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Error Exponents in MIMO communications and Matrix Integrals

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December 15, 2018

XIV Brunel-Bielefeld Workshop on RMT

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Content

An instantiation of Multi-input Multi-output Wireless link

Multiantenna Wireless Channels

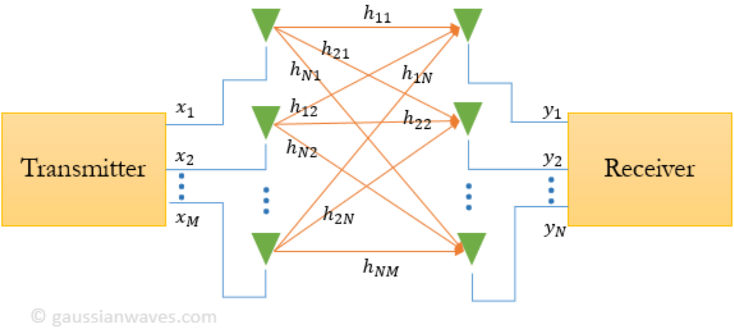
- The input-output relationship on a multiantenna point-to-point radio channel fed with common transmit signal-to-noise ratio (SNR) γ, affected by thermal (AWG) Noise can be written as

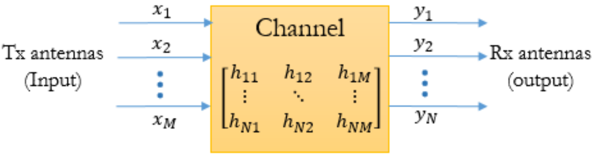
$$\mathbf{y} = \sqrt{\gamma}\mathbf{H}\mathbf{x} + \mathbf{z}; \tag{1}$$

- Above, **y** is the output vector, i.e. the noisy and filtered version of the transmitted input vector **x**. **z** represents the AWGN on the receive antenna array, while the generic entry $h_{i,j}$ of the matrix **H** denotes the channel gain from the j-th transmit antenna to the i-th receive antenna.

Multiantenna Wireless Channels

Multiple Input Multiple Output (MIMO) System





MIMO from channel perspective

Information content

Mutual Information

A channel like (1) is characterized by an information flow (measured in bit/s) given by the so-called *mutual information*.

$$\mathcal{I}(\gamma) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) \,, \tag{2}$$

between the input and the output vectors.

Differential entropy and conditional entropy

In (2),

$$h(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}[-\log\, p(\mathbf{x})], \tag{3}$$

while

$$h(\mathbf{x}|\mathbf{y}) = \mathbb{E}_{(\mathbf{x},\mathbf{y})}[-\log\, p(\mathbf{x}|\mathbf{y})]. \tag{4}$$

Gaussian Environment

Mutual Information with Gaussian Input and Noise

$$\mathcal{I}(\gamma) = \log_2 \det\left(\mathbf{I} + \gamma \mathbf{H}^\dagger \mathbf{H}\right) = \sum_{\ell=1}^{N} \log_2(1 + \gamma \lambda_\ell)\,. \tag{5}$$

We do not generate nor send out an *actually Gaussian* signal, but such a workinig assumption leads to the maximally achievable rate (in bit/s).

Coding Theorem

Error-free (i.e. with vanishing error probability) communication can take place at any speed below a characteristic value, depending on the channel structure/statistics, and denoted as *channel capacity*, if the signal is encoded over a *sufficiently long* sequence.

Capacity of a MIMO channel with informed receiver and uninformed transmitter

In the (ideal) case when a single transmitted codeword/packet experiences all (random) fading states of the channel, the communication speed is characterized by the expected value of $\mathcal{I}(\gamma)$. This is often the easiest metric to be evaluated in closed form, once the statistics of non-zero squared singular values of $\mathbf{H}$ are known.

Error Exponent

Bounding the Error Probability

The average error probability achievable with maximum likelihood decoding can be bounded as

$$P_e \leq K \exp\left(-n_b n_c E(p, R, n_c)\right),$$

if the realization of **H** stays fixed over n_c channel uses; in this case the input-output relationship over a single code block of length n_b reads as

$$\mathbf{Y} = \sqrt{\gamma}\mathbf{H}\mathbf{X} + \mathbf{Z}$$

.

Error Exponent for MIMO

$$E(p(\mathbf{X}), R, n_c) = \max_{0\leq\rho\leq1} \left\{ \max_{r\geq0} -\frac{\ln \mathcal{E}_c}{n_c} - \rho R \right\}, \tag{6}$$

where

$$\mathcal{E}_c = \mathbb{E}_{\mathbf{H}} \left[\int_{\mathbb{C}^{R\times n_c}} \mathbb{E}_{\mathbf{X}} \left[p(\mathbf{Y}|\mathbf{X},\mathbf{H})^{\frac{1}{1+\rho}} \exp\{r(\|\mathbf{X}\|^2 - n_c\mathcal{P})\} \right]^{1+\rho} \mathrm{d}\mathbf{Y} \right]$$

with average input-power constraint,

$$\mathbb{E}_{\mathbf{X}}[\|\mathbf{X}\|^2] \leq n_c\mathcal{P}. \tag{7}$$

Error Exponent related metrics

Cutoff Rate

$$R_0 = -\frac{\log \mathcal{E}_c}{n_c}\Big|_{r=0,\rho=1},$$

Channel Capacity

$$C = \partial_\rho \mathcal{E}_c|_{r=0,\rho=0} = \mathbb{E}\left[\mathcal{I}(\gamma)\right]$$

Considerations

Error exponent(s) evaluation requires on one hand explicit expressions for the input, the output and the transition densities; on the other hand, specific assumptions on the whole transceiver chain are to be made.

Working assumptions

Input Density

$$p_{\mathbf{X}}(\mathbf{X}) \propto \exp\left(-\frac{\|\mathbf{X}\|^2}{\gamma}\right) ,$$

with uniform power allocation across transmit antennas.

Transition Probability

$$p(\mathbf{Y}|\mathbf{X}, \mathbf{H}) \propto \exp\left(-\|\mathbf{Y} - \sqrt{\gamma}\mathbf{H}\mathbf{X}\|^2\right) . \tag{8}$$

A Matrix Integral for the Error Exponent

$$\mathbb{E}_{\mathbf{X}}\left[p(\mathbf{Y}|\mathbf{X},\mathbf{H})^{\frac{1}{1+\rho}}\exp\{r(\|\mathbf{X}\|^2 - n_c\mathcal{P})\}\right]^{1+\rho} \propto$$

$$\int \exp \mathrm{Tr}\left\{-\mathbf{L}\mathbf{X}\mathbf{X}^{\dagger} + \frac{\sqrt{\gamma}}{1+\rho}\left(\mathbf{H}^{\dagger}\mathbf{Y}\mathbf{X}^{\dagger} + \mathbf{X}\mathbf{Y}^{\dagger}\mathbf{H}\right)\right\}\mathrm{d}\mathbf{X} \qquad (9)$$

where

$$\mathbf{L} = \left(\frac{T - r\mathcal{P}}{\mathcal{P}}\right)\mathbf{I} + \frac{\gamma}{1+\rho}\mathbf{H}^{\dagger}\mathbf{H}\,, \qquad (10)$$

The value of the Error Exponent depends in general on the singular values, and on the left and right singular vectors of $\mathbf{H}$.

Unitarily invariant channel

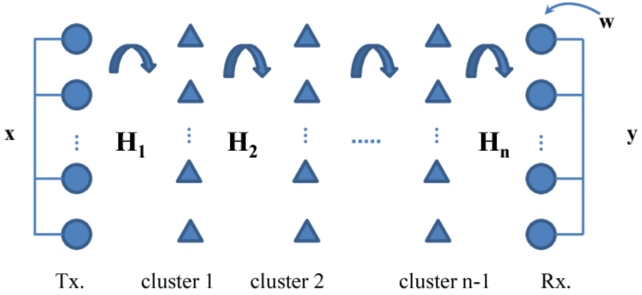
For a block-fading channel, fed by a Gaussian input with uniform power allocation, the RCEE related to an observation window of n_b independent fading blocks of n_c channel uses each, can be expressed as

$$\mathcal{E}_c \propto \int p_{\mathbf{\Lambda}}(\mathbf{\Lambda}) \left|\mathbf{I} + \beta\mathbf{\Lambda}\right|^{-n_c\rho} \mathrm{d}\mathbf{\Lambda}, \tag{11}$$

where $\mathbf{H}^\dagger\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\dagger$.

Let us conclude by making an explicit assumption on $\mathbf{\Lambda}$.

MIMO multiple-clustered scattering



MIMO multiple-clustered scattering

- Objects of noticeable size or groups of objects can be seen as sources of *dominant* contributions in the *multipath fading* process.
- The presence of cluster of dominant scatterers inherently assimilates the channel model, to a cascade of systems.
- The overall impact of multiple-clusters of scatterers is suitably modeled via a product of (independent) Ginibre matrices

$$\mathbf{H} = \mathbf{G}_n \dots \mathbf{G}_2 \mathbf{G}_1 \, . \tag{12}$$

- Spatial correlation at either ends of the link and the presence of LOS paths makes the model more involved, though it should be taken into account (not reported here, but spectral properties in presence of correlation have been derived by Kuijlaars-Stivigny (2014) and Wei (2015)).

Error Exponent with Multiple Scattering

For a block-fading channel with progressive scattering, the RCEE is given by (6) where

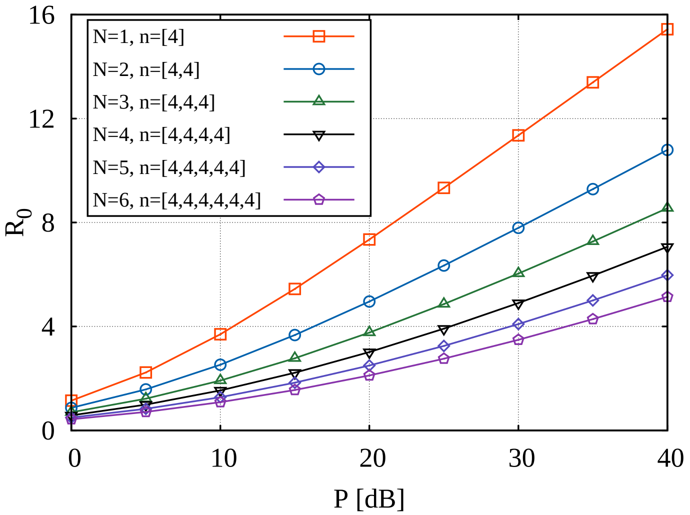
$$\mathcal{E} \propto \left| \mathbf{Z}\left(\frac{(1-pr)(1+\rho)}{\gamma p} \right) \right| . \tag{13}$$

The elements of the $T \times T$ matrix $\mathbf{Z}(x)$ are given by

$$[\mathbf{Z}(x)]_{i,j} = \frac{G^{n+1,1}_{1,n+1}\left(\begin{matrix} 1 \\ n_c\rho, \nu_n + j, \dots, \nu_2 + j, \nu_1 + i + j - 1 \end{matrix} \middle| x \right)}{\Gamma(n_c\rho)} ,$$

where $\nu_i = n_i - T$, $i = 1, \dots, n$.

Cut-off rate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N=2, n=[4,4]
N=3, n=[4,4,4]
N=4, n=[4,4,4,4]
N=5, n=[4,4,4,4,4]
N=6, n=[4,4,4,4,4,4]
R_0
0
4
8
12
16
0
10
20
30
40
P [dB]

Open Problems

- Error Exponents expressions depend on coding scheme, and channel state availability;
- If the receiver is not informed, $p(\mathbf{Y}|\mathbf{X})$ appears (and it is not Gaussian in general);
- For $\mathbf{H}$ non unitarily invariant, the expression is more involved;
- A more general expression of EE corresponding to a *realistic* channel dynamic is to be provided.
- 10.1109/TIT.2018.2803046

Thank you!