Statistics of eigenvectors in the deformed Gaussian unitary ensemble of random matrices

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Overview

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Gaussian Unitary Ensemble (GUE)

- $N \times N$ Hermitian random matrices $H = H^\dagger$
- $\langle H_{ij} \rangle = 0, \quad \left\langle |H_{ij}|^2 \right\rangle = 1/N$
- Probability density $P(H) = C_N e^{-\frac{N}{2}\mathrm{Tr}H^2}$

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Non-trivial statistics of the eigenvalues

Statistics of the eigenvectors in GUE

Unitary invariance of $P(H) = P(UHU^\dagger) \Rightarrow$

- Matrix of the eigenvectors U is uniformly distributed over the unitary group
- In the limit $N \to \infty$, all the eigenvector components ψ_n become independent
- Distribution function of $y = N|\psi_n|^2$ is $P(y) = e^{-y}$
- Inverse participation ratio $I_2 \equiv \sum_{n=1}^{N} \left\langle |\psi_n|^4 \right\rangle \propto 1/N$

Anderson localisation and transition

Anderson model

$$(H\psi)_i = v_i\psi_i + \sum_{\langle ij\rangle} \psi_j, \quad \langle v_i\rangle = 0,\ \langle v_i v_k\rangle = W^2\delta_{ik}$$

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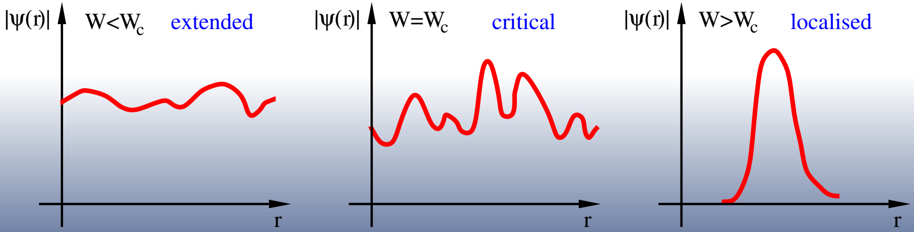

Moments of the eigenvectors

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Extended states: $d_q = 1 \leftrightarrow$ GUE

Localised states: $d_q = 0$

Critical states: $0 < d_q < 1 \leftrightarrow$ multifractal eigenvectors

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Model II

$$H = WVW$$

$$(W)_{ij} = w_i \delta_{ij},\ w_i \text{ - deterministic}, \quad V \in \mathrm{GUE}$$

Supersymmetry approach

Green's functions at the energy $E \pm \mathrm{i}\epsilon \leftrightarrow$ integrals over the supermatrix Q with the action

$$S[Q] = \frac{N}{2}\mathrm{Str}\, Q^2 + \sum_{i=1}^{N} \mathrm{Str} \ln \left[E - d_i - Q + \mathrm{i}\epsilon\Lambda\right]$$

$$\Lambda = \mathrm{diag}(1, 1, -1, -1)$$

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In the limit $N \to \infty$ the integral is dominated by the saddle-points

$$Q = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{E - d_i - Q}$$

Saddle-point solution

$$Q_{s.p.} = t\,I - \mathrm{i}sT^{-1}\Lambda T$$

$T^{-1}\Lambda T$ parametrises the saddle-point manifold in the absence of H_0

$s \neq 0$ and t are two real parameters satisfying the simultaneous equations

$$t = \frac{1}{N}\sum_{i}^{N} \frac{E - t - d_i}{(E - t - d_i)^2 + s^2},$$

$$1 = \frac{1}{N}\sum_{i}^{N} \frac{1}{(E - t - d_i)^2 + s^2}$$

Density of states and moments of the eigenvectors

$$\rho(E) = \frac{s}{\pi}, \quad I_q(n) = \frac{1}{N^q}\left[\frac{1}{(E-t-d_n)^2+s^2}\right]^q \Gamma(q+1)$$

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If $d_i \neq 0 \Rightarrow$

- $I_q(n)$ depends explicitly on d_n and implicitly on all d_i; non-perturbative result
- $I_q \propto N^{-(q-1)}$ – extended eigenvectors for any strength of the perturbation
- I_q/I_q^{GUE} can be arbitrary large

Numerical results

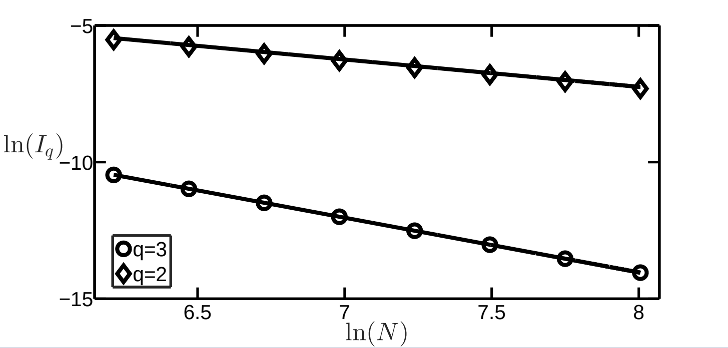

Figure: The moments of the eigenvectors $I_q = \sum_{n=1}^{N} I_q(n)$ for $d_i = -1 + \frac{2}{N}(i-1)$.

Model I with random H_0

$$H = H_0 + V, \quad (H_0)_{ij} = d_i \delta_{ij}, \quad V \in \mathrm{GUE}$$

d_i are independent Gaussian variables, $\langle d_i \rangle = 0$ and $\langle d_i^2 \rangle = \sigma^2$

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$\sigma \gg 1$ – weak perturbation
$\sigma \ll 1$ – strong perturbation

$$t = \frac{1}{N} \sum_i^N \frac{E - t - d_i}{(E - t - d_i)^2 + s^2},$$

$$1 = \frac{1}{N} \sum_i^N \frac{1}{(E - t - d_i)^2 + s^2}$$

t and s random and self-averaging

Equations for $\langle s \rangle$ and $\langle t \rangle$

Averaging over d_i:

$$\langle t \rangle = \left\langle \frac{E - \langle t \rangle - d}{(E - \langle t \rangle - d)^2 + \langle s \rangle^2} \right\rangle_d ,$$

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Using that d is Gaussian distributed:

$$\langle t \rangle = -\mathrm{i}\sqrt{\frac{\pi}{8}}\frac{1}{\sigma} e^{-\frac{(E - \langle t \rangle + \mathrm{i}\langle s \rangle)^2}{2\sigma^2}} F_{-}\left(\frac{E - \langle t \rangle}{\sqrt{2}\sigma}, \frac{\langle s \rangle}{\sqrt{2}\sigma}\right),$$

$$1 = \sqrt{\frac{\pi}{8}}\frac{1}{\sigma \langle s \rangle} e^{-\frac{(E - \langle t \rangle + \mathrm{i}\langle s \rangle)^2}{2\sigma^2}} F_{+}\left(\frac{E - \langle t \rangle}{\sqrt{2}\sigma}, \frac{\langle s \rangle}{\sqrt{2}\sigma}\right),$$

$$F_{\pm}(x, y) = 1 \pm e^{4\mathrm{i}xy}(1 - \mathrm{erf}(\mathrm{i}x + y)) + \mathrm{erf}(\mathrm{i}x - y).$$

The density of states: $\hat{\rho}(E) = \frac{\langle s \rangle}{\pi}$.

Density of states

$\sigma \to 0$	$\sigma \to \infty$
$\hat{\rho}(E) \approx \frac{1}{\pi}\sqrt{1-\frac{E^2}{4}}$	$\hat{\rho}(E) \approx \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{E^2}{2\sigma^2}}$

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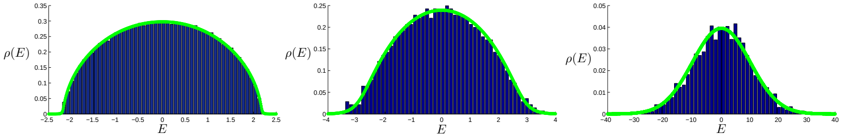

Figure: The histograms for the density of states, for $\sigma = 0.4, 1, 10$, calculated for $N = 3000$ and compared with the analytical predictions.

Moments of the eigenvectors

$$\hat{I}_q = \frac{q}{N^{q-1}} \left[\left(-\frac{1}{2y} \frac{d}{dy} \right)^{q-1} G(y) \right]_{y=\langle s \rangle},$$

$$G(y) = \sqrt{\frac{\pi}{8}} \frac{1}{\sigma y} e^{-\frac{(E-\langle t \rangle + \mathrm{i}y)^2}{2\sigma^2}} F_+ \left(\frac{E - \langle t \rangle}{\sqrt{2}\sigma}, \frac{y}{\sqrt{2}\sigma} \right).$$

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$\sigma \to 0$	$\sigma \to \infty$
$\hat{I}_q \approx \hat{I}_q^{GUE}$	$\hat{I}_q \approx q(2q-3)!! \left(\frac{\sigma^2}{\pi N} \right)^{q-1}$, $\hat{I}_q / \hat{I}_q^{GUE} \propto \sigma^{2(q-1)} \gg 1$

Moments: numerical results

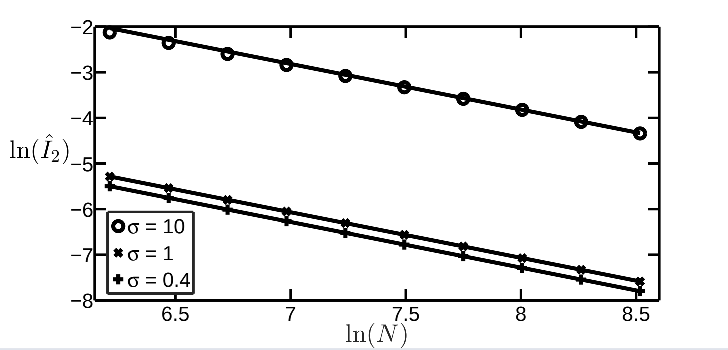

Figure: $\hat{I}_2$ calculated for the three different values of $\sigma = 0.4, 1, 10$ and for $N = 500$ to $N = 3000$.

The Rosenzweig-Porter model

$$H = H_0 + V, \quad (H_0)_{ij} = d_i \delta_{ij}, \quad V \in \mathrm{GUE}$$

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The two-point spectral correlation function: transition from the Wigner-Dyson to the Poisson form at $\gamma = 2$.

Recent results: new phase transition at $\gamma = 1$ separating the ergodic ($\gamma < 1$) and non-ergodic ($1 < \gamma < 2$) states.

$$\hat{I}_q \propto N^{-d_q(q-1)}, \quad 0 < d_q < 1$$

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- For $\gamma > 2$, σ-model breaks down, but the moments can be computed perturbatively.

The Rosenzweig-Porter model: numerical results

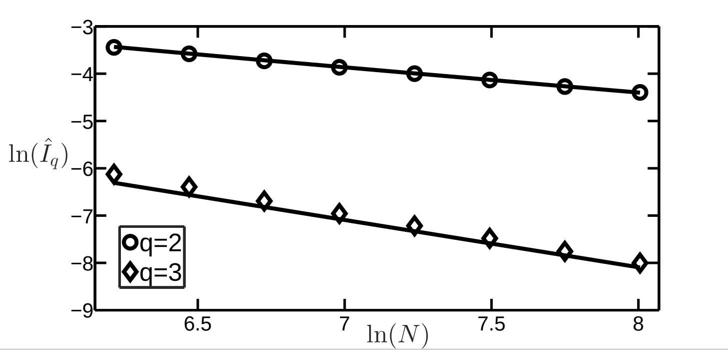

Figure: Numerical simulation for the Rosenzweig-Porter model for $\gamma = 1.5$ and N ranging from 500 to 3000.

Definition of the model and the saddle-point solution

Model II

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Density of states and moments of the eigenvectors

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$$I_q(n) = \frac{1}{(\pi \rho(E) N)^q} \left[\frac{s v_n}{(E - v_n t)^2 + s^2 v_n^2} \right]^q \Gamma(q+1)$$

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- If $v_i = 0 \ \forall i \Rightarrow$ GUE results
- For $E = 0$ the equations for s and t can be solved for $\forall v_i \Rightarrow$

$$\rho(0) = \frac{1}{\pi N} \sum_{i=1}^{N} \frac{1}{v_i}, \quad I_q(n) = \frac{v_n^{-q}}{(\sum_i \frac{1}{v_i})^q} \Gamma(q+1)$$

Example: extended, localized and critical eigenvectors

$$v_n = C_N \left(\frac{1}{n}\right)^p$$

1. $p > 0,\ I_q \propto N^{1-q}$ – GUE scaling

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3. $p < -1,\ I_q \propto \begin{cases} \text{const}, & q > -\frac{1}{p} \\ N^{pq+1}, & q < -\frac{1}{p} \end{cases} \Rightarrow d_q = \begin{cases} 0, & q > -\frac{1}{p} \\ \frac{pq+1}{1-q}, & q < -\frac{1}{p} \end{cases}$

Numerical results for $p = -0.5$

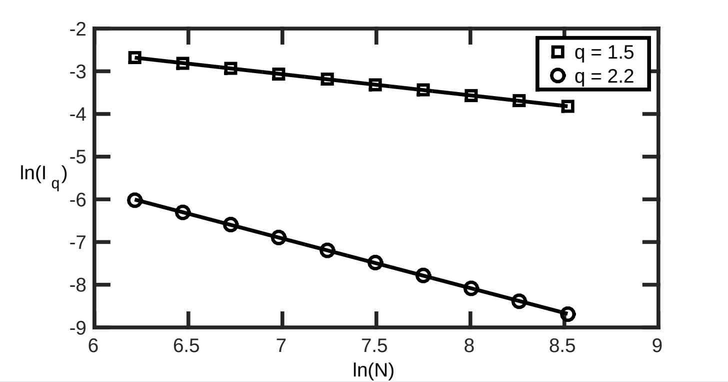

Figure: The numerical values of the gradients of the lines are -0.49 and -1.16 for $q = 1.5$ and $q = 2.2$ respectively. The corresponding analytical results are -0.5 and -1.1

Conclusions

- Non-trivial eigenvectors statistics in the deformed Gaussian unitary ensembles, including extended, localized and critical eigenstates
- Their properties can be described analytically using supersymmetric σ-model
- They can coexist with the Wigner-Dyson statistics of the eigenvalues

K. Truong, AO, J. Phys. A: Math. Theor. **49**, 145005 (2016)
K. Truong, AO, arXiv:1609.03467, accepted in Europhys. Lett.