Transition from random matrix to Poisson statistics

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A few examples

- Eigenvalues of band matrices (Fyodorov and Mirlin; Sodin; Erdős and Knowles; Spencer)
- Heavy tail random matrices (Soshnikov; Auffinger, Ben Arous and Péché; Bordenave and Guionnet)
- Dyson Brownian motion (Duits andJohansson)
- Free fermions at positive temperature (Moshe, Neuberger and Shapiro; Johansson; Dean, Le Doussal, Majumdar and Schehr; Johansson and L.)
- Bohigas-Pato model of *incomplete spectra* (Bothner, Deift, Its and Krasovsky; Charlier and Claeys; Berggren and Duits; L.)

Free fermions and the Gaussian Unitary Ensemble

Zero temperature wave function

Consider a spinless particle confined in an external field $V(x)$, the wave functions φ_n for its position solve the equation:

$$-\nabla^2\varphi_n + V(x)\varphi_n = \varepsilon_n\varphi_n \qquad \text{with} \qquad \varepsilon_0 < \varepsilon_1 < \cdots .$$

When $V(x) = x^2$, this equation has explicit solutions:

$$\varphi_n(x) = \frac{(-1)^n}{\sqrt{n!2^n\sqrt{\pi}}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \qquad \text{and} \qquad \varepsilon_n = 2n+1.$$

If we consider N non-interacting fermions confined by $V(x) = x^2$ at temperature $T = 0$, their positions are described by the ground state wave function:

$$\Phi_N(\mathbf{x}) = \frac{1}{\sqrt{N!}} \det\left[\varphi_n(x_j)\right]_{\substack{n=0,\dots,N-1 \\ j=1,\dots,N}}.$$

Determinantal structure

It means that the joint density of the N particles is given by

$$\rho_N^N(\mathbf{x}) = |\Phi_N(\mathbf{x})|^2 = \frac{1}{N!}\left(\det\left[\varphi_n(x_j)\right]\right)^2. \tag{1}$$

If we let

$$K_0^N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y),$$

using the identity $\det(A^2) = (\det A)^2$, we can rewrite formula (1) as

$$\rho_N(\mathbf{x}) = \frac{1}{N!}\det\left[K_0^N(x_i,x_j)\right]_{i,j=1,\ldots,N}.$$

$\Rightarrow$ the distribution of the free fermions at $T=0$ is a **determinantal point process** on $\mathbb{R}$ with **correlation kernel** K_0^N.

Eigenvalues statistics for the GUE

We consider the system confined in the external potential $V(x) = 2Nx^2$ at temperature $T = 0$ and look at the empirical measure

$$\nu_N^{\alpha,0} = \sum_{n=1}^{N} \delta_{(\lambda_n - x_0)N^{1-\alpha}}$$

for any $0 \le \alpha \le 1$ and $|x_0| < 1$.

Theorem

Let $\Lambda_\varrho^{\sin}$ be the determinantal process on $\mathbb{R}$ with correlation kernel

$$K_\varrho^{\sin}(x, y) = \frac{\sin\left(\pi\varrho(x-y)\right)}{\pi(x-y)}.$$

Then, as $N \to \infty$,

$$\nu_N^{0,0} \Rightarrow \Lambda_{\varrho_{sc}(x_0)}^{\sin}.$$

Mesoscopic fluctuations

Theorem (Fyodorov–Khoruzhenko–Simm, L.)

$\forall 0 < \alpha < 1$, $\forall f \in H^{1/2}$ with compact support, as $N \to \infty$,

$$\nu_N^{\alpha,0}(f) - \mathbb{E}\left[\nu_N^{\alpha,0}(f)\right] \Rightarrow \mathcal{N}\left(0, \|f\|_{H^{1/2}}^2\right).$$

Generalizations by

- Breuer and Duits
- Bourgade, Erdős, Yau, and Yin; Bekerman-Lodhia
- Boutet de Monvel and Khorunzhy; Lodhia and Simm; He and Knowles

Transition at mesoscopic scale

Free fermions at positive temperature

The joint density of N fermions at temperature $T > 0$ is given by

$$p_{N,T}(x_1, \ldots, x_N) = \frac{1}{Z_N(T)N!} \sum_{n_1 < \cdots < n_N} \left| \det \left[\varphi_{n_i}(x_j) \right]_{i,j=1,\ldots,N} \right|^2 \exp\left(-\frac{1}{T} \sum_{i=1}^{N} \varepsilon_{n_i} \right) .$$

This p.d.f. does not define a determinantal point process. However, it is known that the corresponding **grand-canonical ensemble** is determinantal with correlation kernel

$$K_T^N(x, y) = \sum_{n=0}^{\infty} \frac{1}{e^{(\varepsilon_n - \mu)/T} + 1} \varphi_n(x) \varphi_n(y) .$$

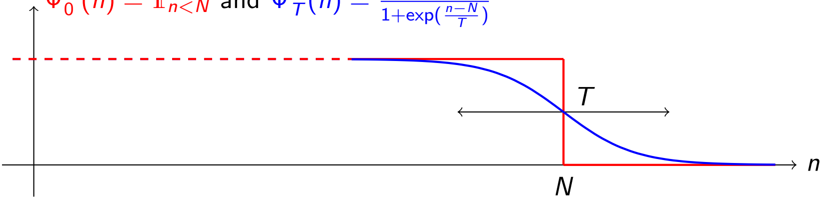
The chemical potential μ is chosen so that the expected number of fermions is $\mathbb{E}\left[\#\right] = N$.

Heuristics

For any $N, T > 0$,

$$K_T^N(x,y) = \sum_{n=0}^{\infty} \Psi_T^N(n)\varphi_n(x)\varphi_n(y) \ .$$

$\Psi_0^N(n) = \mathbb{1}_{n<N}$ and $\Psi_T^N(n) = \frac{1}{1+\exp(\frac{n-N}{T})}$



Let $J_0, J_1, \ldots$ be independent Bernoulli random variables with parameters $\mathbb{E}[J_n] = \Psi_T^N(n)$. An equivalent correlation kernel is given by

$$\widehat{K}_T^N(x,y) = \sum_{n=0}^{\infty} J_n\varphi_n(x)\varphi_n(y) \ .$$

From GUE to Poisson statistics

It turns out that the right scaling to study the transition from GUE statistics to Poisson is

$$T = 2\tau N^{\eta} \qquad \text{where } 0 < \eta < 1 \text{ and } \tau > 0$$
$$\mu = 2N + 1 \ .$$

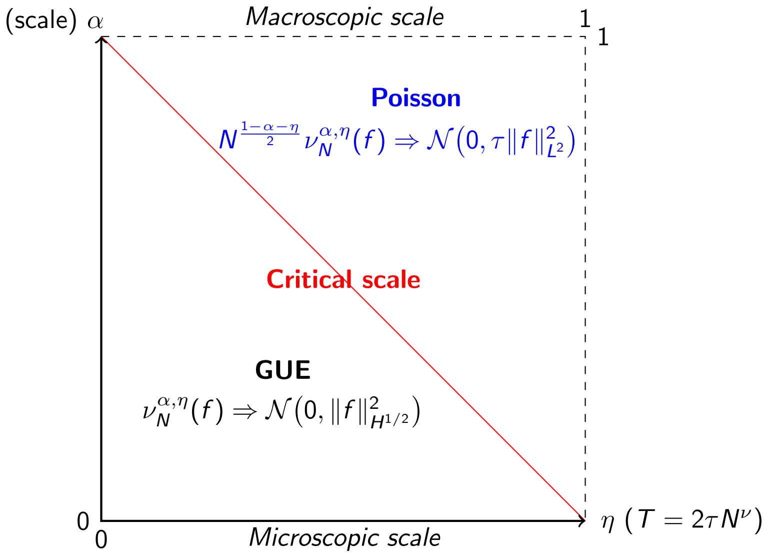
The transition depends on the scaling of the process. If f is a compactly supported function and $0 \leq \alpha \leq 1$, we consider the random variable

$$\nu_N^{\alpha,\eta}(f) = \sum_{k=1}^{\#} f(\lambda_k N^{1-\alpha}) - \mathbb{E}\left[\sum_{k=1}^{\#} f(\lambda_k N^{1-\alpha})\right],$$

where $(\lambda_1, \dots, \lambda_\#)$ are distributed according to the determinantal point process with correlation kernel

$$K_T^N(x,y) = \sum_{n=0}^{\infty} \frac{1}{e^{(n-N)/\tau N^{\eta}} + 1} \varphi_n(x)\varphi_n(y).$$

The transition



The Critical regime

Theorem (Johansson-L)

For any bounded function $f \in H^{1/2}(\mathbb{R})$ with compact support and for any $0 < \alpha < 1$, the linear statistic $\nu_N^{\alpha,\alpha}(f)$ converges in distribution as $N \to \infty$ to a random variable $X_\tau(f)$ whose cumulants are given by

$$C^n\left[X_\tau(f)\right] = 2\sum_{\mathbf{k}\vdash n}\Upsilon_0(\mathbf{k})\int_{u_1+\cdots+u_n=0} d^{n-1}u \int_{x_1<\cdots<x_n} d^n x \;\Re\left\{\prod_{i=1}^{n}\frac{\hat{f}(u_i)e^{x_i}}{(1+e^{x_i})^2}\right\}\mathbf{G}_\tau^{\mathbf{k}}(u,x).$$

where $G_\tau^{\mathbf{k}}(u,x)$ is a complicated function coming from the combinatorics of cumulants...

Cumulants method

if Z is a real-valued random variable, we define its cumulants $\mathrm{C}^n[Z]$ by

$$\Psi_Z(t) = \log \mathbb{E}\left[e^{tZ}\right] = \sum_{n=1}^{\infty} \mathrm{C}^n[Z] \frac{t^n}{n!}$$

If Λ is a determinantal process on $(\mathfrak{X}, \mu)$ with correlation kernel K, then for any function $g \in L^\infty(\mathfrak{X} \to \mathbb{R})$ with compact support in A,

$$\begin{aligned}\mathbb{E}\left[\prod_{\lambda \in \Lambda} (1 + g(\lambda))\right] &= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{A^k} \prod_{i=1}^{k} g(x_i) \det_{k \times k} (K(x_i, x_j))\, d\mu^k(\mathbf{x}) \\ &= \det\left(\mathrm{I} + gK\right)_{L^2(A,\mu)}.\end{aligned}$$

Taking $g(\lambda) = e^{f(\lambda)} - 1$, we obtain an explicit formula for the cumulant generating function of the linear statistics $\Lambda(f) := \sum_{\lambda \in \Lambda} f(\lambda)$.

Cumulants method

If the kernel K is reproducing, i.e. $\int K(x,y)K(y,z)d\mu(y) = K(x,z)$, then we obtain

$$C^n\left[\Lambda(f)\right] = -\int_{x_0=x_n} \Upsilon_0^n[f](\mathbf{x}) \prod_{1\le j\le n} K(x_j, x_{j-1})d\mu^n(\mathbf{x}).$$

where for all $\mathbf{x}\in\mathfrak{X}^n$,

$$\Upsilon_0^n[f](\mathbf{x}) = \sum_{k_1+\cdots+k_\ell=n} \frac{(-1)^\ell}{\ell}\frac{n!}{k_1!\cdots k_\ell!}\prod_{1\le j\le \ell} f(x_j)^{k_j}.$$

Then, to prove that

$$\Lambda_N(f) - \mathbb{E}\left[\Lambda_N(f)\right] \Rightarrow \mathcal{N}\left(0, \|f\|_H^2\right) \quad \text{as } N\to\infty,$$

it suffices to show that for there exists $m>2$ so that

$$\lim_{N\to\infty} C^n\left[\sum_{\lambda\in\Lambda_N} f(\lambda)\right] = \begin{cases} \|f\|_H^2 & \text{if } n=2 \\ 0 & \text{for all } n\ge m \end{cases}.$$

Incomplete determinantal processes

Bohigas and Pato incomplete model

Consider the random configuration Λ of a point process and let $\widetilde{\Lambda}$ be the configuration obtained after performing an independent Bernoulli percolation with parameter $0 < p < 1$ on Λ.

Proposition

If Λ is a determinantal process with correlation kernel K, then $\widetilde{\Lambda}$ is also determinantal with kernel $\widetilde{K}_p(x, y) = pK(x, y)$.

If the kernel K is reproducing and $q = 1 - p$, we obtain

$$\mathrm{C}^n\left[\widetilde{\Lambda}(f)\right] = -\sum_{m=0}^{n}(-q)^m \int_{x_0=x_n} \Upsilon^n_m[f](\mathbf{x}) \prod_{1\le j\le n} K(x_j, x_{j-1})d\mu^n(\mathbf{x}).$$

where for all $m \in [n]$ and $\mathbf{x} \in \mathfrak{X}^n$,

$$\Upsilon^n_m[f](\mathbf{x}) = \sum_{\substack{k_1+\cdots+k_\ell=n\\ m\le\ell\le n}} \frac{(-1)^\ell}{\ell}\binom{\ell}{m}\frac{n!}{k_1!\cdots k_\ell!}\prod_{1\le j\le \ell} f(x_j)^{k_j}.$$

Numerical simulations

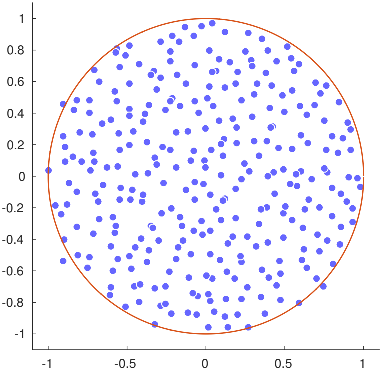


Figure: Sample of the Ginibre ensemble (302 points)

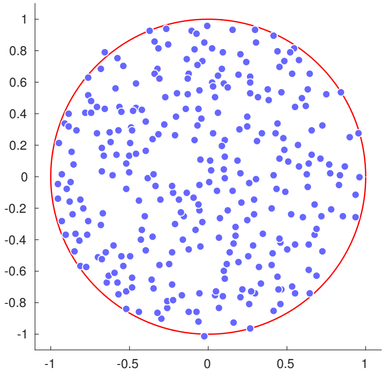
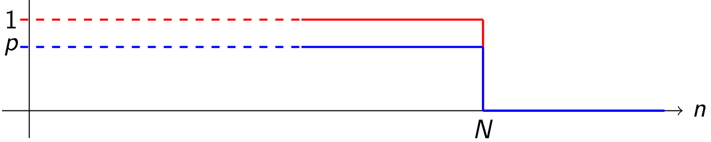


Figure: Sample of the incomplete Ginibre ensemble with p=0.6 (302 points)

Heuristics

e.g. for the GUE,

$$\widetilde{K}_p(x,y) = \sum_{n=0}^{N-1} p\, \varphi_n(x)\varphi_n(y).$$



Let $\mathrm{J}_0, \mathrm{J}_1, \ldots$ be independent Bernoulli random variables with parameters $\mathbb{E}\left[\mathrm{J}_n\right] = p$. An equivalent correlation kernel is given by

$$\widehat{K}_p(x,y) = \sum_{n=0}^{N-1} \mathrm{J}_n \varphi_n(x)\varphi_n(y) .$$

β-ensembles at $\beta = 2$

Consider the Hamiltonian on $\mathfrak{X}^N$

$$\mathcal{H}_V^N(\mathbf{x}) = \sum_{1\le i<j\le N} \log|x_i - x_j|^{-1} + N\sum_{1\le i\le N} V(x_i)$$

and the Boltzmann-Gibbs measure with density $\rho_N(\mathbf{x}) \propto e^{-\beta\mathcal{H}_V^N(\mathbf{x})}$.
If $\beta = 2$,

$$\rho_N(\mathbf{x}) = \frac{1}{N!}\det\left[K_V^N(x_i, x_j)\right]_{i,j=1,\dots,N}$$

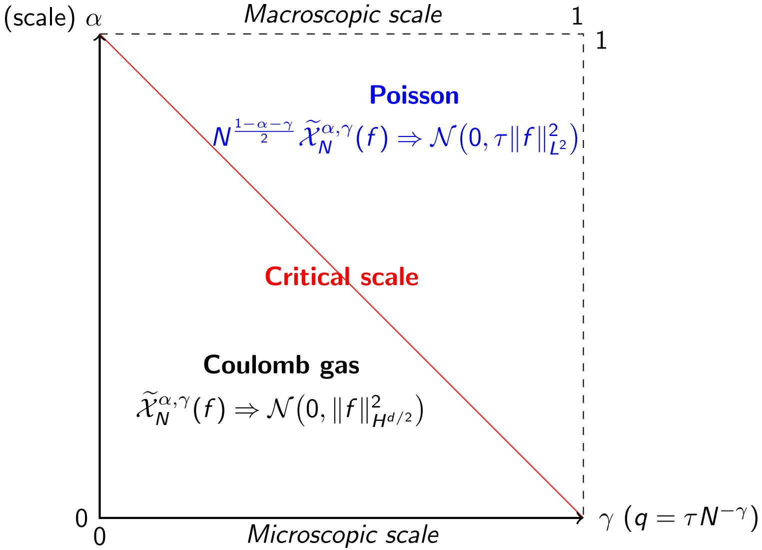
where $K_V^N(x,y) = \sum_{n=0}^{N-1} \varphi_n(x)\varphi_n(y)$ with $\varphi_n(x) = P_n(x)e^{-NV(x)}$ and

$$\int P_n(x)P_m(x)e^{-2NV(x)}d\mu(x) = \delta_{n,m}.$$

We let $\Lambda_N = \{\lambda_1, \dots, \lambda_N\}$ with joint density ϱ_N and $\widetilde{\Lambda}_N$ be the incomplete process with $q = \tau N^{-\gamma}$ and consider the linear statistic:

$$\widetilde{\mathcal{X}}_N^{\alpha,\gamma}(f) = \sum_{\lambda\in\widetilde{\Lambda}_N} f\left((\lambda - x_0)N^{\frac{1-\alpha}{d}}\right) - \mathbb{E}\left[\sum_{\lambda\in\widetilde{\Lambda}_N} f\left((\lambda - x_0)N^{\frac{1-\alpha}{d}}\right)\right].$$

The transition in dimension $d = 1, 2$



The critical regime

Theorem

Suppose that the potential V is real analytic on $\mathbb{R}^d$. Let $f \in C_c^3(\mathbb{R}^d)$ and $0 < \alpha < 1$. If $\lim_{N\to\infty} qN^{\alpha-1} = \tau$, then

$$\lim_{N\to\infty} \mathbb{E}\left[e^{\widetilde{\mathcal{X}}_N^{\alpha,\gamma}(f)}\right] = \exp\left\{\|f\|^2_{H^{d/2}(\mathbb{R}^d)} + \tau\varrho_{\text{eq}}(x_0)\int_{\mathbb{R}^d}\left(e^{-f(x)} - 1 - f(x)\right)dx\right\}.$$

Strategy of the proof

Recall that we have

$$C^n\left[\widetilde{\Lambda}_N(f)\right] = C^n\left[\Lambda_N(f)\right] - \sum_{m=1}^{n}(-q)^m \int_{x_0=x_n} \Upsilon_m^n[f](\mathbf{x}) \prod_{1\le j\le n} K_N(x_j, x_{j-1})d^n\mathbf{x}$$

where $K_V^N(x,y) = \sum_{n=0}^{N-1}\varphi_n(x)\varphi_n(y)$. It suffices to show that there exists numbers Γ_m^n ad $\kappa > 0$ so that for any nice test function f, we have

$$\left|\int_{x_0=x_n} \Upsilon_m^n[f](\mathbf{x}) \prod_{1\le j\le n} K_N(x_j, x_{j-1})d^n\mathbf{x} - \Gamma_m^n \int f(x)^n dx\right| \ll (\log N)^\kappa.$$

It turns out that the correct choice is given by

$$\Gamma_m^n = \Upsilon_m^n[1] = \sum_{\substack{k_1+\cdots+k_\ell=n \\ m\le\ell\le n}} \frac{(-1)^\ell}{\ell}\binom{\ell}{m}\frac{n!}{k_1!\cdots k_\ell!}.$$

In particular, we obtain $\Gamma_0^n = \mathbb{1}_{n=1}$ and $\Gamma_1^n = (-1)^n$.

Thank you!

K. Johansson, G. L. – Gaussian and non-Gaussian fluctuations for mesoscopic linear statistics in determinantal processes. arXiv:1504.06455

G. L. – Incomplete determinantal processes: from random matrix to Poisson statistics. arXiv:1612.00806