\kappa + \frac{1}{\kappa}\right), \qquad z_c = 1 - \frac{1}{\kappa^2}}$$

- For both $\kappa^2 < 1$ and $\kappa^2 > 1$

$$\boxed{\mu = \beta N \frac{\kappa^2 + 1}{2}}$$

Final answer

- Saddle-point equations $\frac{\partial F(E,r)}{\partial E} = 0$ $\frac{\partial F(E,r)}{\partial r} = 0$ plus mean Green function $\longrightarrow$

$$\boxed{E_c = \sigma\sqrt{N}\left(\kappa + \frac{1}{\kappa}\right), \qquad z_c = 1 - \frac{1}{\kappa^2}}$$

- For both $\kappa^2 < 1$ and $\kappa^2 > 1$

$$\boxed{\mu = \beta N \frac{\kappa^2 + 1}{2}}$$

Local Porter-Thomas distribution

For $-2\sigma\sqrt{N} \leq E \leq 2\sigma\sqrt{N}$ and all κ

$$P_\beta(x, E) = \frac{1}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right)$$

$$I(E) \equiv N\langle|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}} E\right)^{-1}$$

Final answer

- Saddle-point equations $\frac{\partial F(E,r)}{\partial E} = 0$ $\frac{\partial F(E,r)}{\partial r} = 0$ plus mean Green function $\longrightarrow$

$$\boxed{E_c = \sigma\sqrt{N}\left(\kappa + \frac{1}{\kappa}\right), \qquad z_c = 1 - \frac{1}{\kappa^2}}$$

- For both $\kappa^2 < 1$ and $\kappa^2 > 1$

$$\boxed{\mu = \beta N \frac{\kappa^2 + 1}{2}}$$

Local Porter-Thomas distribution

For $-2\sigma\sqrt{N} \le E \le 2\sigma\sqrt{N}$ and all κ

$$P_\beta(x, E) = \frac{1}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right)$$

$$I(E) \equiv N\langle|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

$$N\sum_{E_\alpha}\langle|\Psi_1(E_\alpha)|^2\rangle \approx \int_{-2\sigma\sqrt{N}}^{2\sigma\sqrt{N}} \rho_W(E)\left[\int_0^\infty z\, P_\beta(z, E_\alpha)\,\mathrm{d}z\right]\mathrm{d}E = \begin{cases} 1 & \kappa^2 < 1 \\ 1/\kappa^2 & \kappa^2 > 1 \end{cases}$$

Final answer

- Saddle-point equations $\frac{\partial F(E,r)}{\partial E} = 0$ $\frac{\partial F(E,r)}{\partial r} = 0$ plus mean Green function $\longrightarrow$

$$\boxed{E_c = \sigma\sqrt{N}\left(\kappa + \frac{1}{\kappa}\right), \qquad z_c = 1 - \frac{1}{\kappa^2}}$$

- For both $\kappa^2 < 1$ and $\kappa^2 > 1$

$$\boxed{\mu = \beta N \frac{\kappa^2 + 1}{2}}$$

Local Porter-Thomas distribution

For $-2\sigma\sqrt{N} \le E \le 2\sigma\sqrt{N}$ and all κ

$$P_\beta(x, E) = \frac{1}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right)$$

$$I(E) \equiv N\langle|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

$$N\sum_{E_\alpha}\langle|\Psi_1(E_\alpha)|^2\rangle \approx \int_{-2\sigma\sqrt{N}}^{2\sigma\sqrt{N}} \rho_W(E)\left[\int_0^\infty z\, P_\beta(z, E_\alpha)\,\mathrm{d}z\right]\mathrm{d}E = \begin{cases} 1 & \kappa^2 < 1 \\ 1/\kappa^2 & \kappa^2 > 1 \end{cases}$$

Cf. *H. A. Weidenmüler, PRL* **105**, *232501 (2010)* :
$I(E)$ is associated with the existence of a virtual or weakly bound state near threshold

Another approach based on the averaged Green function

- The exact Schur complement formula $(\hat{G}(E) \equiv (E - M)^{-1})$

$$\hat{G}_{11}(E) = (E - Z - \sum_{j,k\neq 1} M_{1j}\tilde{G}_{jk}M_{k1})^{-1}$$

- $\tilde{G}_{jk}$ = the Green function of $(N-1)\times(N-1)$ matrix obtained from matrix M by removing line 1 and row 1 = standard $(N-1)\times(N-1)$ random matrix, $\tilde{H}$
- For G^{β} matrices $M_{1j}M_{k1} \underset{N\to\infty}{\longrightarrow} \sigma^2\delta_{jk}$
- Therefore

$$G_{11}(E) \underset{N\to\infty}{\longrightarrow} (E - Z - \sigma^2 G_0(E))^{-1}$$

- $G_0 = \mathrm{Tr}\,(E - \tilde{H})^{-1}$. For $N \to \infty$

$$G_0(E) \approx \frac{E - \sqrt{E^2 - 4\sigma^2 N}}{2\sigma^2}$$

- Eigenfunction expansion :

$$G_{ij}(E) = \sum_{\alpha} \frac{\Psi_i(\alpha)\Psi_j^*(\alpha)}{E - E_{\alpha}} \longrightarrow \langle|\Psi_1(E)|^2\rangle \approx \frac{\mathrm{Im}\, G_{11}(E - \mathrm{i}0)}{\pi\rho_W(E)}$$

- After some algebra :

$$I(E) \equiv N\langle|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

Generalisation for finite rank-interaction is straightforward but (i) outlier interaction requires more careful calculations and (ii) the Gaussian distribution cannot be proved

Large-window distribution

- Local Gaussian distribution only for $\Psi_1(E)$ in a small energy window $|\delta E| \ll \sigma\sqrt{N}$
- If $E_1 < E_\alpha < E_2$

$$\langle z_\alpha^q \rangle_{[E_1,E_2]} = \frac{c_\beta(q)}{\delta N} \int_{E_1}^{E_2} \rho_W(E) \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}} E \right)^{-q} \mathrm{d}E,\ \delta N = \int_{E_1}^{E_2} \rho_W(E)\mathrm{d}E$$

- $\rho_W(E) =$ Wigner's spectral density, $c_\beta(q) =$ the Gaussian moments
  $c_1(q) = \frac{2^q \Gamma(q+1/2)}{\sqrt{\pi}}, \qquad c_2(q) = \Gamma(q+1)$
- The full distribution $=$ weighted integral of local PT distributions

$$\mathcal{P}_\beta(x) = \frac{1}{\delta N} \int_{E_1}^{E_2} \frac{\rho_W(E)}{(2\pi x)^{1-\beta/2} (I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)} \right) \mathrm{d}E,\ I(E) = \frac{1}{\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}E}}$$

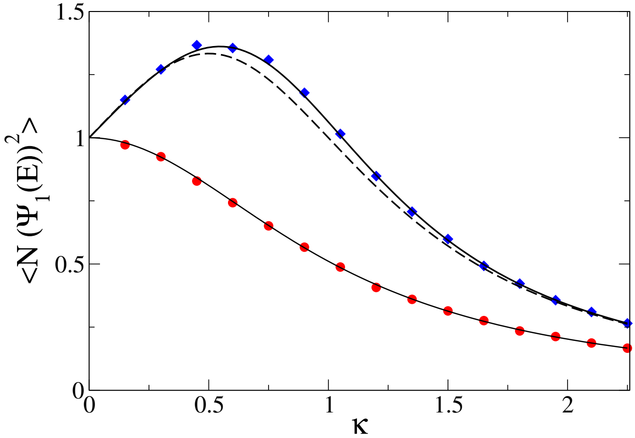
- If all states are included, $E_1 = -2\sigma\sqrt{N}$, $E_2 = 2\sigma\sqrt{N}$
- For $\beta = 1$ (GOE)

$$\mathcal{P}_1(x) = \sqrt{\frac{2}{\pi^3 x}} \int_0^\pi \mathrm{d}\phi\, \sin^2\phi \sqrt{\kappa^2 + 1 - 2\kappa\cos\phi}\, \mathrm{e}^{-\frac{1}{2}(\kappa^2 - 2\kappa\cos\phi + 1)x}$$

- For $\beta = 2$ (GUE)

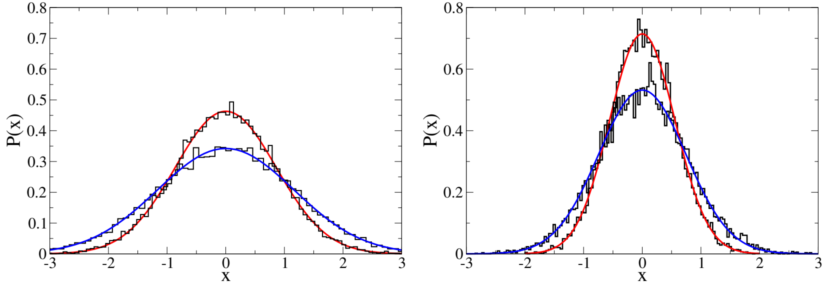
$$\mathcal{P}_2(x) = \frac{I_1(2\kappa x)}{\kappa x} \mathrm{e}^{-(\kappa^2+1)x}$$

Numerics : mean values of $N\langle(\Psi_1(E))^2\rangle$ for different κ



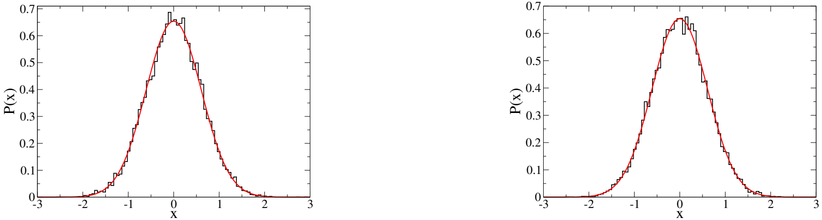
- Red circles are mean values for energies in the interval $[-\sqrt{N}/2, \sqrt{N}/2]$
- Blue diamonds are the same but for energies in the interval $[\sqrt{N}/2, 3\sqrt{N}/2]$
- Solid black lines = large-window theoretical predictions
- Dashed black line is the small-window predictions
- $N = 1000$, $\sigma = 1$, and each point is averaged over 50 random realisations

Numerics : distribution of $x = \sqrt{N}\Psi_1(E)$ for $\beta = 1$ (GOE)

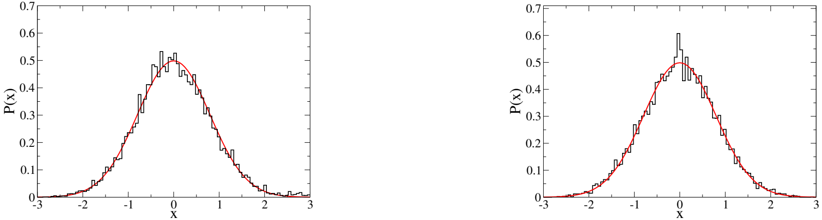


- Left : states with energies in $[-\sqrt{N}/2, \sqrt{N}/2]$ (lower curve) and $[\sqrt{N}/2, 3\sqrt{N}/2]$ (upper curve) for $\kappa = 0.6$
- Right : the same but for $\kappa = 1.5$
- Solid lines are the Gaussian fits with zero mean whose variance agrees well with the theoretical prediction

Numerics : distribution of $x = \sqrt{N}\,\mathrm{Re}\,\Psi_1(E)$ (left) and $x = \sqrt{N}\,\mathrm{Im}\,\Psi_1(E)$ (right) for $\beta = 2$ (GUE) and $\kappa = 0.6$

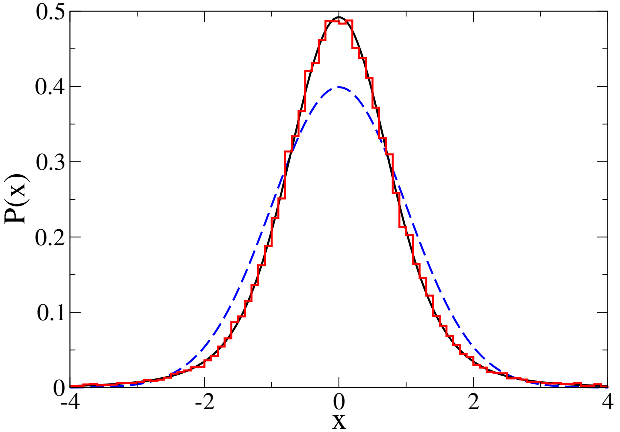


Energies in the interval $[-\sqrt{N}/2, \sqrt{N}/2]$



Energies in the interval $[\sqrt{N}/2, 3\sqrt{N}/2]$

Numerics : distribution of $x = \sqrt{N}\Psi_1(E)$ for all states, $\kappa = .8$, $\beta = 1$



- Blue dashed line is the PT distribution (Gaussian) : $P(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$
- Black solid line is theoretical prediction

$$P(x) = \sqrt{\frac{2}{\pi^3}} \int_0^{\pi} \mathrm{d}\phi \sin^2\phi \sqrt{\kappa^2 + 1 - 2\kappa\cos\phi}\, \mathrm{e}^{-\frac{1}{2}(\kappa^2 - 2\kappa\cos\phi + 1)x^2}$$

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\Big(-\frac{x}{2I}\Big), \quad P_2(x) = \frac{1}{I} \exp\Big(-\frac{x}{I}\Big), \quad x = N|\Psi|^2, \quad I = 1$$

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right), \quad P_2(x) = \frac{1}{I} \exp\left(-\frac{x}{I}\right), \quad x = N|\Psi|^2, \quad I = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$I(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}} E\right)^{-1}$$

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi l x}} \exp\Big(-\frac{x}{2l}\Big), \quad P_2(x) = \frac{1}{l}\exp\Big(-\frac{x}{l}\Big), \quad x = N|\Psi|^2, \quad l = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$l(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \Big(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\Big)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi l x}} \exp\Big(-\frac{x}{2l}\Big), \quad P_2(x) = \frac{1}{l} \exp\Big(-\frac{x}{l}\Big), \quad x = N|\Psi|^2, \quad l = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$l(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \Big(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\Big)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\Big(-\frac{x}{2I}\Big), \quad P_2(x) = \frac{1}{I} \exp\Big(-\frac{x}{I}\Big), \quad x = N|\Psi|^2, \quad I = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$I(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \Big(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}} E\Big)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions
- In the limit $N \to \infty$ all other components of eigenfunctions (except $\Psi_1(E)$) remain distributed according to the usual PT distribution

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x)=\frac{1}{\sqrt{2\pi I x}}\exp\Big(-\frac{x}{2I}\Big),\quad P_2(x)=\frac{1}{I}\exp\Big(-\frac{x}{I}\Big),\quad x=N|\Psi|^2,\quad I=1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x=N|\Psi_1(E)|^2$ has the same functional form but with

$$I(E)\equiv\langle N\,|\Psi_1(E)|^2\rangle=\Big(\kappa^2+1-\frac{\kappa}{\sigma\sqrt{N}}E\Big)^{-1}$$

- When $\kappa^2>1$ there exists one collective state whose mean energy is $E_c=\sigma\sqrt{N}(\kappa+\kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle=1-\kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions
- In the limit $N\to\infty$ all other components of eigenfunctions (except $\Psi_1(E)$) remain distributed according to the usual PT distribution
- Experimentally one has to fit a width distribution for a resonance in a small energy window to the local PT formula thus finding the $I(E)$ from the fit
- Taking into account together different resonances with different energies is not a sensitive way to investigate this phenomenon

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right), \quad P_2(x) = \frac{1}{I}\exp\left(-\frac{x}{I}\right), \quad x = N|\Psi|^2, \quad I = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$I(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions
- In the limit $N \to \infty$ all other components of eigenfunctions (except $\Psi_1(E)$) remain distributed according to the usual PT distribution
- Experimentally one has to fit a width distribution for a resonance in a small energy window to the local PT formula thus finding the $I(E)$ from the fit
- Taking into account together different resonances with different energies is not a sensitive way to investigate this phenomenon
- Introduction of rank-one interaction does not change local spectral statistics

Conclusion

- Standard PT distribution = eigenvectors of large random matrix are *i.i.d.r.v.*

$$P_1(x) = \frac{1}{\sqrt{2\pi I x}} \exp\left(-\frac{x}{2I}\right), \quad P_2(x) = \frac{1}{I}\exp\left(-\frac{x}{I}\right), \quad x = N|\Psi|^2, \quad I = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$I(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions
- In the limit $N \to \infty$ all other components of eigenfunctions (except $\Psi_1(E)$) remain distributed according to the usual PT distribution
- Experimentally one has to fit a width distribution for a resonance in a small energy window to the local PT formula thus finding the $I(E)$ from the fit
- Taking into account together different resonances with different energies is not a sensitive way to investigate this phenomenon
- Introduction of rank-one interaction does not change local spectral statistics
- Experimentally for resonances $\Delta_3(L)$ at small distances = RM prediction but deviates from it at $L \approx 40 - 70\ \bar{d}$ as for dynamical systems