Polynomial Ensembles of Derivative Type

Holger Kösters

(joint work with Mario Kieburg)

Bielefeld University

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Overview

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Motivation

Standing Assumption:
Let $\mathbf{X}_n$ be a *bi-invariant* (*bi-unitarily invariant*) (*isotropic*) random matrix with values in $G := \mathrm{GL}(n, \mathbb{C})$.

$$\mathbf{X}_n \text{ bi-invariant } :\Leftrightarrow \text{ for any } V, W \in K := \mathrm{U}(n), \;\; V\mathbf{X}_n W^* \stackrel{d}{=} \mathbf{X}_n$$

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Associated Densities:

- (bi-invariant) matrix density: $f_G(g)$ on $\mathrm{GL}(n, \mathbb{C})$
  jpdf of the matrix entries w.r.t. *Lebesgue measure* on $\mathbb{C}^{n \times n}$
- (symmetric) singular value density: $f_{\mathrm{SV}}(a) = f_{\mathrm{SV}}(a_1, \ldots, a_n)$ on $\mathbb{R}_+^n$
  jpdf of the *squared* singular values
- (symmetric) eigenvalue density: $f_{\mathrm{EV}}(z) = f_{\mathrm{EV}}(z_1, \ldots, z_n)$ on $\mathbb{C}_*^n$
  jpdf of the eigenvalues

Basic Question:
What can we say about these densities (and the relation between them)?

Motivation

Ginibre Matrix

$$f_{\mathrm{SV}}(a) \propto |\Delta_n(a)|^2 \prod_{j=1}^n e^{-a_j}$$

Fisher (1939), Hsu (1939), Roy (1939), ...

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n e^{-|z_j|^2}$$

Ginibre (1965)

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$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n e^{-|z_j|^2}$ Ginibre (1965)

Product of *p* Ginibre Matrices

$f_{SV}(a) \propto \Delta_n(a) \det \left[(-a_k \frac{\partial}{\partial a_k})^{j-1} w_{p,0}(a_k) \right]$ Akemann–Kieburg–Wei (2013)

$f_{EV}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w_{p,0}(|z_j|^2)$ Akemann–Burda (2012)

where $w_{p,q}(x) = G^{p,q}_{q,p}\left(\begin{matrix} -n,\dots,-n \\ 0,\dots,0 \end{matrix} \middle| \, x \right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma^p(s)\,\Gamma^q(1+n-s)\,x^{-s}\,ds$

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Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det \left[(-a_k \tfrac{\partial}{\partial a_k})^{j-1} w_{p,q}(a_k) \right]$$ Forrester (2014)

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w_{p,q}(|z_j|^2)$$ Adhikari–Reddy–Reddy–Saha (2013)

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Common Structure:

$$f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det \left[(-a_k \tfrac{\partial}{\partial a_k})^{j-1} w(a_k)\right]$$

$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w(|z_j|^2)$$

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Several Examples:

- products of independent [inverse] Ginibre matrices
- products of independent [inverse] truncated unitary matrices
- mixed products

Akemann–Burda (2012), Akemann–Strahov (2013), Akemann–Kieburg–Wei (2013), Akemann–Ipsen–Kieburg (2013), Adhikari–Reddy–Reddy–Saha (2013), Ipsen–Kieburg (2014), Akemann–Burda–Kieburg–Nagao (2014), Forrester (2014), Akemann–Ipsen–Strahov (2014), Kuijlaars–Zhang (2014), Kuijlaars–Stivigny (2014), Kieburg–Kuijlaars–Stivigny (2015), Kuijlaars (2015), Akemann–Ipsen (2015), Claeys–Kuijlaars–Wang (2015), ...

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Main Questions:

- Can we find more examples with this structure?
- Can we show that this structure is preserved under multiplication?
- Can we give a unifying proof?

Overview

Polynomial Ensembles of Derivative Type

Polynomial Ensemble Kuijlaars–Stivigny (2014)

$\mathbf{X}_n$ is from a *polynomial ensemble* if it is bi-invariant and

$$f_{\mathsf{SV}}(a) \propto \Delta_n(a) \det \left(w_j(a_k) \right)_{j,k=1,\dots,n}$$

for some weight functions $w_1, \dots, w_n$ (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$.

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Theorem (Transfer Law) Kuijlaars–Stivigny (2014)

If $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim$ Ginibre are independent,
then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast w_{Gin}, \dots, w_n \circledast w_{Gin})$, where $w_{Gin}(x) := e^{-x}$.

Kuijlaars–Stivigny (2014), Kieburg–Kuijlaars–Stivigny (2015), Kuijlaars (2015), Claeys–Kuijlaars–Wang (2015), ...

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Polynomial Ensemble of Derivative Type Kieburg–K. (2016)

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Examples from RMT

- induced Wishart–Laguerre ensemble: $w_0(a) = a^\nu e^{-a}$
- induced Jacobi ensemble: $w_0(a) = a^\nu (1-a)^{\mu-1} \mathbf{1}_{(0,1)}(a)$
- induced Cauchy–Lorentz ensemble: $w_0(a) = a^\nu (1+a)^{-\nu-\mu-1}$

- products of such random matrices: $w_0(a) =$ Meijer-G-function

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- Muttalib–Borodin ensemble (of Wishart–Laguerre type)

  Muttalib (1995), Borodin (1999), Cheliotis (2014), Forrester–Liu (2014), Forrester–Wang (2015), Zhang (2015), ...

  (a) $f_{\mathrm{SV}}(a) \propto \Delta_n(a)\, \Delta_n(a^\theta)\, (\det a)^\nu\, e^{-\operatorname{tr} a^\theta}$ $\qquad w_0(a) = a^\nu e^{-\alpha a^\theta}$

  (b) $f_{\mathrm{SV}}(a) \propto \Delta_n(a)\, \Delta_n(\ln a)\, (\det a)^\nu\, e^{-\operatorname{tr}(\ln a)^2}$ $\qquad w_0(a) = a^\nu e^{-\alpha(\ln a)^2}$

Overview

Main Results

If $\mathbf{X}_n \sim DPE(w_0)$, then $f_{\mathrm{SV}}(a) \propto \Delta_n(a) \det\left(\left(-a_k \frac{\partial}{\partial a_k}\right)^{j-1} w_0(a_k)\right)_{j,k=1,\ldots,n}$.

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Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where

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More generally, let $\mathbf{X}_n \sim PE(w_1,\dots,w_n)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
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Theorem (Transfer Law) Kieburg–K. (2016)
Let $\mathbf{X}_n \sim DPE(w_0)$. Then $\mathbf{X}_n^{-1} \sim DPE(\widetilde{w}_0)$, where $\widetilde{w}_0(x) = w_0(x^{-1})\, x^{-n-1}$.

Overview

Key Tool: Spherical Transform

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Univariate Situation

X : r.v. with values in $\mathbb{C}$
and a rotation-invariant density f_X

dist. of $X \longleftrightarrow$ dist. of $|X|^2$

$|X|^2$: r.v. with values in $\mathbb{R}_+$
and a density $f_{|X|^2}$

Mellin Transform

$$\mathcal{M}_X(s) = \int_{\mathbb{R}_+} f_{|X|^2}(y)\, y^s \, \frac{dy}{y}$$
$$= \int_{\mathbb{C}} f_X(x)\, |x|^{2s} \, \frac{dx}{|x|^2}$$

for suitable $s \in \mathbb{C}$

Uniqueness Theorem
$\mathcal{M}_{X_1} = \mathcal{M}_{X_2} \Rightarrow X_1 \overset{d}{=} X_2$

Multiplication Theorem
X_1, X_2 ind. $\Rightarrow \mathcal{M}_{X_1 X_2} = \mathcal{M}_{X_1} \cdot \mathcal{M}_{X_2}$

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Multivariate Situation

$\mathbf{X}$: r.v. with values in $\mathrm{GL}(n, \mathbb{C})$
and a bi-invariant density $f_{\mathbf{X}}$

dist. of $\mathbf{X} \longleftrightarrow$ dist. of $\mathbf{X}^*\mathbf{X}$

$\mathbf{X}^*\mathbf{X}$: r.v. with values in $\mathrm{Pos}(n, \mathbb{C})$
and a conjugation-invariant density $f_{\mathbf{X}^*\mathbf{X}}$

Spherical Transform

$$\mathcal{S}_{\mathbf{X}}(s) = \int_{\mathrm{Pos}(n,\mathbb{C})} f_{\mathbf{X}^*\mathbf{X}}(y)\, \varphi_s(y) \, \frac{dy}{(\det y)^n} = \int_{\mathrm{GL}(n,\mathbb{C})} f_{\mathbf{X}}(x)\, \varphi_s(x^* x) \, \frac{dx}{|\det x|^{2n}}$$

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Key Tool: Spherical Transform

Spherical Transform

$$\mathcal{S}_{\mathbf{X}}(s) = \int_{\mathrm{GL}(n,\mathbb{C})} f_{\mathbf{X}}(x)\,\varphi_s(x^*x)\,\frac{dx}{|\det x|^{2n}} \qquad (s \in \mathbb{C}^n)$$

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Spherical Function

Let $x \in \mathrm{GL}(n,\mathbb{C})$, let $\mathbf{X}_n \in \mathrm{GL}(n,\mathbb{C})$ be a bi-invariant random matrix with the same singular values as x, and let $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$ be its *QR decomposition*.

$$\varphi_s(x^* x) := \mathbb{E}\Big(\prod_{j=1}^{n} \mathbf{R}_{jj}^{2(s_j + \varrho_j - 1)} \Big) \qquad (\text{where } \varrho_j := j - \tfrac{n+1}{2})$$

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Spherical Function for $\mathrm{GL}(n,\mathbb{C})$ Gelfand–Naimark (1950)

$$\varphi_s(x^*x) = C_n \frac{\det[(\lambda_j(x^*x))^{s_k+(n-1)/2}]_{j,k=1,\dots,n}}{\Delta_n(s)\Delta_n(\lambda(x^*x))} \qquad (x \in \mathrm{GL}(n,\mathbb{C}))$$

where $\lambda(x^*x)$ is the vector consisting of the eigenvalues of x^*x

Some Ideas from the Proofs

Proposition (Spherical Transform) Kieburg–K. (2016)

If $\mathbf{X}_n \sim DPE(w_0)$, then $\mathcal{S}_{\mathbf{X}_n}(s) \propto \prod_{k=1}^{n} (\mathcal{M} w_0)(s_k - \frac{n-1}{2})$.

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Proof:

Set $Df(x) := (-x)f'(x)$ and note that $\mathcal{M}(Df)(s) = s\,\mathcal{M}f(s)$.

$$\mathcal{S}_{\mathbf{X}}(s) = \int_{\mathrm{GL}(n,\mathbb{C})} f_{\mathbf{X}}(x)\,\varphi_s(x^*x)\,\frac{dx}{|\det x|^{2n}} = \int_{(0,\infty)^n} f_{SV}(\lambda)\,\varphi_s(\lambda)\,\frac{d\lambda}{(\det\lambda)^n}$$

$$\propto \int_{(0,\infty)^n} \Delta_n(\lambda)\,\det\left(D^{j-1}w_0(\lambda_k)\right)\frac{\det\left((\lambda_j)^{s_k+(n-1)/2}\right)}{\Delta_n(s)\,\Delta_n(\lambda)}\,\frac{d\lambda}{(\det\lambda)^n}$$

$$\propto \frac{1}{\Delta_n(s)}\det\left(\int_0^\infty D^{j-1}w_0(\lambda)\lambda^{s_k-(n+1)/2}\,d\lambda\right)$$

$$= \frac{\det\left((s_k - \frac{n-1}{2})^{j-1}\,(\mathcal{M}w_0)(s_k - \frac{n-1}{2})\right)}{\Delta_n(s)} = \prod_{k=1}^{n}(\mathcal{M}w_0)(s_k - \tfrac{n-1}{2})\,.$$

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Theorem (Transfer Law for the Product) Kieburg–K. (2016)
Let $\mathbf{X}_n \sim DPE(w_0)$ and $\mathbf{Y}_n \sim DPE(v_0)$ be independent.
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Proof:

$$\begin{aligned} & \mathbf{X}_n \sim DPE(w_0)\,,\ \mathbf{Y}_n \sim DPE(v_0) \\ \Rightarrow\quad & \mathcal{S}_{\mathbf{X}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M} w_0(s_k - \tfrac{n-1}{2})\,,\ \mathcal{S}_{\mathbf{Y}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M} v_0(s_k - \tfrac{n-1}{2}) \\ \Rightarrow\quad & \mathcal{S}_{\mathbf{X}_n\mathbf{Y}_n}(s) \propto \prod_{k=1}^{n} \mathcal{M}(w_0 \circledast v_0)(s_k - \tfrac{n-1}{2}) \\ \Rightarrow\quad & \mathbf{X}_n\mathbf{Y}_n \sim DPE(w_0 \circledast v_0) \end{aligned}$$

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$$f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \left(\prod_{j=1}^{n} |z_j|^{2n-2j} \right) \int_T f_G(zt)\, dt$$

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$$f_{(\mathbf{R}_{11},\ldots,\mathbf{R}_{nn})}(r_1, \ldots, r_n) \propto \left(\prod_{j=1}^{n} |z_j|^{2n-2j} \right) \int_T f_G(rt)\, dt$$

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Thus, since $\mathcal{S}_{\mathbf{X}}$ is essentially the (componentwise) Mellin transform of $f_{(\mathbf{R}_{11},\ldots,\mathbf{R}_{nn})}$, we get $f_{(\mathbf{R}_{11},\ldots,\mathbf{R}_{nn})}$, and hence f_{EV}, from $\mathcal{S}_{\mathbf{X}}$ by (componentwise) Mellin inversion:

$$\mathcal{S}_{\mathbf{X}_n}(s) \propto \prod_{j=1}^{n} \mathcal{M} w_0(s_j - \tfrac{n-1}{2}) \quad \Rightarrow \quad f_{\mathrm{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^{n} w_0(|z_j|^2)$$

Overview

Relation between Singular Value and Eigenvalue Densities

$L^{1,K}_{\text{prob}}$: set of all *bi-invariant* probability densities on $G = \mathrm{GL}(n, \mathbb{C})$
$L^{1,SV}_{\text{prob}}$: set of all *induced* singular value densities
$L^{1,EV}_{\text{prob}}$: set of all *induced* eigenvalue densities

Theorem Kieburg–K. (2016)
The map $f_{\text{SV}} \longmapsto f_{\text{EV}}$ establishes a bijection between $L^{1,SV}_{\text{prob}}$ and $L^{1,EV}_{\text{prob}}$.

$$\frac{f_{\text{EV}}(z)}{|\Delta_n(z)|^2} = C'_n \int_{\mathbb{R}^n} \int_{(0,\infty)^n} \frac{f_{\text{SV}}(a)}{\Delta_n(a)} \frac{\det[a_j^{k+\imath s_k}]}{\Delta_n(\varrho + \imath s)} \frac{da}{a} \operatorname{perm}\left[|z_j|^{-2(k+\imath s_k)}\right] ds$$

$$\frac{f_{\text{SV}}(a)}{\Delta_n(a)} = C''_n \int_{\mathbb{R}^n} \int_{(0,\infty)^n} \frac{f_{\text{EV}}(\sqrt{r})}{|\Delta_n(\sqrt{r})|^2} \operatorname{perm}[r_j^{k+\imath s_k}] \frac{dr}{r} \Delta_n(\varrho + \imath s) \det[a_j^{-(k+\imath s_k)}] ds$$

Related Work Fyodorov–Khoruzhenko (2007), Wei–Fyodorov (2008), ...
description of the map $g_{\text{SV}} \mapsto g_{\text{EV}}$ between the one-point densities for *deterministic* singular value configurations

Interpolation between Product Ensembles I

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\mathrm{SV}}^{(p,q)}(a) \propto \Delta_n(a) \det\left[(-a_k \tfrac{\partial}{\partial a_k})^{j-1} w_{p,q}(a_k)\right]$$

$$f_{\mathrm{EV}}^{(p,q)}(z) \propto |\Delta_n(z)|^2 \textstyle\prod_{j=1}^{n} w_{p,q}(|z_j|^2)$$

where $w_{p,q}(x) = \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \Gamma^p(s)\, \Gamma^q(1+n-s)\, x^{-s}\, ds$

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Problem: Is it possible to interpolate between these "product ensembles" ?

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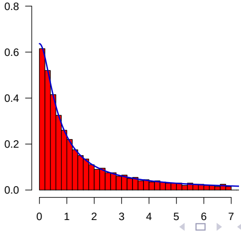
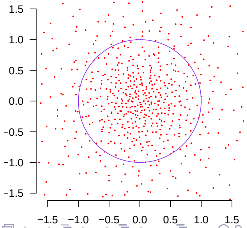
Motivation: *heavy-tailed* one-point densities (in the limit as $n \to \infty$)

$$g_{\mathrm{SV}}^{(p,q)}(a) \asymp a^{-\frac{q+3}{q+1}} \quad (a \to \infty)$$

$$g_{\mathrm{EV}}^{(p,q)}(z) \asymp |z|^{-\frac{2q+2}{q}} \quad (z \to \infty)$$





Interpolation between Product Ensembles II

Product of p Ginibre Matrices and q Inverse Ginibre Matrices

$$f_{\mathrm{SV}}^{(p,q)}(a) \propto \Delta_n(a) \det \left[(-a_k \tfrac{\partial}{\partial a_k})^{j-1} w_{p,q}(a_k) \right]$$

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Problem: Is it possible to interpolate between these "product ensembles"?

Solution: *Formally* define $w_{p,q}$, $f_{\mathrm{SV}}^{(p,q)}$, $f_{\mathrm{EV}}^{(p,q)}$ as above, for any $p, q \geq 0$.

- $f_{\mathrm{SV}}^{(p,q)}$ is a probability density if and only if

$$(p \in \mathbb{N}_0 \text{ or } p > n-1) \quad \text{and} \quad (q \in \mathbb{N}_0 \text{ or } q > n-1).$$

- $f_{\mathrm{EV}}^{(p,q)}$ is a probability density for any $p, q \geq 0$.

Interpolation between Product Ensembles II

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Consequently, in general, one may not let $n \to \infty$ for fixed $p, q \geq 0$.

Connection to Pólya Frequency Functions

Pólya Frequency Function

A function $f : \mathbb{R} \to \mathbb{R}$ is called *Pólya frequency function* of order n (PF_n) if

$$\det(f(x_j - y_k))_{j,k=1,\ldots,m} \geq 0$$

for any $m = 1, \ldots, n$ and any $x_1 < \ldots < x_m$, $y_1 < \ldots < y_m$.

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Proposition Kieburg–K. (2016)

If $f \in \mathrm{PF}_n$ (with suitable differentiability and integrability properties), then $w_0 := f \circ \log$ gives rise to a random matrix $\mathbf{X}_n \sim DPE(w_0)$.

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Related Work

- Pólya frequency functions are also connected to several classical distributions in *Hermitian* random matrix theory (GUE, LUE).
  Karlin (1968), Pickrell (1990), Olshanski–Vershik (1994), Faraut (2002), ...
- The theory of DPE's may also be developed for *Hermitian* random matrices endowed with additive convolution.
  Kuijlaars–Román (2016), ...

Overview

Summary and Outlook

Polynomial Ensembles of Derivative Type

- singular value and eigenvalue densities with special *determinantal* structure
- special structure is preserved under independent products ($\rightsquigarrow$ transfer laws)
- key tool: spherical transform
- many examples (from RMT and via Pólya frequency functions)

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Open Problems **(partly work in progress)**

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Thank you very much for your attention!