Optimal global rigidity estimates in unitary invariant ensembles

Tom Claeys

UCLouvain

joint work with Benjamin Fahs, Gaultier Lambert and Christian Webb

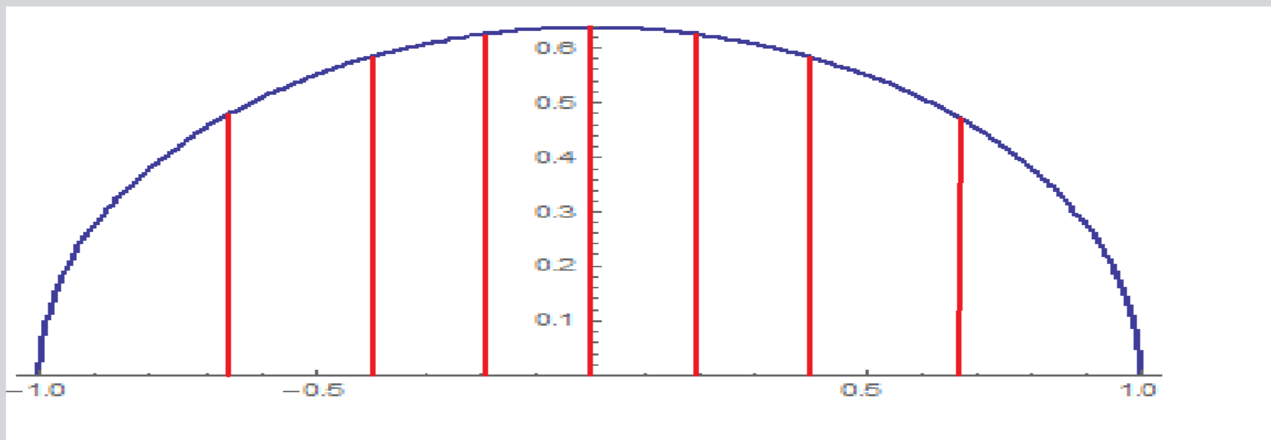
XIV Brunel–Bielefeld Workshop on Random Matrix Theory, December 14, 2018

Global rigidity in the GUE

Classical GUE eigenvalue locations

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the ordered eigenvalues of a GUE matrix of size $N \times N$, normalized such that the eigenvalue distribution converges to a semi-circle law on $[-1, 1]$. We define the classical locations $\kappa_1, \kappa_2, \ldots, \kappa_N \in [-1, 1]$ of the eigenvalues by

$$\frac{2}{\pi} \int_{-1}^{\kappa_j} \sqrt{1 - x^2} dx = \frac{j}{N}, \qquad j = 1, \ldots, N.$$



Global rigidity in the GUE

Global rigidity

What can we say for large N about the distribution of the normalized maximal fluctuation of eigenvalues (cf. Bourgade-Erdos-Yau)

$$M_N := \max_{j=1,\dots,N}\left\{\frac{2}{\pi}\sqrt{1-\kappa_j^2}|\lambda_j-\kappa_j|\right\}?$$

Global rigidity in the GUE

Upper bound for generalized Wigner matrices (Erdos-Yau-Yin '12)

$$\mathbb{P}\left(M_N \geq \frac{(\log N)^{\alpha \log\log N}}{N}\right) \leq C \exp\left(-c(\log N)^{\alpha' \log\log N}\right)$$

Lower bound for GUE (Gustavsson '05)

$$2\sqrt{2}\sqrt{1-\kappa_j^2}\,\frac{N}{\sqrt{\log N}}(\lambda_j - \kappa_j) \to \mathcal{N}(0,1)$$

for $\delta \leq j \leq (1-\delta)N$, which implies (non-optimal) lower bounds for M_N

.

Global rigidity in the GUE

Theorem (C-Fahs-Lambert-Webb '18)

For any $\epsilon > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\left((1-\epsilon)\frac{\log N}{\pi N} < M_N < (1+\epsilon)\frac{\log N}{\pi N} \right) = 1.$$

Unitary invariant ensembles

A similar result holds for unitary invariant ensembles with eigenvalue distribution

$$\frac{1}{Z_N} \prod_{1\leq i<j\leq N} \left|\lambda_i - \lambda_j\right|^2 \prod_{1\leq j\leq N} e^{-NV(\lambda_j)}$$

for real analytic V with sufficient growth at $\pm\infty$.

Equilibrium measure and classical locations

Semi-circle law is then replaced by the equilibrium measure μ_V minimizing

$$\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|^{-1} d\mu(x)d\mu(y) + \int_{\mathbb{R}} V(x)d\mu(x).$$

We assume that μ_V is one-cut regular, and that the support is $[-1,1]$ for convenience.

The classical locations $\kappa_1, \ldots, \kappa_N \in [-1,1]$ are now defined by $\int_{-1}^{\kappa_j} d\mu_V(x) = j$.

Global rigidity in unitary invariant ensembles

Theorem (C-Fahs-Lambert-Webb '18)

For any $\epsilon > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\left((1-\epsilon)\frac{\log N}{\pi N} < \max_{j=1,\dots,N} \left\{ \frac{d\mu_V}{dx}(\kappa_j)|\lambda_j - \kappa_j| \right\} < (1+\epsilon)\frac{\log N}{\pi N} \right) = 1.$$

Global rigidity in unitary invariant ensembles

Eigenvalue counting function

We prove this result by estimating the extrema of the normalized eigenvalue counting function

$$h_N(x) = \sqrt{2}\pi\left(\sum_{1\le j\le N} \mathbf{1}_{\lambda_j\le x} - N\int_{-1}^{x} d\mu_V \right), \qquad x\in\mathbb{R}.$$

Namely, we prove that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[(1-\delta)\sqrt{2}\log N \le \max_{x\in\mathbb{R}} \{ \pm h_N(x) \} \le (1+\delta)\sqrt{2}\log N \right] = 1.$$

Heuristically, we expect $h_N(\lambda_j) = \int_{\lambda_j}^{\kappa_j} d\mu_V(x) \approx \frac{d\mu_V}{dx}(\kappa_j)(\kappa_j - \lambda_j)$, which explains the connection between global rigidity and the maximum of the normalized eigenvalue counting function.

Extreme of log-correlated fields

The problem is then to study extrema of the log-correlated field h_N.
Extrema of such processes have been studied in other models:

- ✓ Riemann ζ function and CUE (Fyodorov-Hiary-Keating '12, Arguin-Belius-Bourgade '16, Chhaibi-Madaule-Najnudel '16)
- ✓ Circular Beta Ensemble and Sine Beta process (Chhaibi-Madaule-Najnudel '16, Paquette-Zeitouni '16, Holcomb-Paquette '18)
- ✓ Characteristic polynomial in unitary invariant ensembles (Fyodorov-Simm '14, Lambert-Paquette '18)

Global rigidity in unitary invariant ensembles

Multiplicative chaos

Powerful tools to study such extrema come from the theory of multiplicative chaos

- ✓ General theory (Kahane '85, Rhodes-Vargas '14, Berestycki '15)
- ✓ Applied to Circular Unitary Ensemble (Fyodorov-Keating '14, Webb '15, Berestycki-Webb-Wong '18, Lambert-Ostrovsky-Simm '18)

Upper bound estimates

The upper bound for $\max_{x\in I}\{\pm h_N(x)\}$ can be obtained using an elementary one-moment method.

1. $$\max_{x\in I}\{\pm h_N(x)\} \leq \max_{j:\kappa_j\in I}\{\pm h_N(\kappa_j)\} + 1.$$

2. By a union bound and Markov's inequality,

$$\mathbb{P}\left(\max_{j:\kappa_j\in I}\{h_N(\kappa_j)\} > Y\right) \leq \sum_{j:\kappa_j\in I}\mathbb{P}\left(h_N(\kappa_j) > Y\right)$$

$$\leq \sum_{j:\kappa_j\in I}\frac{\mathbb{E}e^{\gamma h_N(\kappa_j)}}{e^{\gamma Y}}.$$

Upper bound estimates

3. $\mathbb{E}e^{\gamma h_N(x)}$ is a Hankel determinant with discontinuous weight $e^{-NV(\lambda)}\, e^{\gamma \mathbf{1}_{\lambda \leq x}}$, and large N asymptotics for such Hankel determinants are known for $x \in (-1+\delta, 1-\delta)$ (Its-Krasovsky '08 for GUE, Charlier '18 for one-cut regular unitary invariant ensembles).

Upper bound estimates

4. Choosing γ, Y conveniently and substituting the Hankel asymptotics, we get the upper bound

$$\lim_{N\to\infty} \mathbb{P}\left[\max_{x\in[-1+\delta,1-\delta]} h_N(x) \leq (1+\delta)\sqrt{2}\log N\right] = 1.$$

5. To extend the upper bound to $\max_{x\in\mathbb{R}}$, we prove estimates for $\mathbb{E}e^{\gamma h_N(x)}$ for x close to ± 1, or in other words for Hankel determinants with a jump discontinuity close to a soft edge,

$$\mathbb{E}e^{\gamma h_N(x)} = O\left((1-x^2)^{\frac{3\gamma^2}{4}} N^{\frac{\gamma^2}{2}}\right)$$

as $N \to \infty$, uniformly for $|x| \leq 1 - MN^{-2/3}$.

Global rigidity in unitary invariant ensembles

Lower bound estimates

Optimal lower bound estimates are much harder to obtain, and require to investigate the log-correlated structure of h_N.

Log-correlated structure

It is well-known (Johansson '98) that $h_N(x)$ behaves for large N like a Gaussian process $X(x)$ with logarithmic covariance kernel

$$\Sigma(x,y) := \log\left|\frac{1-xy+\sqrt{1-x^2}\sqrt{1-y^2}}{x-y}\right|.$$

Maximum of the eigenvalue counting function

For studying the maximum of h_N, we prove that the random measure

$$d\mu_N^\gamma = \frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}dx, \qquad \gamma \in \mathbb{R}$$

converges weakly in distribution to a multiplicative chaos measure which can be formally written as (cf. Kahane '85, Rhodes-Vargas '10, Berestycki '17, Berestycki-Webb-Wong '17)

$$d\mu^\gamma(x) = \frac{e^{\gamma X(x)}}{\mathbb{E}e^{\gamma X(x)}}dx.$$

Multiplicative chaos

Extreme values

It will turn out that the extreme values of the limiting measure μ^γ will lead us to estimates for extreme values of h_N.

Heuristics

Heuristically, the random measure $d\mu_N^\gamma(x) = \frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}dx$ is expected to be dominated for $\gamma > 0$ by x-values where $h_N(x)$ is exceptionally large, and it is natural to expect that the multiplicative chaos measure μ^γ will give us information about large values of $h_N(x)$.
For $|\gamma| > \sqrt{2}$, $\mu^\gamma = 0$.

Multiplicative chaos

Multiplicative chaos and γ-thick points

Consider the set of γ-thick points

$$\mathscr{T}_N^{\pm\gamma} = \{x \in [-1,1] : \pm h_N(x) \geq \pm\gamma \log N\}\,.$$

This set contains points where $h_N(x)$ is of the order of its variance rather than its standard deviation. It follows from the multiplicative chaos convergence that for any $\gamma \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$, in probability,

$$\lim_{N\to\infty} \frac{\log|\mathscr{T}_N^{\gamma}|}{\log N} = -\frac{\gamma^2}{2}.$$

Multiplicative chaos

Freezing transition

Another consequence of the multiplicative chaos convergence is that

$$\lim_{N\to\infty} \frac{1}{\log N} \log\left(\int_{-1}^{1} e^{\gamma h_N(x)}\,dx\right) = \begin{cases} \gamma^2/2 & \text{if } \gamma \le \sqrt{2} \\ \sqrt{2}\gamma - 1 & \text{if } \gamma \ge \sqrt{2} \end{cases},$$

in probability.

In the physics literature, this is called a freezing transition of the random energy landscape h_N (cf. Fyodorov-Bouchaud '08, Fyodorov-Le Doussal-Russo '12, Fyodorov-Keating '14 for CUE).

Exponential moment estimates

Convergence to multiplicative chaos

The key technical input to prove convergence of μ_N^γ to μ consists of detailed asymptotic estimates as $N \to \infty$ for exponential moments of the form

$$\mathbb{E} e^{\gamma_1 h_N(x) + \gamma_2 h_N(y) + \sum_{j=1}^N W(\lambda_j)}.$$

These can also be written as Hankel determinants

$$D_N(x, y; \gamma_1, \gamma_2; W) = \det \left(\int_{\mathbb{R}} \lambda^{i+j} f(\lambda; x, y; \gamma_1, \gamma_2; W) d\lambda \right)_{i,j=0}^{N-1},$$

with $f(\lambda; x, y; \gamma_1, \gamma_2; W) = e^{\sqrt{2}\pi\gamma_1 \mathbf{1}_{\{\lambda \leq x\}} + \sqrt{2}\pi\gamma_2 \mathbf{1}_{\{\lambda \leq y\}} + W(\lambda) - NV(\lambda)}$.

Asymptotics are known (Charlier '18) for $x \neq y \in (-1, 1)$ fixed and for W independent of N.

Exponential moment estimates

Two merging singularities

$$\log D_N(x_1,x_2;\gamma_1,\gamma_2;0) = \log D_N(x_1;\gamma_1+\gamma_2;0) + \sqrt{2}\pi\gamma_2 N\int_{x_1}^{x_2} d\mu_V$$
$$-\gamma_1\gamma_2\max\{0,\log(|x_1-x_2|N)\} + \mathcal{O}(1),$$

as $N\to\infty$, where the error term is uniform for $-1+\delta < x_1 < x_2 < 1-\delta, 0 < x_2 - x_1 < \delta$ for δ sufficiently small.

Method of proof

We prove this using a similar method than the one used for Toeplitz determinants with merging Fisher-Hartwig singularities (C-Krasovsky '15) and Hankel determinants with merging root singularities (C-Fahs '16), based on a Riemann-Hilbert method.

N-dependent W

Assume that $W = W_N$ is a sequence of functions which are analytic and uniformly bounded on a suitable domain which does not shrink too fast with N.

$$\log D_N(x_1, x_2; \gamma_1, \gamma_2; W_N) = \log D_N(x_1, x_2; \gamma_1, \gamma_2; 0)$$
$$+ N \int W_N d\mu_V + \frac{1}{2}\sigma(W_N)^2 + \sum_{j=1}^{2} \frac{\gamma_j}{\sqrt{2}} \sqrt{1 - x_j^2}\,\mathcal{U}W_N(x_j) + o(1),$$

as $N \to \infty$, uniformly for (x_1, x_2) in any fixed compact subset of $(-1, 1)^2$, where

$$\sigma(f)^2 = \iint_{\S^2} f'(x) f'(y) \frac{\Sigma(x, y)}{2\pi^2} dx dy, \ (\mathcal{U}w)(x) = \frac{1}{\pi} \mathrm{P.V.} \int_{-1}^{1} \frac{w(t)}{x - t} \frac{dt}{\sqrt{1 - t^2}}.$$

Exponential moment estimates

Finally, we need also asymptotics for Hankel determinants with one singularity tending to the edge ± 1. This is needed for the upper bound estimate for the maximum of h_N.

Singularity close to the edge

$$\log \frac{D_N(x;\gamma;0)}{D_N(x;0;0)} = \sqrt{2}\pi\gamma N \int_{-1}^{x} d\mu_V(\xi) + \frac{\gamma^2}{2}\log N + \frac{3\gamma^2}{4}\log(1-x^2) + \mathcal{O}(1),$$

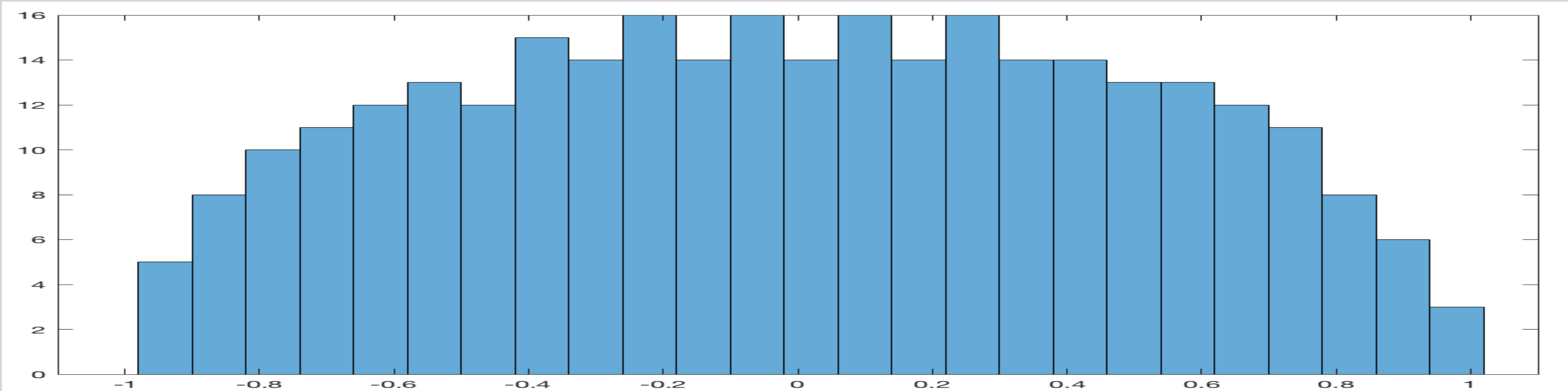
as $N \to \infty$, with the error term uniform for all $|x| \leq 1 - MN^{-2/3}$, with M sufficiently large.

Summary of the method

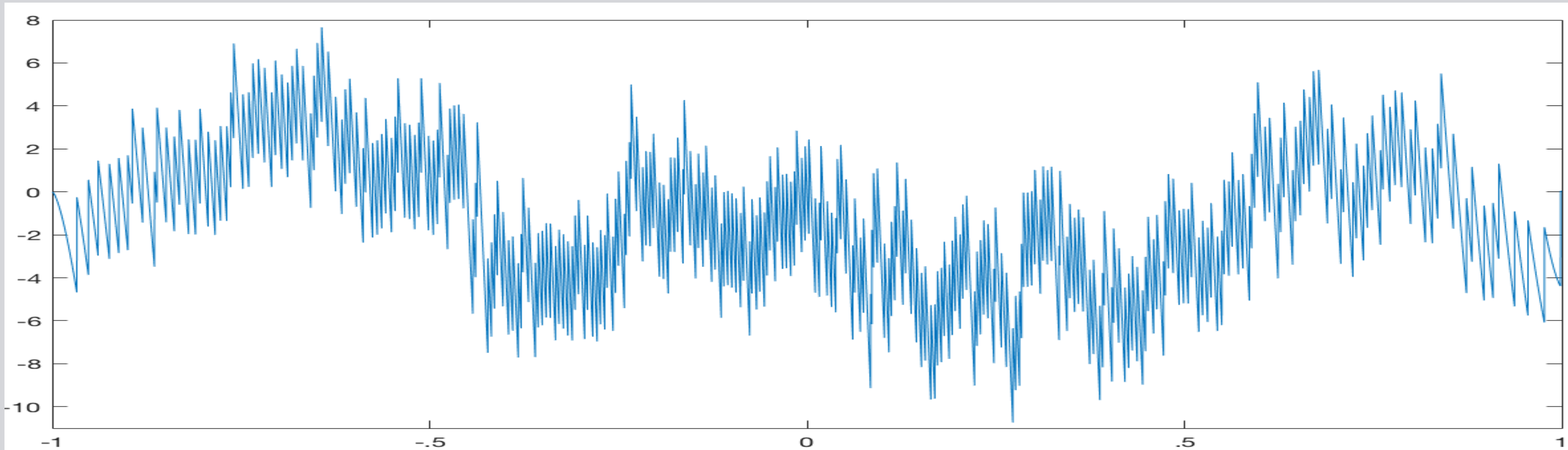
1. Hankel determinant asymptotics
   $\implies$ Convergence of $\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}dx$ to a multiplicative chaos measure μ^γ
   $\implies$ Estimates for γ-thick points
   $\implies$ Estimates for the lower bound of $\max h_N$
2. Hankel determinant asymptotics
   $\implies$ Estimates for the upper bound of $\max h_N$ via one-moment method
3. Estimates for extrema of h_N
   $\implies$ Estimates for global rigidity of eigenvalues

Simulations

Histogram of GUE eigenvalues for $N = 300$

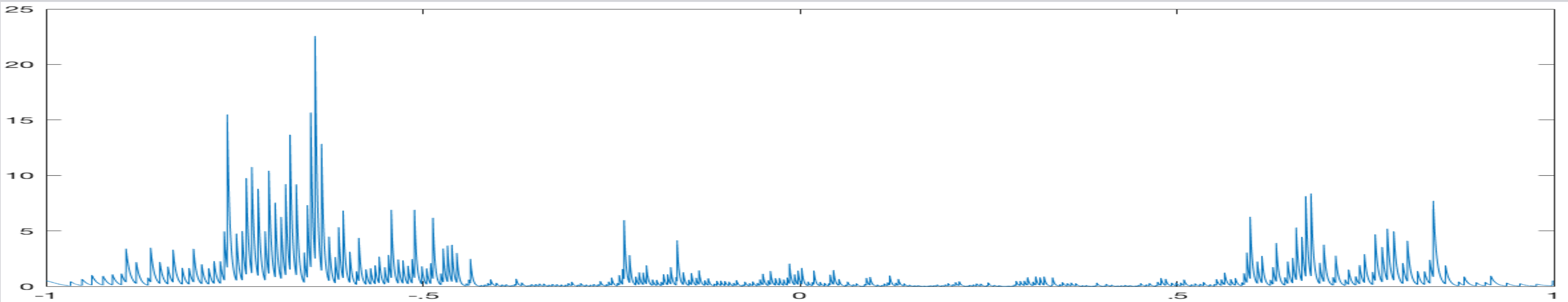


Normalized eigenvalue counting function h_N for $N = 300$.

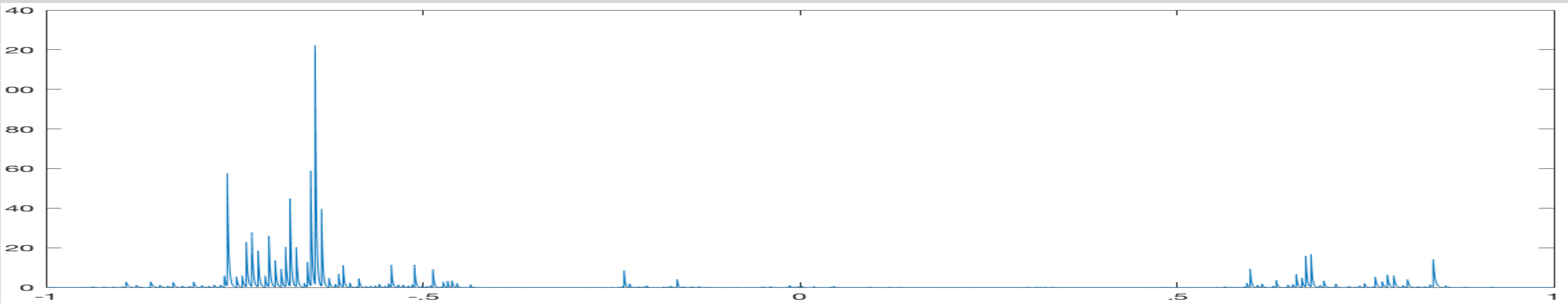


Simulations

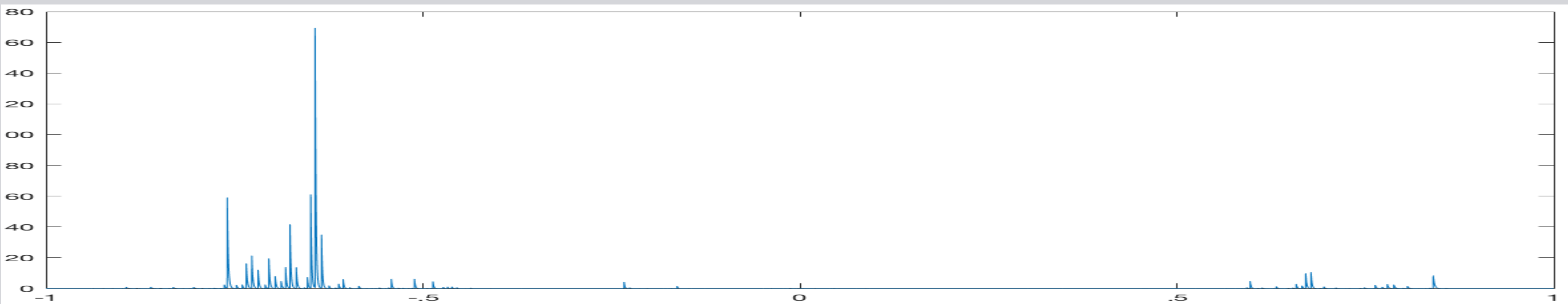
$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 0.5$



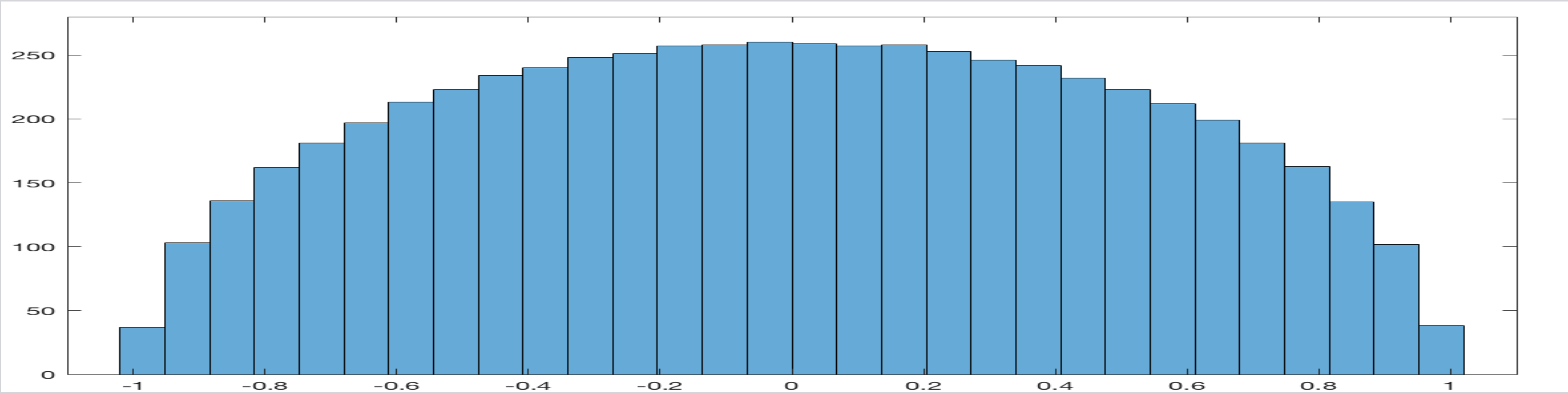
$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 1.0$



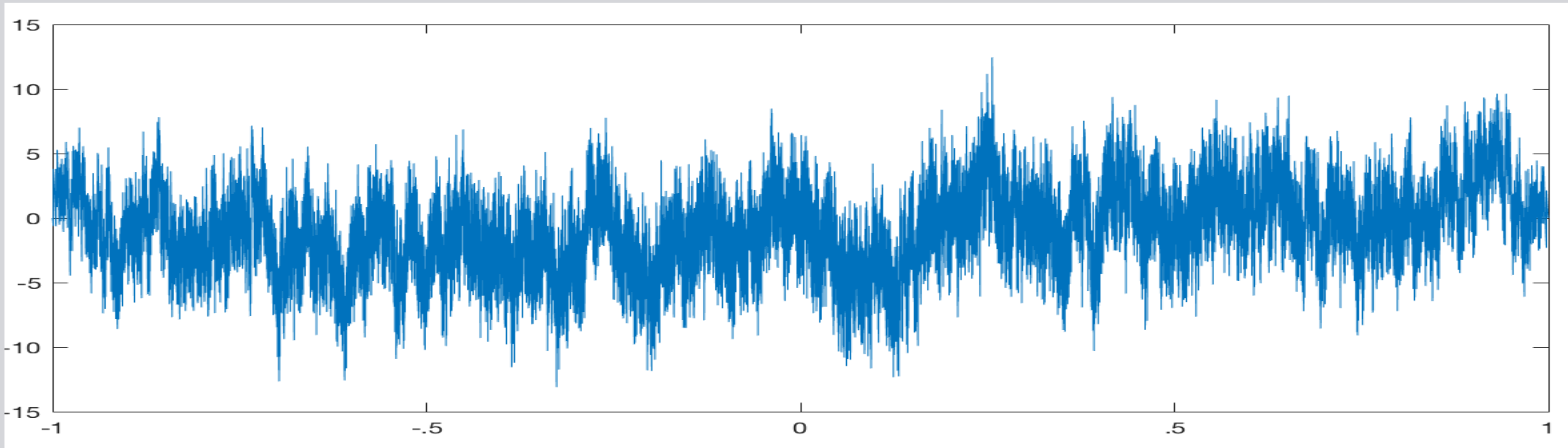
$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ **with** $\gamma = 1.4$



Histogram of GUE eigenvalues for $N = 300$

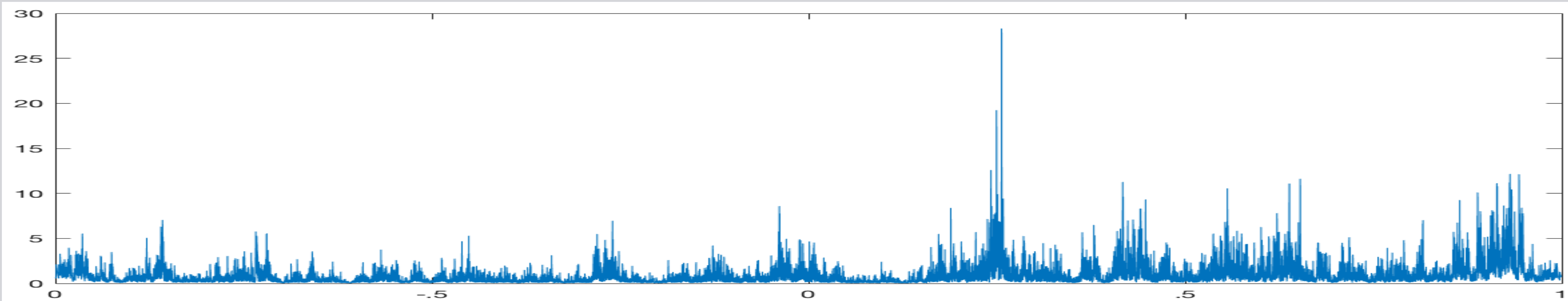


Normalized eigenvalue counting function h_N for $N = 300$.

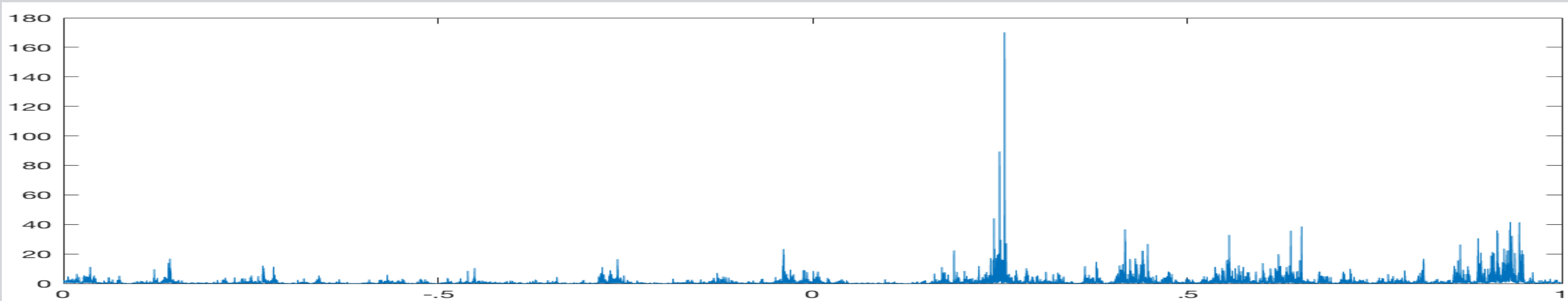


Simulations

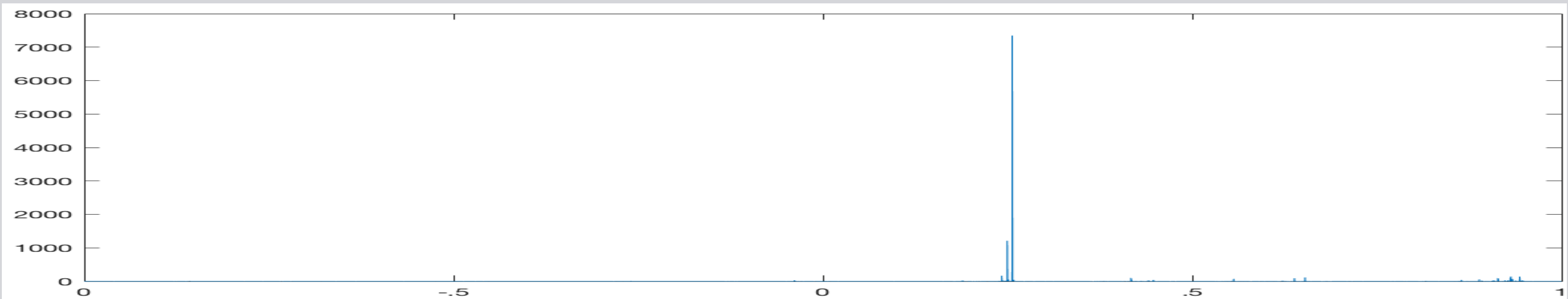
$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 0.3$



$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 0.5$



$\frac{e^{\gamma h_N(x)}}{\mathbb{E}e^{\gamma h_N(x)}}$ with $\gamma = 1.4$



Thank you for your attention!