

DISCRETE BETA ENSEMBLES

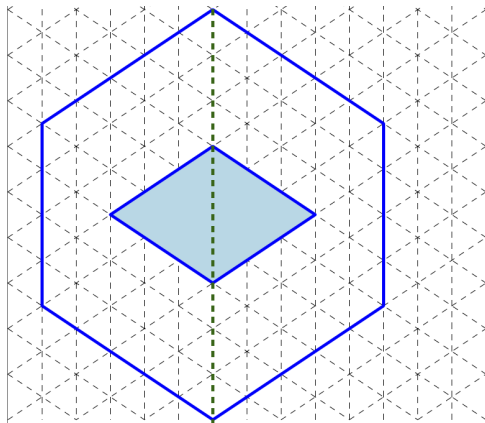
ALICE GUIONNET

CNRS and ÉNS Lyon

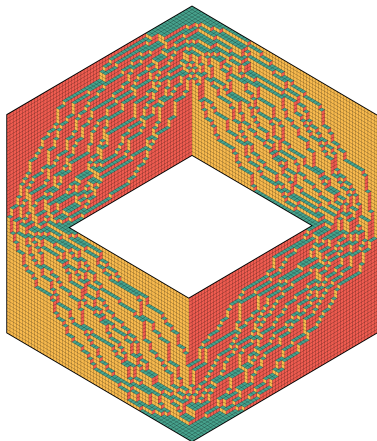
RMT workshop@Brunel

Joint works with A. Borodin, G. Borot, V. Gorin

Lozenges tiling of a 0

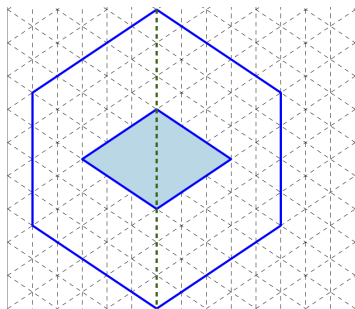


Lozenges tiling of a 0



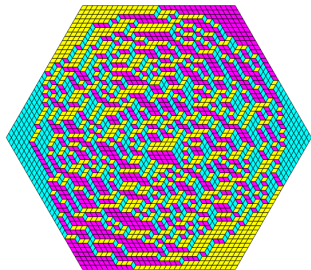
Outlook of the results

Take the tiling uniformly at random in a 0. Consider a vertical line in the middle of the tiling: it meets either horizontal tiles or the border of a tile in one of the two other directions. Let $\ell_1, \dots, \ell_N$ be the positions of the horizontal tiles.



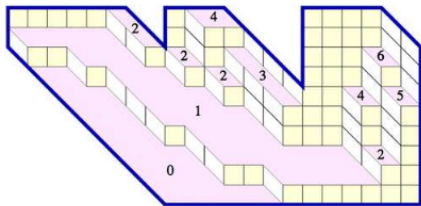
- ▶ $\frac{1}{N} \sum_{i=1}^N f(\ell_i/N) \mapsto \int f(x) d\mu(x)$ a.s for all f continuous.
- ▶ Assume the number of horizontal tiles in each connected component is given. Take f analytic.
 $\sum_{i=1}^N (f(\ell_i/N) - \mathbb{E}[f(\ell_i/N)])$ converges towards a centered Gaussian, with covariance as given by random matrix analogue.

Previous results



Petrov (2012) generalizes this result to a family of polygons (Gelfand-Tsetlin patterns), which includes the hexagon. Facets are allowed.

Kenyon (2004). Assume that there is no facets ($0 < \frac{d\mu}{dx} < 1$) [for the hexagon, the boundary is taken to be the limiting arctic circle (to avoid facets)]. Then, the fluctuations of the surface is given up to isomorphism by the Gaussian Free Field.



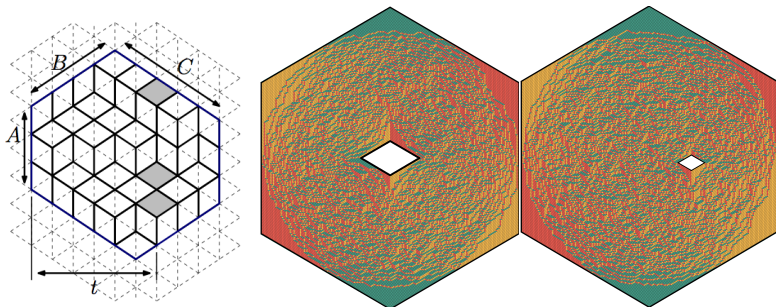
Tiling distribution

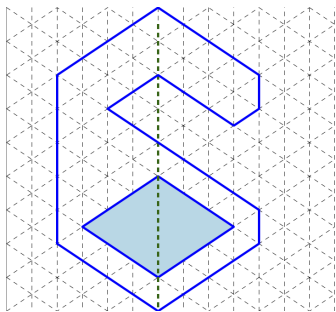
Let $\ell_i^h, 1 \leq i \leq N_h$, be the positions of the horizontal tiles in the h -th connected component.

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{1 \leq h \leq h' \leq K} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i < j}} |\ell_i^h - \ell_j^{h'}|^2 \prod_{i=1}^N w_h(\ell_i^h)$$

where $\ell_i^h \geq \ell_{i-1}^h + 1$.

For some w , this is the distribution of the N horizontal lozenges in a vertical line of a uniformly distributed tiling:



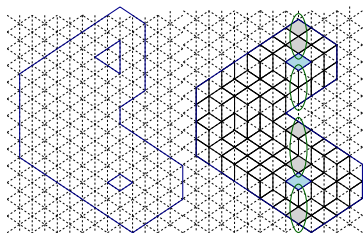


Let N_i be the number of horizontal tiles in the i -th connected component starting from bottom. The distribution of the $N = N_1 + N_2 + N_3$ horizontal lozenges in the central vertical line of a uniformly distributed tiling of the 6 is

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'} \\ i < j}} |\ell_j^{h'} - \ell_i^h|^{\theta_{h,h'}} \prod_{\substack{1 \leq i \leq N_h \\ h}} w_h(\ell_i^h)$$

where $\theta_{h,h'} = 2$ if $h = h'$ or $h = 1, h' = 2$, $\theta_{h,h'} = 1$ if h or $h' = 3$.

More examples



Let N_i be the number of horizontal tiles in the i -th connected component starting from bottom. The geometry dictates:
 $N_1 + N_2 = 3$ and $N_3 + N_4 = 2$. With $\theta_{h,h'} = (1_{h-h' \leq 1} + 1)/2$:

$$\begin{aligned} \mathbb{P}_w(\ell) &= \frac{1}{Z_N} \prod_{h,h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i \neq j}} (\ell_j^{h'} - \ell_i^h)^{\theta_{h,h'}} \\ &\times \prod_{h=1,2} \prod_{1 \leq i \leq N_h} (\ell_i^h - 9)(\ell_i^h - 8)(\ell_i^h - 2)(\ell_i^h + 1) \\ &\times \prod_{h=3,4} \prod_{1 \leq i \leq N_h} (\ell_i^h - 9)(\ell_i^h - 8)(\ell_i^h - 2) \end{aligned}$$

Results

Consider for $\ell_{i+1}^h \geq \ell_i^h + 1$, $\ell^h \in [a_h, b_h]$, $a_{h+1} \geq b_h + cN$, $c > 0$, the probability measure:

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{1 \leq h \leq h' \leq K} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i < j}} |\ell_i^h - \ell_j^{h'}|^2 \prod_{h=1}^k \prod_{i=1}^{N_h} w_h(\ell_i^h)$$

Theorem (Saff-Totic 97, ...)

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Theorem (Saff-Totic 97, ...)

Assume $N_h/N \mapsto \varepsilon_h$ $w_h(x) \simeq e^{-NV_h(x/N) + O(\log N)}$, V_h smooth enough, and

$$\lim_{|x| \rightarrow \infty} V_h(x) / \log |x| \varepsilon_h > 1.$$

Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N_h} \delta_{\ell_i^h/N} \rightarrow \mu_\varepsilon^h \quad \text{a.s.}$$

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$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i < j}} |\ell_i^h - \ell_j^{h'}|^2 \prod_{h=1}^k \prod_{i=1}^{N_h} w_h(\ell_i^h)$$

Theorem (Breuer-Duits 13', Borodin-Gorin-G 15')

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$-\frac{w_h(x)}{w_h(x-1)} = \frac{\phi_{N,h}^+(x)}{\phi_{N,h}^-(x)}$, $\phi_{N,h}^\pm$ analytic, $\phi_{N,h}^\pm = \phi_h^\pm + \frac{1}{N}\phi_{1,h}^\pm + o(\frac{1}{N})$.

Then

$$\left(\sum_{i=1}^{N_h} (f_h(\ell_i^h/N) - \mathbb{E}[f_h(\ell_i^h/N)]) \right)_h \Rightarrow N(0, \Sigma(f)) \quad \forall f \text{ real analytic.}$$

Discrete and Continuous covariance

Take V smooth going to infinity fast enough and

$$dP_V^N := \frac{1}{Z_V^N} \prod_{i < j} |\lambda_i - \lambda_j|^{2\theta} e^{-N \sum V(\lambda_i)} \prod d\lambda_i.$$

Then,

$$\frac{1}{N} \sum_{i=1}^N f(\lambda_i) \mapsto \mu_V(f) \quad a.s.$$

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If μ_V has a connected support (Johansson 97')

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The covariance is the same than in the discrete case if the support of the equilibrium measures are the same:

$$\Sigma((z - \cdot)^{-1}, (w - \cdot)^{-1}) :=$$

$$\frac{1}{(z - w)^2} \left(1 - \frac{zw - \frac{1}{2}(\alpha_1 + \beta_1)(z + w) + \alpha_1\beta_1}{\sqrt{(z - \alpha_1)(z - \beta_1)}\sqrt{(w - \alpha_1)(w - \beta_1)}} \right).$$

Results: Discrete β -ensembles ($\beta = 2\theta$)

$$\ell_{i+1}^h - \ell_i^h - \theta_{h,h} \in \mathbb{N}, \quad v_p(N_1, \dots, N_p) = r_p, \quad 1 \leq p \leq n.$$

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N'_h, i < j}} \frac{\Gamma(\ell_j^{h'} - \ell_i^h + 1) \Gamma(\ell_j^{h'} - \ell_i^h + \theta_{h,h'})}{\Gamma(\ell_j^{h'} - \ell_i^h) \Gamma(\ell_j^{h'} - \ell_i^h + 1 - \theta_{h,h'})} \prod w_h(\ell_i^h)$$

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Theorem (Borot-Gorin-G 16')

Assume

$$-N_h/N \mapsto \varepsilon_h, \quad w_h(x) \simeq e^{-NV_h(x/N)},$$

$$-(\theta_{h,h'})_{h,h'} \geq 0, \quad \theta_{h,h} > 0, \quad \langle x, \theta x \rangle = \langle v(x), \theta' v(x) \rangle, \quad \theta' \geq 0. \quad \text{Then}$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{N_k} \delta_{\ell_i^h/N} \rightarrow \mu_\varepsilon^h \quad \text{a.s.},$$

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Theorem (Borodin, Gorin-G 15', Borot-Gorin-G 16')

Same assumptions as before and $\{0 < \frac{d\mu_\varepsilon^h}{dx} < \theta_{hh}^{-1}\}$ connected. w is

off critical. $\frac{w_h(x)}{w_h(x-1)} = \frac{\phi_{N,h}^+(x)}{\phi_{N,h}^-(x)}, \quad \phi_{N,h}^\pm$ analytic,

$$\phi_{N,h}^\pm = \phi_h^\pm + \frac{1}{N} \phi_{1,h}^\pm + o\left(\frac{1}{N}\right).$$

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Work in progress[Borot-Gorin-G 16']

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being the distribution of the vertical line D_N in the tiling of a region Ω_N , $\frac{1}{N}\Omega_N$ (resp. $\frac{1}{N}D_N$) converging to Ω (resp. D). Ω planar.

Then, $\Sigma(f)$ is the covariance of the linear statistics of f under the Gaussian Free field in Ω (with metric dictated by local densities), with Dirichlet conditions at the boundary of the liquid region.

Work in progress[Borot-Gorin-G 16'] : Discrete β -ensembles

$$\mathbb{P}_w(\ell) = \frac{1}{Z_N^w} \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N'_h, i < j}} \frac{\Gamma(\ell_j^{h'} - \ell_i^h + 1) \Gamma(\ell_j^{h'} - \ell_i^h + \theta_{h,h'})}{\Gamma(\ell_j^{h'} - \ell_i^h) \Gamma(\ell_j^{h'} - \ell_i^h + 1 - \theta_{h,h'})} w_h(\ell_i^h)$$

Same hypotheses as before and $\phi_N^\pm = \sum_{k=0}^2 \frac{1}{N^2} \phi_k^\pm + o(N^{-K})$.

Then

$$\frac{1}{N^2} \ln \frac{Z_N^w}{Z_N^{w'}} = \sum_{k=0}^2 N^{-k} c_k + o(N^{-2})$$

c_k are defined recursively. The expansion is different from its continuous analogue.

Work in progress[Borot-Gorin-G 16'] : Discrete β -ensembles

$$\mathbb{P}_w(\ell) \propto \prod_{h \leq h'} \prod_{\substack{1 \leq i \leq N_h \\ 1 \leq j \leq N_{h'}, i < j}} \frac{\Gamma(\ell_j^{h'} - \ell_i^h + 1) \Gamma(\ell_j^{h'} - \ell_i^h + \theta_{h,h'})}{\Gamma(\ell_j^{h'} - \ell_i^h) \Gamma(\ell_j^{h'} - \ell_i^h + 1 - \theta_{h,h'})} \prod w_h(\ell_i^h)$$

Same hypotheses as before but $\{0 < \frac{d\mu^h}{dx} < \theta_{hh}^{-1}\}$ is not connected or N_h random. Assume $\phi_N^\pm = \sum_{k=0}^2 \frac{1}{N^k} \phi_k^\pm + o(N^{-2})$. Then

$$\sum_{i=1}^{N_h} (f_h(\ell_i^h/N) - \mathbb{E}[f_h(\ell_i^h/N)])$$

converges only under subsequences in general. For instance, if the liquid region has two connected subsets S_1, S_2 , the number of horizontal tiles in S_1 is equivalent to a discrete Gaussian centered at $N\mu_h^\varepsilon(S_1) - \lfloor N\mu_h^\varepsilon(S_1) \rfloor$ (c.f Kriecherbauer-Shcherbina 10', Borot-G 13')

Classical proof of CLT : Stein's method

Let P_n be a family of distribution and L_n some operator on a set $\mathcal{F}$ of functions so that

$$P_n(L_n f) = 0 \quad \forall f \in \mathcal{F}.$$

Assume P_n is tight and L_n goes to L . Assume moreover there exists a unique P so that

$$P(Lf) = 0 \quad \forall f \in \mathcal{F}.$$

Then P_n converges towards P .

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Then P_n converges towards P . If $L_n = L + \frac{1}{n}L_1 + \frac{1}{n^2}L_2 \cdots$, and L invertible, one can hope that

$$P_n(f) = P(f) + \frac{1}{n}P(L_1 L^{-1}f) + \cdots.$$

General strategy to study the fluctuations of

$$M_N(f) = \sum_{i=1}^N f(\lambda_i)$$

- ▶ Prove almost sure convergence of $N^{-1}M_N(f)$ by large deviations or saddle point analysis or Stein's method.

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$$\ln \mathbb{E}[e^{\lambda M_N(f)}] = \int_0^\lambda \mathbb{E}_{tf}[M_N(f)] dt.$$

where $\mathbb{E}_{tf}[g] = \mathbb{E}[ge^{tM_N(f)}] / \mathbb{E}[e^{tM_N(f)}]$.

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 - ▶ getting rid of small terms (e.g. by concentration of measure theory) to get approximately closed equations,
 - ▶ linearizing the equations around the limit and solve the linear equation.

Example: Convergence for β -ensembles

$$dP_{\beta,V}^N(\lambda) = \frac{1}{Z_{\beta,V}^N} \prod_{i < j} |\lambda_i - \lambda_j|^\beta \exp\{-N\beta \sum_{i=1}^N V(\lambda_i)\} \prod d\lambda_i$$

Theorem

$L_N = \frac{1}{N} \sum \delta_{\lambda_i}$ converges $P_{\beta,V}^N$ -almost surely towards the unique minimizer μ_V of

$$I(\mu) = \frac{1}{2} \int \int (V(x) + V(y) - \log |x - y|) d\mu(x) d\mu(y).$$

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Proof.

$$\frac{dP_{\beta,V}^N(\lambda)}{d\lambda_N} \simeq e^{-\beta N^2(I(L_N) - \inf I(\mu))}.$$

Fluctuations for β -ensembles : The Schwinger-Dyson equations [Johansson 97']

$$dP_{\beta,V}^N(\lambda) = \frac{1}{Z_{\beta,V}^N} \prod_{i < j} |\lambda_i - \lambda_j|^\beta \exp\{-N\beta \sum_{i=1}^N V(\lambda_i)\} \prod d\lambda_i$$

The empirical measure $L_N = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ satisfies the Schwinger-Dyson equations

$$\begin{aligned} & \int \left(\frac{1}{2} \iint \frac{f(x) - f(y)}{x - y} dL_N(x) dL_N(y) - \int V'(x) f(x) dL_N(x) \right) dP_{\beta,V}^N \\ &= \frac{1}{N} \left(\frac{1}{2} - \frac{1}{\beta} \right) \int \int f'(x) dL_N(x) dP_{\beta,V}^N. \end{aligned}$$

(this is integration by parts)

Fluctuations for β -ensembles : Linearization

If V is analytic, taking $f(x) = (z - x)^{-1}$, we find that
 $W_N(z) = \int (z - x)^{-1} dL_N(x)$, $W_V(z) =$
 $\int (z - x)^{-1} d\mu_V(x)$, $\Delta W_N = W_N - W_V$ satisfy

$$K\mathbb{E}[\Delta W_N](z) = -\frac{1}{N}\left(\frac{1}{2} - \frac{1}{\beta}\right)\partial_z \mathbb{E}[W_N(z)] - \mathbb{E}[\Delta W_N(z)^2]$$

with

$$Kf(z) = 2W_V(z)f(z) - \oint \frac{d\xi}{2i\pi} \frac{1}{z - \xi} V'(\xi)f(\xi).$$

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By **concentration**, $N\mathbb{E}[\Delta W_N(z)^2]$ is small, hence if K is invertible

$$\lim_{N \rightarrow \infty} N\mathbb{E}[\Delta W_N](z) = -\left(\frac{1}{2} - \frac{1}{\beta}\right)K^{-1}\partial_z W_V(z) =: W_V^1(z)$$

Central limit theorem

For f analytic, we deduce that

$$\begin{aligned}\ln \mathbb{E}_V[e^{\lambda M_N(f)}] &= \int_0^\lambda \mathbb{E}_{V-\frac{t}{N}f}[M_N(f)] dt \\ &= N \int_0^\lambda \mu_{V-\frac{t}{N}f}(f) dt \\ &\quad + \int_0^\lambda \int_C \frac{d\xi}{2i\pi} f(\xi) \mathbb{E}_{V-\frac{t}{N}f}[\Delta W_N(\xi)] dt \\ &= \lambda N \mu_V(f) + \frac{\lambda^2}{2} Q_V(f, f) + \lambda L_V(f) + o(1)\end{aligned}$$

with

$$\begin{aligned}Q_V(f, g) &= -D_V[\mu_V(g)][f] \\ L_V(g) &= \frac{1}{2i\pi} \int_C g(\xi) W_V^1(\xi) d\xi\end{aligned}$$

Generalization to the discrete setting

$$\mathbb{P}(\ell) = \frac{1}{Z_N} \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_j - \ell_i + 1) \Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i) \Gamma(\ell_j - \ell_i + 1 - \theta)} \prod_{i=1}^N w(\ell_i)$$

- The convergence is similar, but the discrete setting imposes $\frac{d\mu}{dx} \leq \frac{1}{\theta}$ (c.f. Feral 08') .

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- Analogues of Schwinger-Dyson equation was given by Nekrasov:
If $\frac{w(x)}{w(x-1)} = \frac{\phi_N^+(x)}{\phi_N^-(x)}$ with $\phi_N^\pm(\xi)$ holomorphic in a domain $\mathcal{M}_N \subset \mathbb{C}$,

$$\phi_N^-(\xi) \cdot \mathbb{E}_{\mathbb{P}} \left[\prod_{i=1}^N \left(1 - \frac{\theta}{\xi - \ell_i} \right) \right] + \phi_N^+(\xi) \cdot \mathbb{E}_{\mathbb{P}} \left[\prod_{i=1}^N \left(1 + \frac{\theta}{\xi - \ell_i - 1} \right) \right]$$

is holomorphic in $\mathcal{M}_N$ (check residues vanish).

Nekrasov equation analysis

Assume $\phi_N^\pm(N\xi) \simeq \phi^\pm(\xi) + \frac{1}{N}\phi_1^\pm(\xi) + o(\frac{1}{N})$ and $\{0 < \frac{d\mu}{dx} < \frac{1}{\theta}\} = [a, b]$.

Nekrasov equation asymptotically becomes with

$$\Delta W_N(z) = \frac{1}{N} \sum (z - \ell_i/N)^{-1} - W(z), \quad W(z) = \int (z - x)^{-1} d\mu(x),$$

$$Q(\xi)\mathbb{E}[\Delta W_N(\xi)] = \frac{1}{N}f(\xi) + R_N(\xi) + o(\frac{1}{N})$$

with $Q(\xi) = \phi^-(\xi)e^{-\theta W(\xi)} - \phi^+(\xi)e^{\theta W(\xi)}$, f known and R_N analytic.

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Assume **off criticality** : $Q(\xi) = H(\xi)\sqrt{(\xi - a)(b - \xi)}$ with H **analytic, non vanishing**. Integrating over a contour around the support of the measure gives

$$\sqrt{(\xi - a)(b - \xi)}\mathbb{E}[\Delta W_N(\xi)] = \frac{1}{N} \frac{1}{2\pi i} \int_C \frac{H(z)^{-1}f(z)}{\xi - z} dz + o(\frac{1}{N}).$$

Conclusion

- ▶ The case where the interaction is given by $\prod_{i \neq j} |\ell_i - \ell_j|^\theta$, $\theta \neq 1, 2$ is open.
- ▶ In the continuous setting, similar analysis allows to build approximate transport and prove universality for local fluctuations. In the discrete setting, Local fluctuations are unknown for $\theta \neq 2$.
- ▶ Nekrasov equations include more general models, cf quivers.