How many stable equilibria will a large complex system have?

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after joint work with Yan Fyodorov (PNAS 2016) and unpublished work with Gerard Ben Arous and Yan Fyodorov

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$$\dot{\mathbf{y}} = -\mu\mathbf{y} + J\mathbf{y}, \qquad J \in \mathbf{R}^{N\times N}.$$

Here $\mu > 0$ sets the relaxation time scale and J_{jk} characterises interaction. In an ecological context, $y_j(t)$ is the variation about equilibrium value in popn dens of species j, and J_{jk} measures per capita effect of species k on species j, hence **community matrix J is asymmetric** (think of sea lions and sardines).

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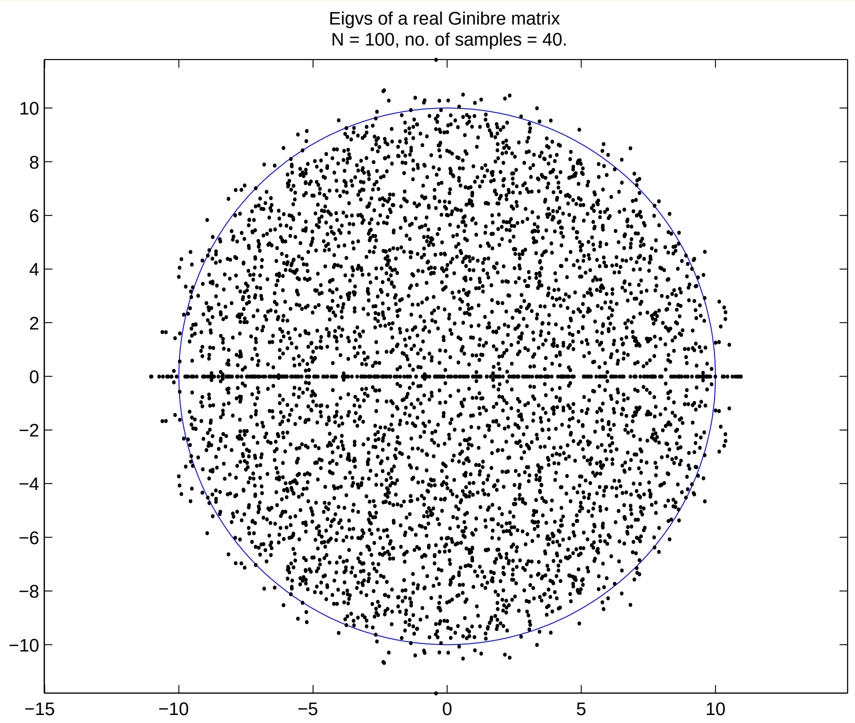

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May-Wigner instability transition

Have Girko's circular law: As $N \to \infty$, EV distribution of $J/\sqrt{N}$ converges to unif distrib on the unit disk, Girko1984, Bai1997, Götze &.Tikhomirov, Tau & Vu 2010.

Also, the rescaled spectral radius of J is $\leq$ 1 in the limit of large N, Geman 1984.

Hence for N large

the linearised system is stable if $\frac{\mu}{\alpha\sqrt{N}} > 1$ and **unstable if** $\frac{\mu}{\alpha\sqrt{N}} < 1$.

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Is there a signature of May-Wigner instability transition on the global scale?

Non-linear systems: setup

A simple model for generic large complex systems: consider

$$\frac{dx_i}{dt} = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N.$$

$\mu > 0$ as before, and now $\mathbf{f}(\mathbf{x})$ is a smooth random field with N components.

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This system may have multiple equilibria depending on the realisation of $\mathbf{f}(\mathbf{x})$.

Near equilibrium $\mathbf{x}_e$ it 'reduces' to May's model with $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$, $J_{jk} = \frac{\partial f_j}{\partial x_k}(\mathbf{x}_e)$.

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Gradient-descent flow, $\mathbf{f} = -\nabla V$, is special (but typical) case. Have

$$\frac{d\mathbf{x}}{dt} = -\nabla L, \quad L(\mathbf{x}) = \frac{\mu|\mathbf{x}|^2}{2} + V(\mathbf{x})$$

[note that $J_{jk} = J_{kj}$ here].

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Helpful for building geometric intuition: $\mathbf{x}(t)$ moves in the direction of the steepest descent, perpendicular to level lines $L(\mathbf{x}) = h$ towards ever smaller values of h.

The term $\mu|\mathbf{x}|^2/2$ represents the globally confining parabolic potential, a deep well on the surface of $L(\mathbf{x})$. The random potential $V(\mathbf{x})$ generates many local minima of $L(\mathbf{x})$ (shallow wells). Have two competing terms...

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$$f_i(\mathbf{x}) = -\frac{\partial V}{\partial x_i} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\partial A_{ij}}{\partial x_j}, \quad A_{ij}(\mathbf{x}) = -A_{ji}(\mathbf{x})\ \forall i, j.$$

This is a fairly general class. If $N = 3$ can find V and $\mathbf{A}$ s.t. $\mathbf{f} = \nabla V + \nabla \times \mathbf{A}$ (Helmholtz decomposition). In these terms, our $\mathbf{f}(\mathbf{x})$ is the sum of 'gradient' (irrotational) and 'solenoidal' parts.

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$$\langle V(\mathbf{x})V(\mathbf{y})\rangle = v^2 F_V\left(|\mathbf{x}-\mathbf{y}|^2\right), \langle A_{ij}(\mathbf{x})A_{nm}(\mathbf{y})\rangle = a^2 F_A\left(|\mathbf{x}-\mathbf{y}|^2\right)(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn})$$

with suitable F_V and F_A:

$$F_{V,A}(s) = \int_0^\infty e^{-st} d\sigma_{V,A}(t) \quad \forall s \geq 0, \quad \text{(Schoenberg,1938)}$$

where $d\sigma_V$ and $d\sigma_A$ have finite mass. Also assume finite 3rd moments, and normalise $d^2 F_{V,A}(s)/ds^2\big|_{s=0} = 1$.

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Let $\tau = \dfrac{v^2}{v^2 + a^2}$. This is a measure of relative strength of the two components: if $\tau = 1$ then $\mathbf{f}(\mathbf{x})$ is purelly gradient, and if $\tau = 0$ then $\mathbf{f}(\mathbf{x})$ is purelly solenoidal.

A signature of the May-Wigner transition on the global scale

Let $\mathcal{N}_{tot}$ be the total number of equilibrium pnts of $\dot{\mathbf{x}} = -\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$.

These are solutions of the system of N equations $-\mu\mathbf{x} + \mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Introduce dimensionless $m = \dfrac{\mu}{2\alpha\sqrt{N}}$, where $\alpha = \sqrt{v^2 + a^2}$ interaction strength.

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Theorem. *[YF and BK 2016] Assume $0 \leq \tau < 1$. To leading order for N large,*

$$\langle \mathcal{N}_{tot} \rangle = \begin{cases} 1 & \text{if } m > 1 \\ \sqrt{\frac{2(1+\tau)}{1-\tau}}\, e^{N\,\Sigma_{tot}(m)} & \text{if } 0 < m < 1 \end{cases}$$

where $\Sigma_{tot}(m) = \frac{m^2-1}{2} - \ln m$. Moreover, the relative width of the crossover region is $N^{-1/2}$ and the crossover profile of $\langle \mathcal{N}_{tot} \rangle$ can be found in closed form.

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Thus, **as the complexity exceeds a critical level, large complex non-linear systems exhibit a sharp transition from having on average one point of equilibrium to having exponentially many equilibria.**

Rice-Kac and reduction to RMT

Want to count zeros of $-\mu\mathbf{x}+\mathbf{f}(\mathbf{x})$. By Kac-Rice,

$$\mathcal{N}_{tot} = \int_{\mathbb{R}^N} \delta\left(-\mu\mathbf{x}+\mathbf{f}(\mathbf{x})\right) \left|\det\left(-\mu\delta_{ij}+\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right)\right| d\mathbf{x}.$$

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$$\langle J_{ij} J_{nm} \rangle = \alpha^2 (\delta_{in}\delta_{jm} + \tau\delta_{jn}\delta_{im} + \tau\delta_{ij}\delta_{nm}) + O(1/N).$$

Thus, $J = \alpha(X + \sqrt{\tau}\xi I)$ where $X \sim$ RealGin(τ) and $\xi \sim N(0,1)$, independent.

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where $x=(m+t\sqrt{\tau})\sqrt{N}$ and $P(X)\propto\exp\left[-\frac{1}{2(1-\tau^2)}\left(\mathrm{Tr}\,XX^T-\tau\,\mathrm{Tr}\,X^2\right)\right]$.

Analytic problem: find the average of the abs value of the characteristic polynomial in the real elliptic Ginibre ensemble.

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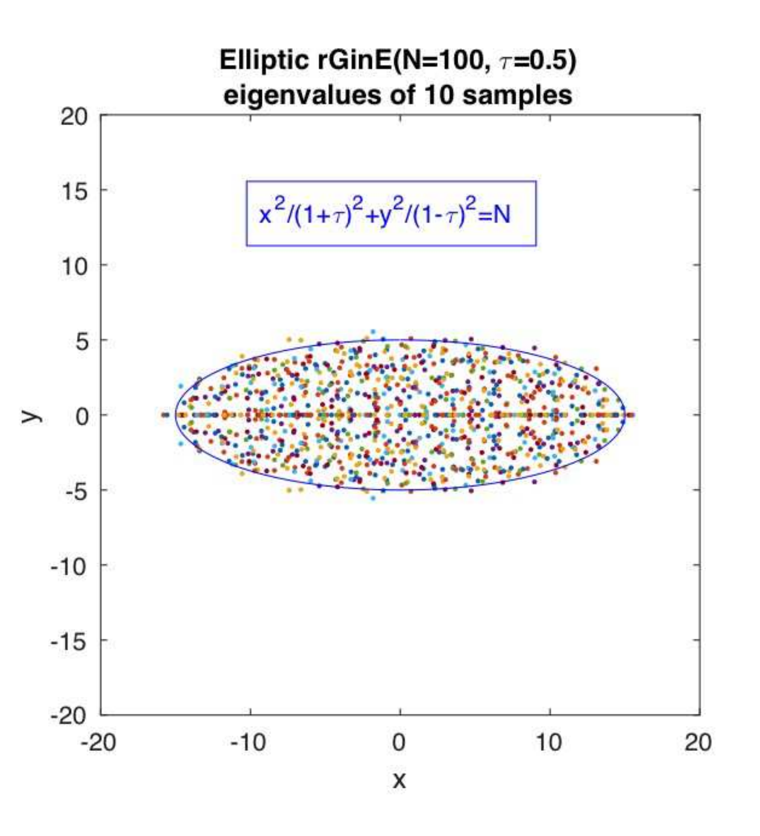

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Edelman-Kostlan-Shub trick

Start with the real elliptic ensemble X_{N+1} of $(N+1) \times (N+1)$ matrices.

Decompose $X_{N+1} = Q \begin{pmatrix} x & \mathbf{w} \\ 0 & X_N \end{pmatrix} Q^T$, where x is a real eigenvalue of X_{N+1}, and Q is an orthogonal matrix that exchanges the corresponding eigenvector and $(1, 0, \ldots, 0)$ (Householder reflection).

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The Jacobian of changing from X_{N+1} to $X_N, Q, x, \mathbf{w}$ is $|\det(xI_N - X_N)|$.

Note: $\mathrm{Tr}\, X_{N+1}X_{N+1}^T = x^2 + \mathbf{w}^T\mathbf{w} + \mathrm{Tr}\, X_N X_N^T$ and $\mathrm{Tr}\, X_{N+1}^2 = x^2 + \mathrm{Tr}\, X_N^2$.

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Therefore, have the relation

$$\rho_{N+1}^{(r)}(x) = \frac{(N-2)!!}{(N-1)!} \frac{e^{-\frac{x^2}{2(1+\tau)}}}{2\sqrt{1+\tau}} \left\langle \left| \det\left(xI - X\right) \right| \right\rangle_{X_N}$$

where $\rho_{N+1}^{(r)}(x)$is the mean density of real eigenvalues in the real elliptic ensemble X_{N+1} of $(N+1) \times (N+1)$ matrices , and the average on the right is over the real elliptic ensemble X_N of $N \times N$.

This relation comes in handy. Forrester and Nagao 2008 found $\rho_{N+1}^{(r)}(x)$ in closed form in terms of Hermite polynomials, the rest is asymptotic analysis.

How many equilibria are stable?

Averaged number of stable equilibria $\langle \mathcal{N}_{st} \rangle$ via Rice-Kac:

$$\langle \mathcal{N}_{st} \rangle = \frac{1}{m^N N^{N/2}} \int_{-\infty}^{\infty} \langle \det(xI - X)\, \chi_x(X) \rangle_X \frac{e^{-\frac{Nt^2}{2}} dt}{\sqrt{2\pi/N}},$$

where $x = (m + t\sqrt{\tau})\sqrt{N}$ and $\chi_x(X) = 1$ if all N EVs X have real parts less than x, and $\chi_x(X) = 0$ otherwise. No need for absolute value because of χ

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This yields $\langle \mathcal{N}_{st} \rangle \to 1$ if $m > 1$, and if $0 < m < 1$ then, to leading order in N, $\langle \mathcal{N}_{st} \rangle \propto e^{N\Sigma_{st}}$, with $0 < \Sigma_{st} < \Sigma_{tot}$, Fyodorov & Nadal 2012.

Thus, for purely gradient dynamics, as the complexity increases, there is an abrupt change from a simple set of equilibria, typically a single stable equilibrium, to a phase portrait dominated by an exponential number of unstable equilibria with an admixture of a smaller, but still exp in N, number of stable equilibria.

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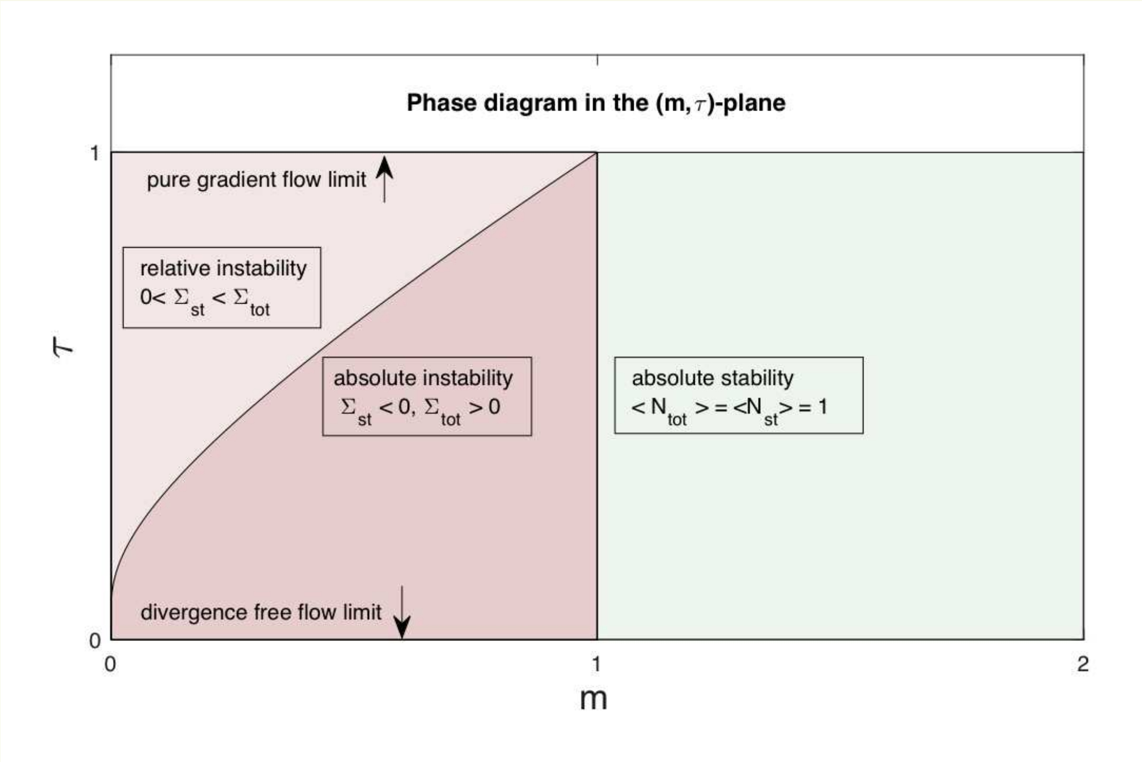
Bouchaud's conjecture: in the general case of non-gradient dynamics, there exists a further phase transition in the plane (m, τ) such that below a certain number $\tau_c(m)$ stable equilibria are no longer exponentially abundant in the limit $N \to \infty$ (i.e. $\Sigma_{st}(m, \tau) \to 0$).

Bouchaud's conjecture verified

Claim (Ben Arous, Fyodorov, Kh, unpublished, work in progress): For $0 < m < 1$,

$$\lim_{N\to\infty} \frac{1}{N} \ln\langle \mathcal{N}_{st} \rangle = m - 1 - \ln m - \frac{(1-m)^2}{2\tau} = \Sigma_{st}(m,\tau).$$

Have three regions in the (m,τ)-plane separated by critical lines $m_c = 1$ and $\tau_c(m) = \frac{1}{2}\frac{(1-m)^2}{m-1-\ln m}, \quad 0 < m < m_c$



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- Our analysis is applicable to complex multi-species communities in which each kind of species on its own becomes extinct and thus interaction is key to persistence of the community.

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THANK YOU