December 15, 2018

Brunel

Periodically weighted tilings and (matrix) orthogonal polynomials

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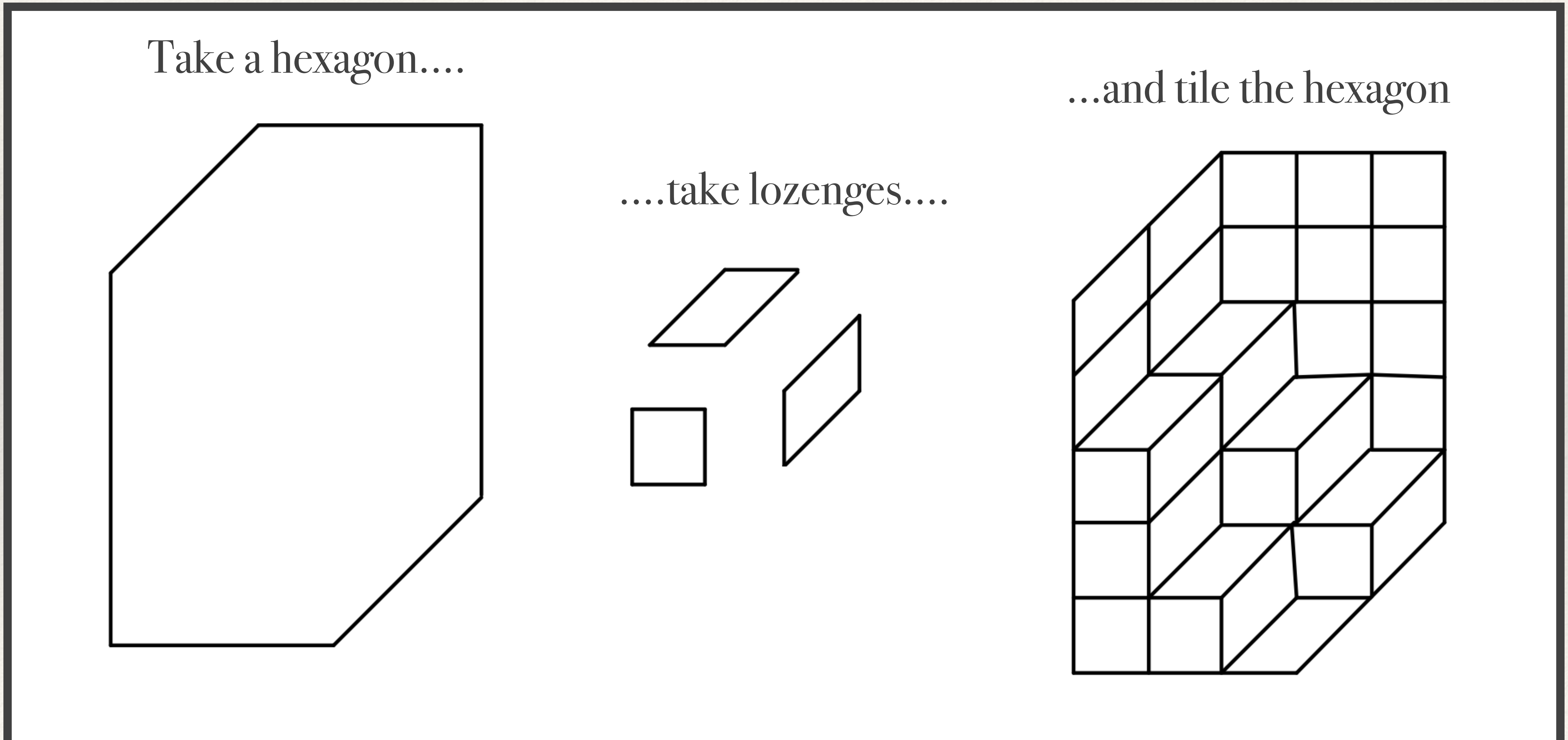
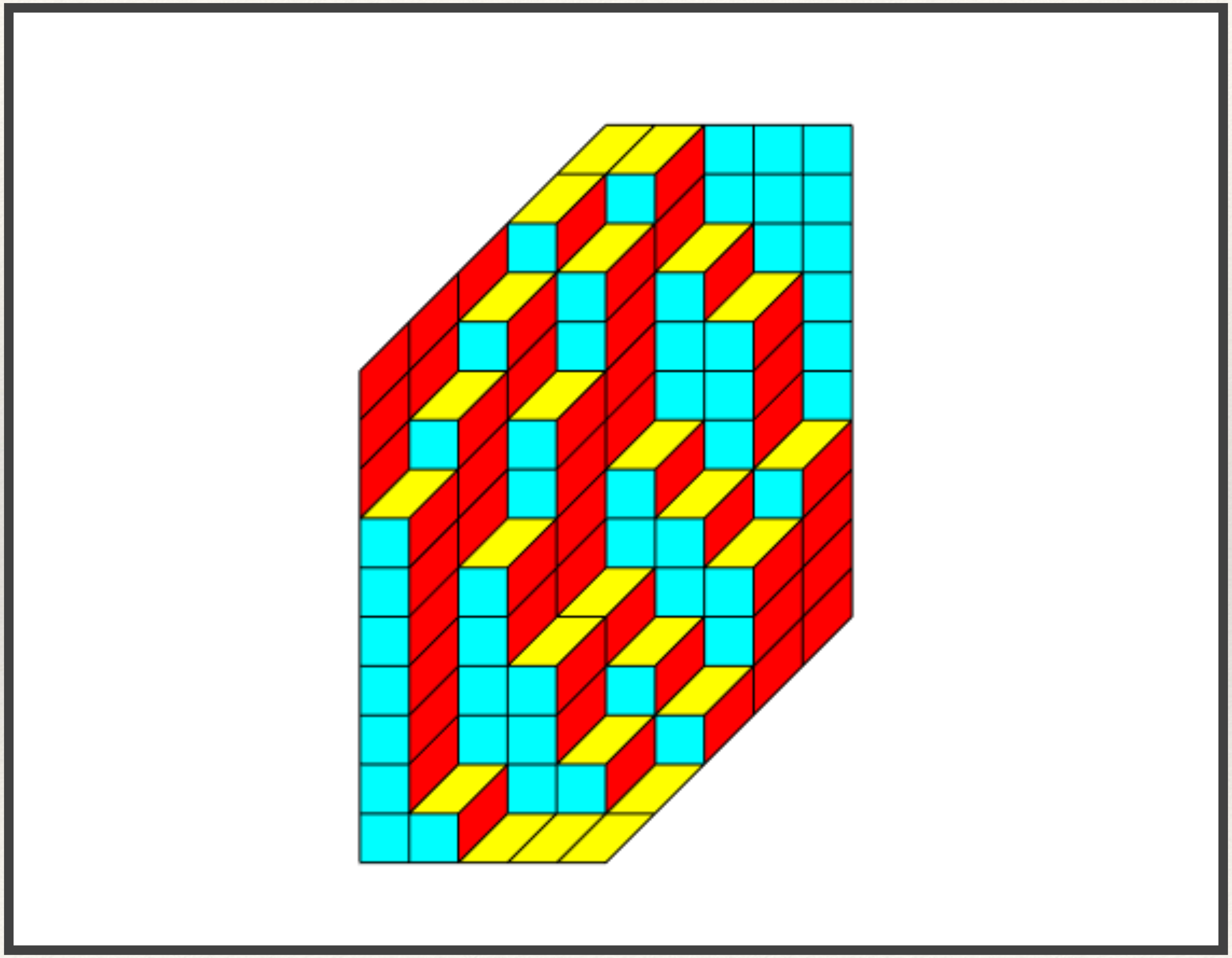
Based on joint works with :

- *A.B.J. Kuijlaars, arXiv:1712.05636*
- *C. Charlier, A.B.J. Kuijlaars and J. Lenells (upcoming)*
- *T. Berggren, (upcoming)*

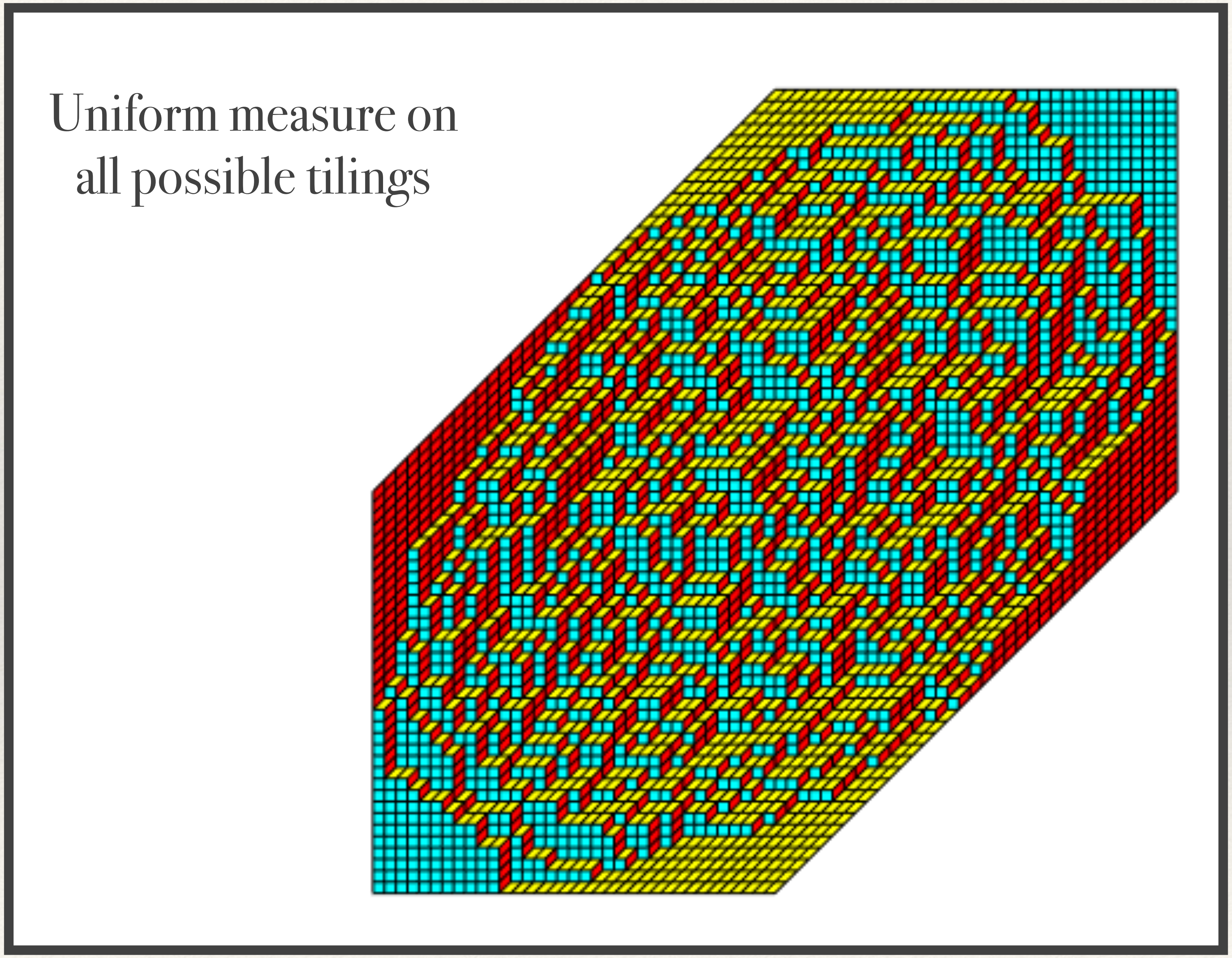
Lozenge tilings of the hexagon



Lozenge tilings of the hexagon

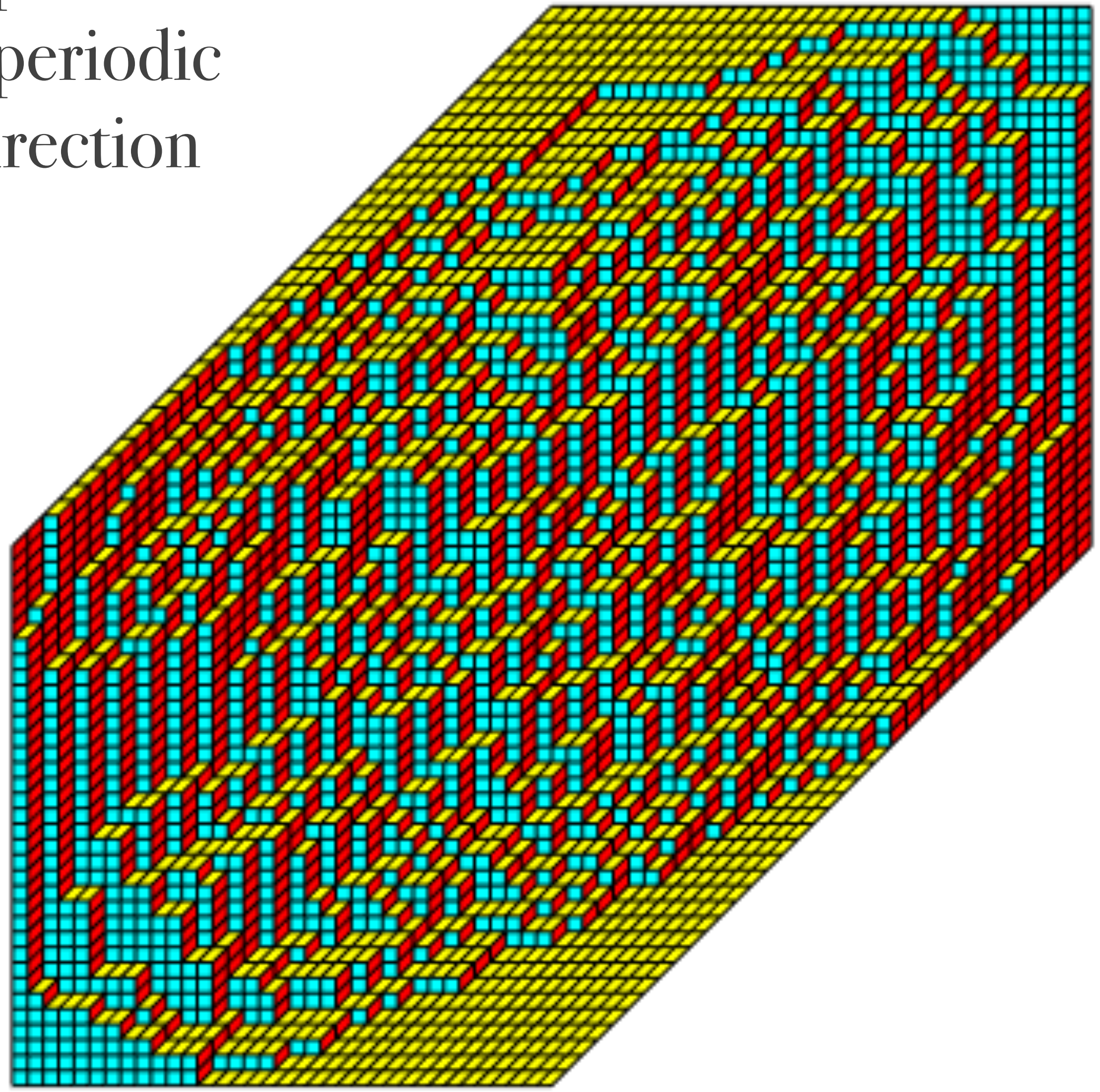


Random lozenge tilings large hexagons



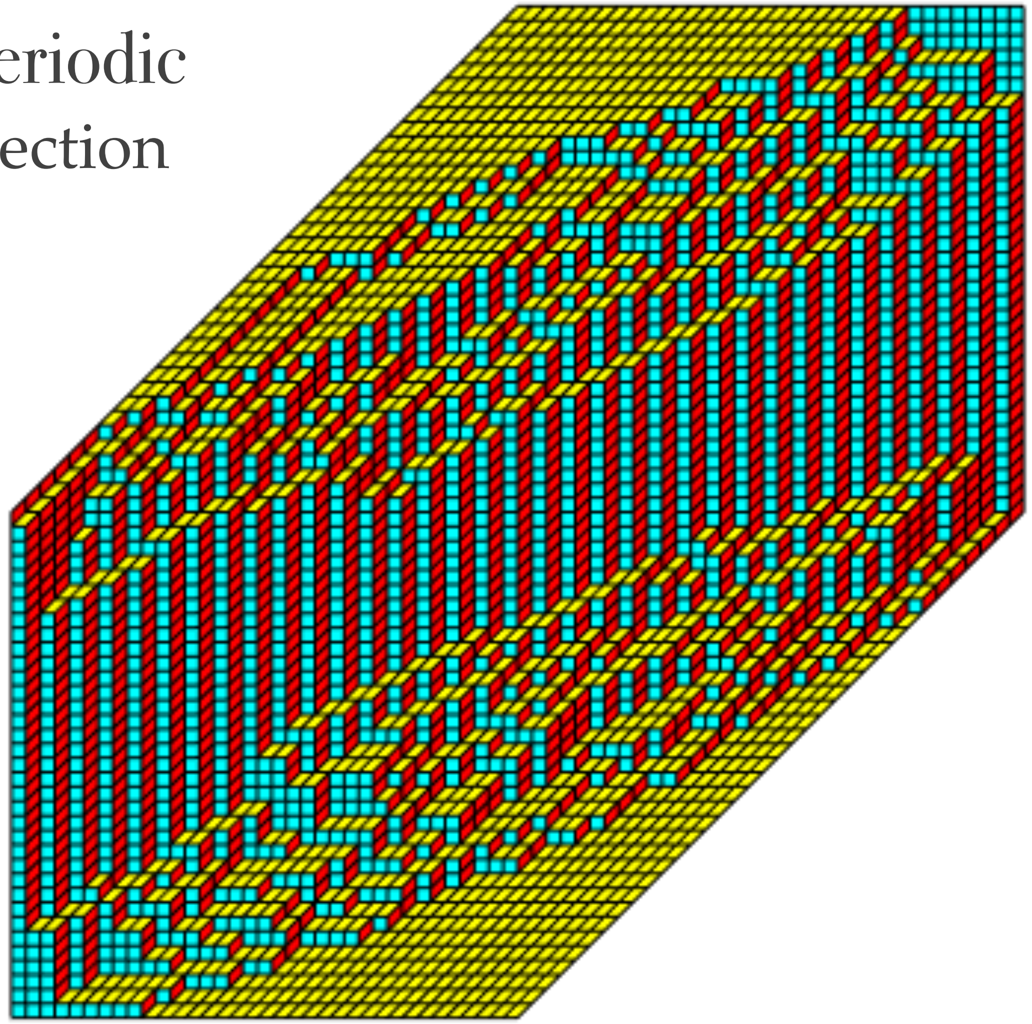
Lozenge tilings of the hexagon

Measure on all possible tilings that is 2-periodic in horizontal direction

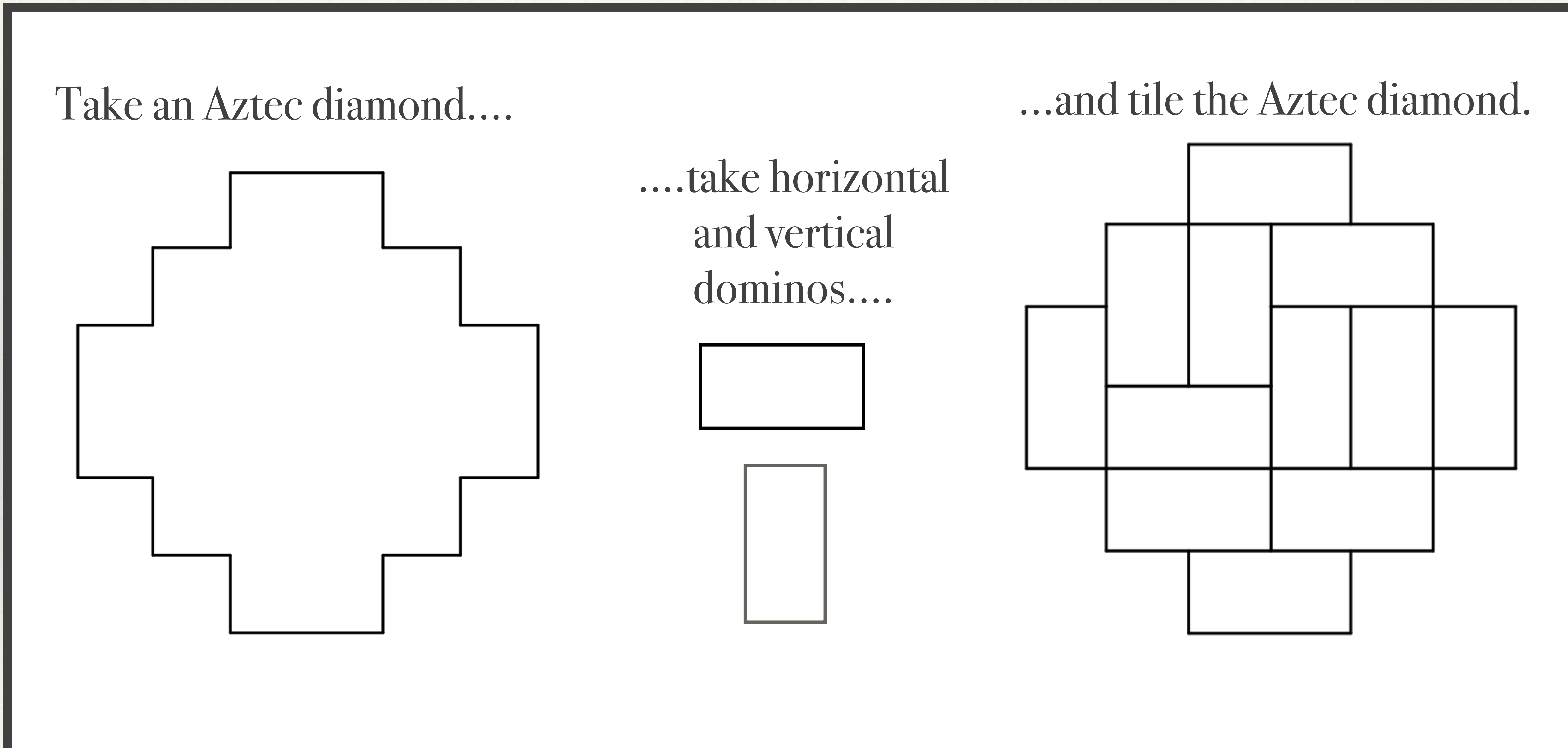


Lozenge tilings of the hexagon

Measure on all possible tilings that is 2-periodic in horizontal direction

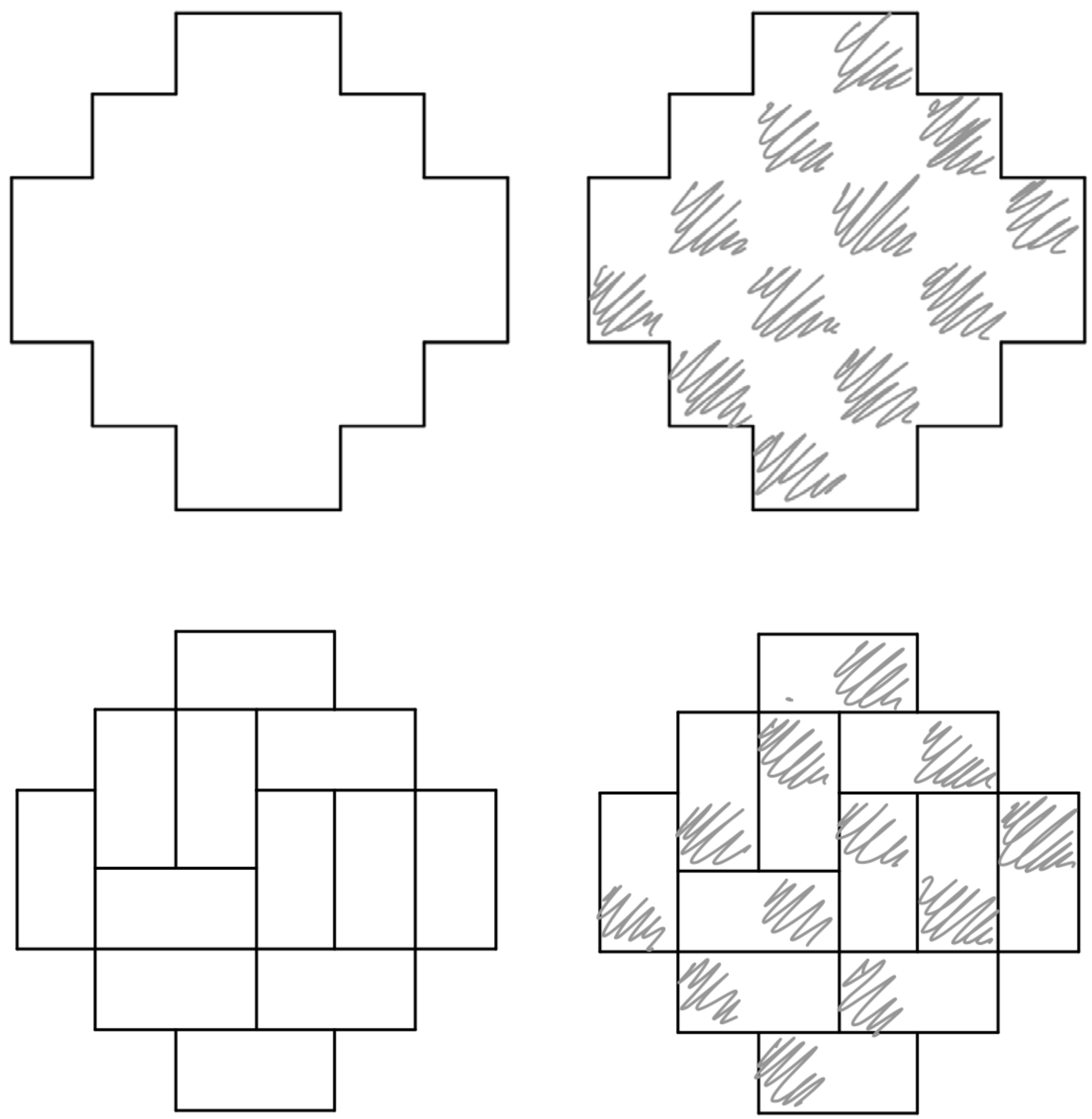


Domino tilings of an Aztec diamond

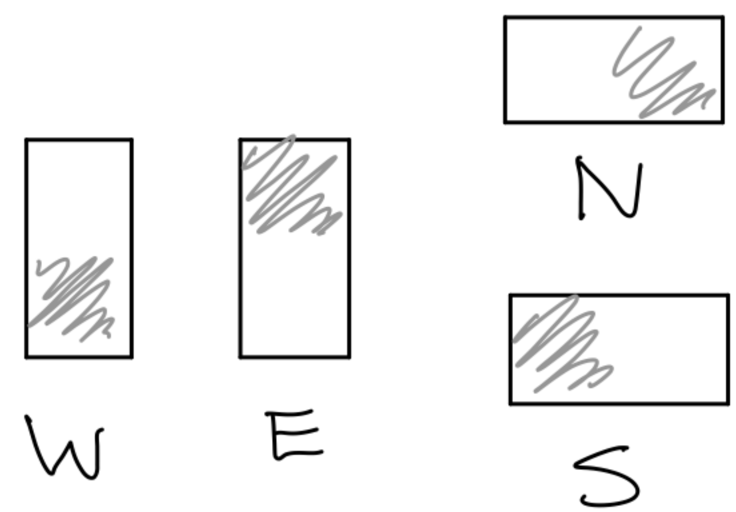


Domino tilings of the hexagon

Draw a checkerboard on the Aztec diamond…

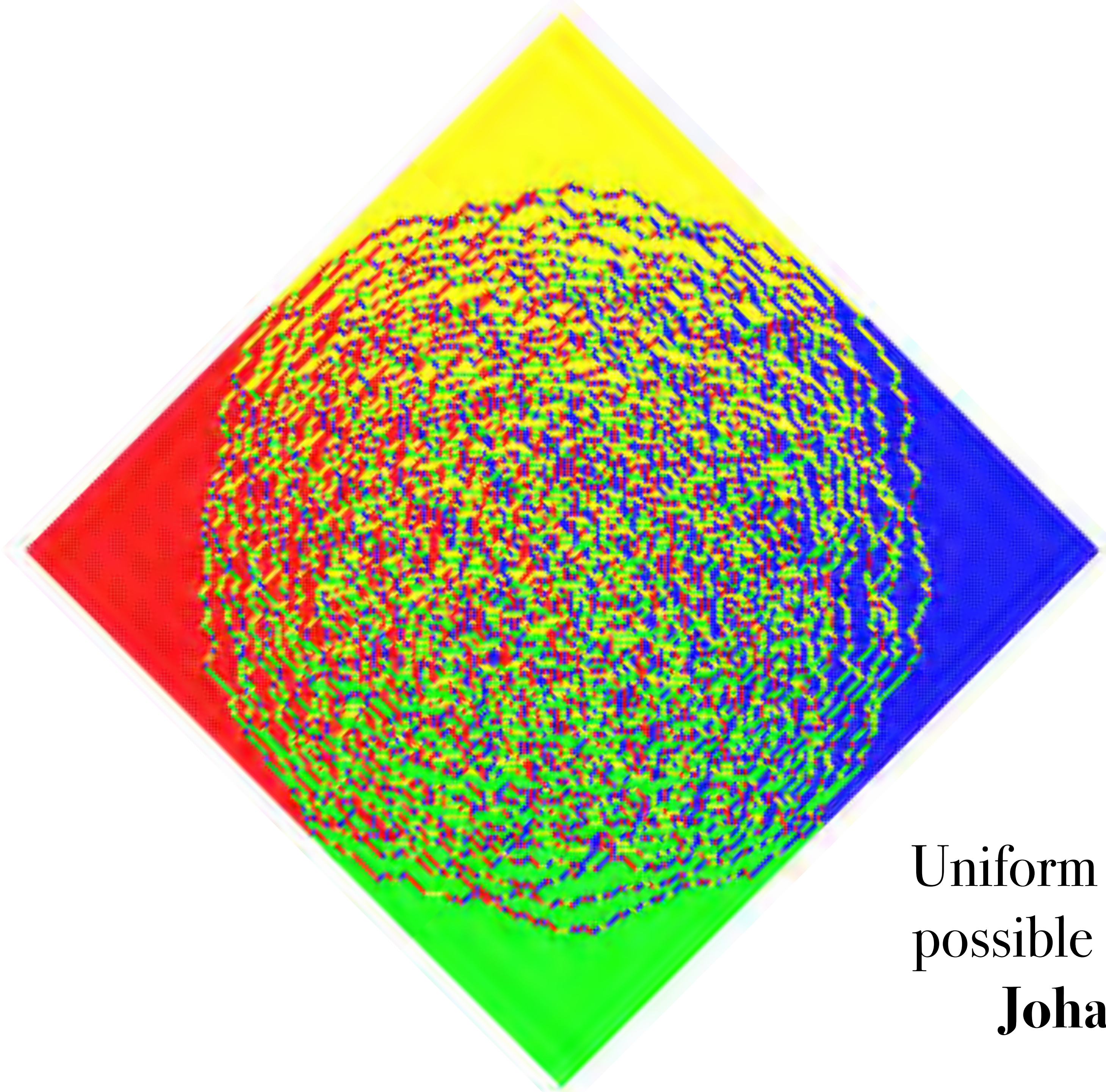


… giving four
type of dominos…



….each will have its own color.

Domino tilings of the hexagon

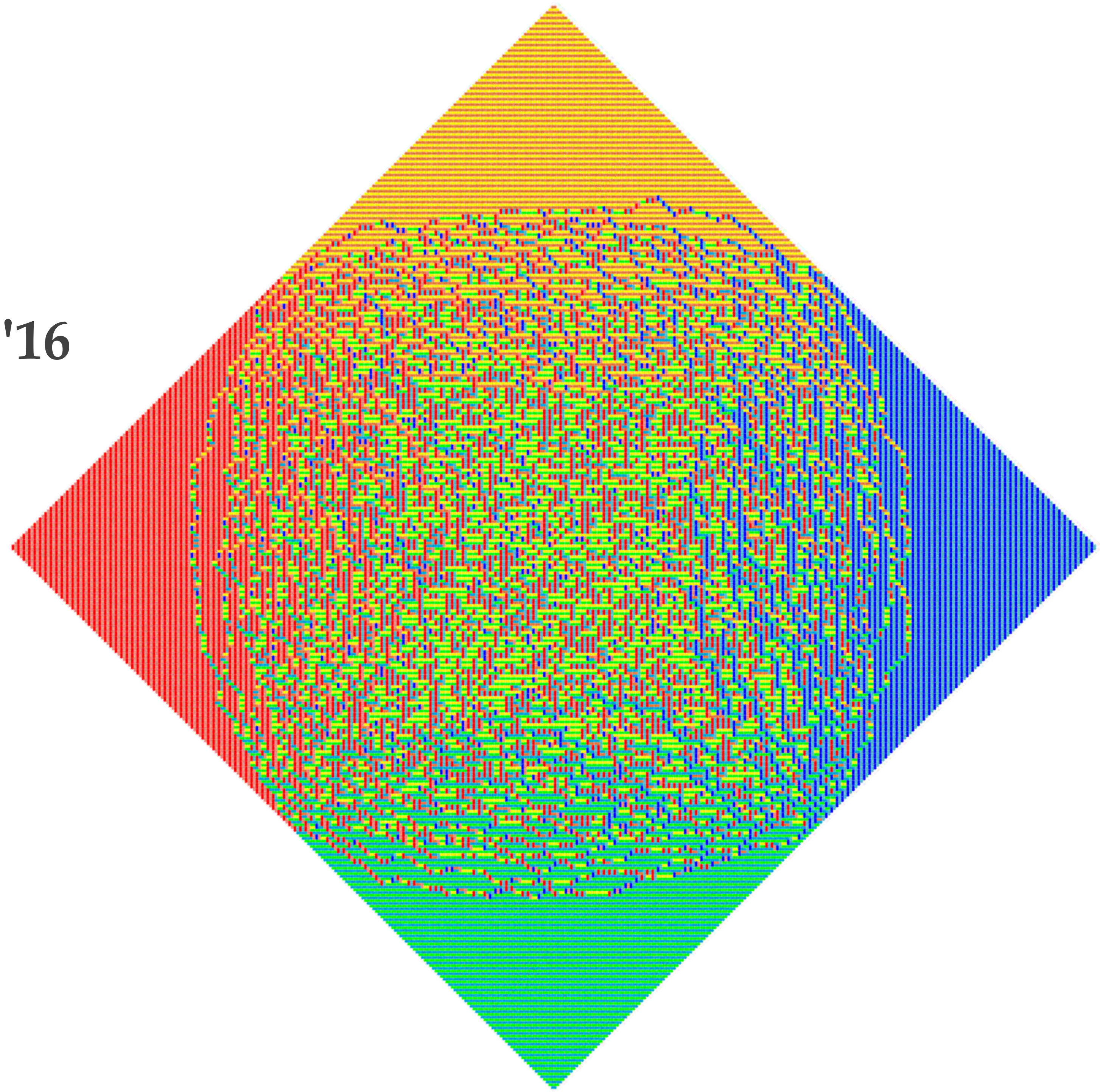


Uniform measure on all possible domino tilings
Johansson '03

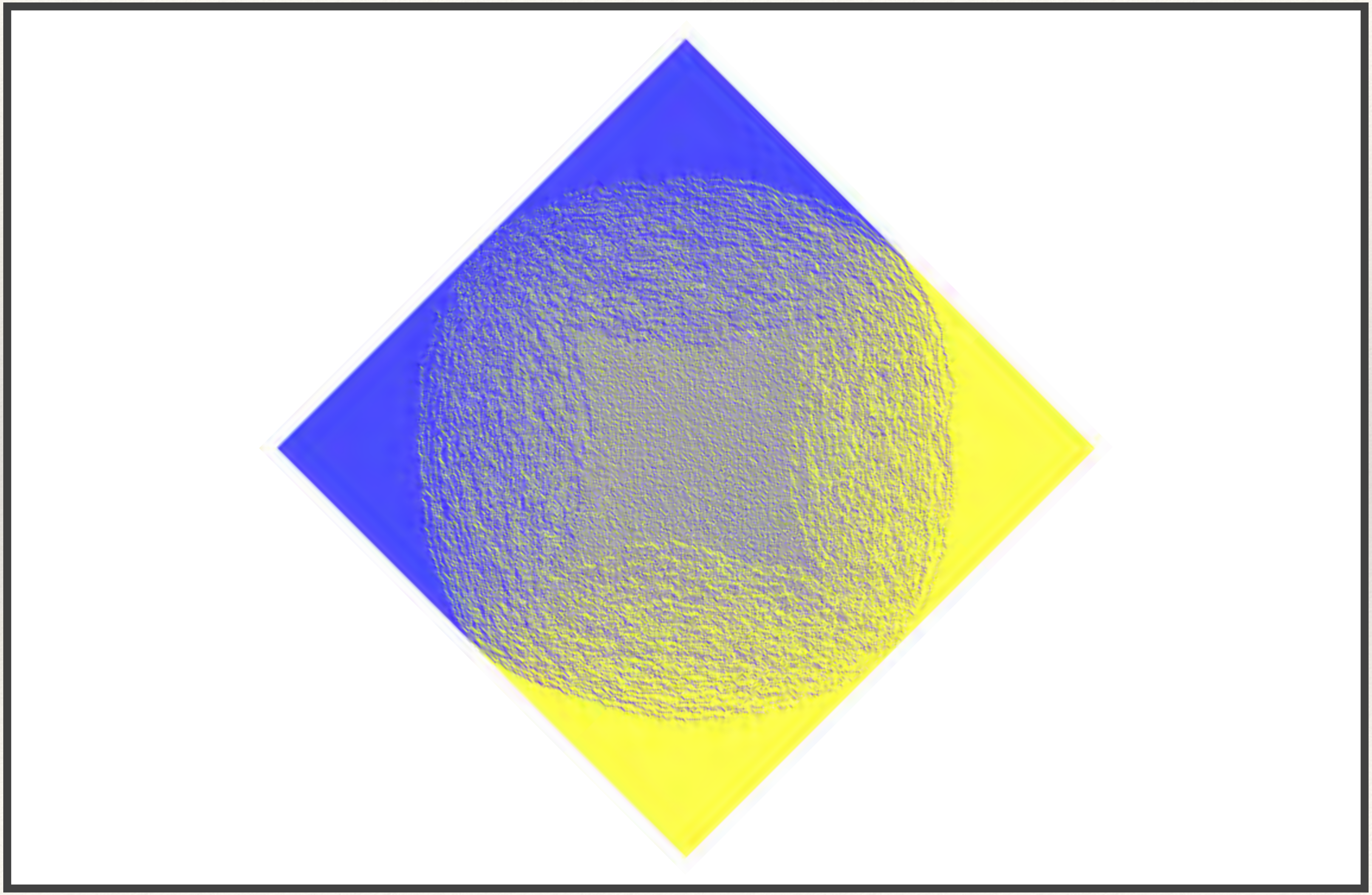
Domino tilings of the Aztec Dimoand

2-periodic weighting

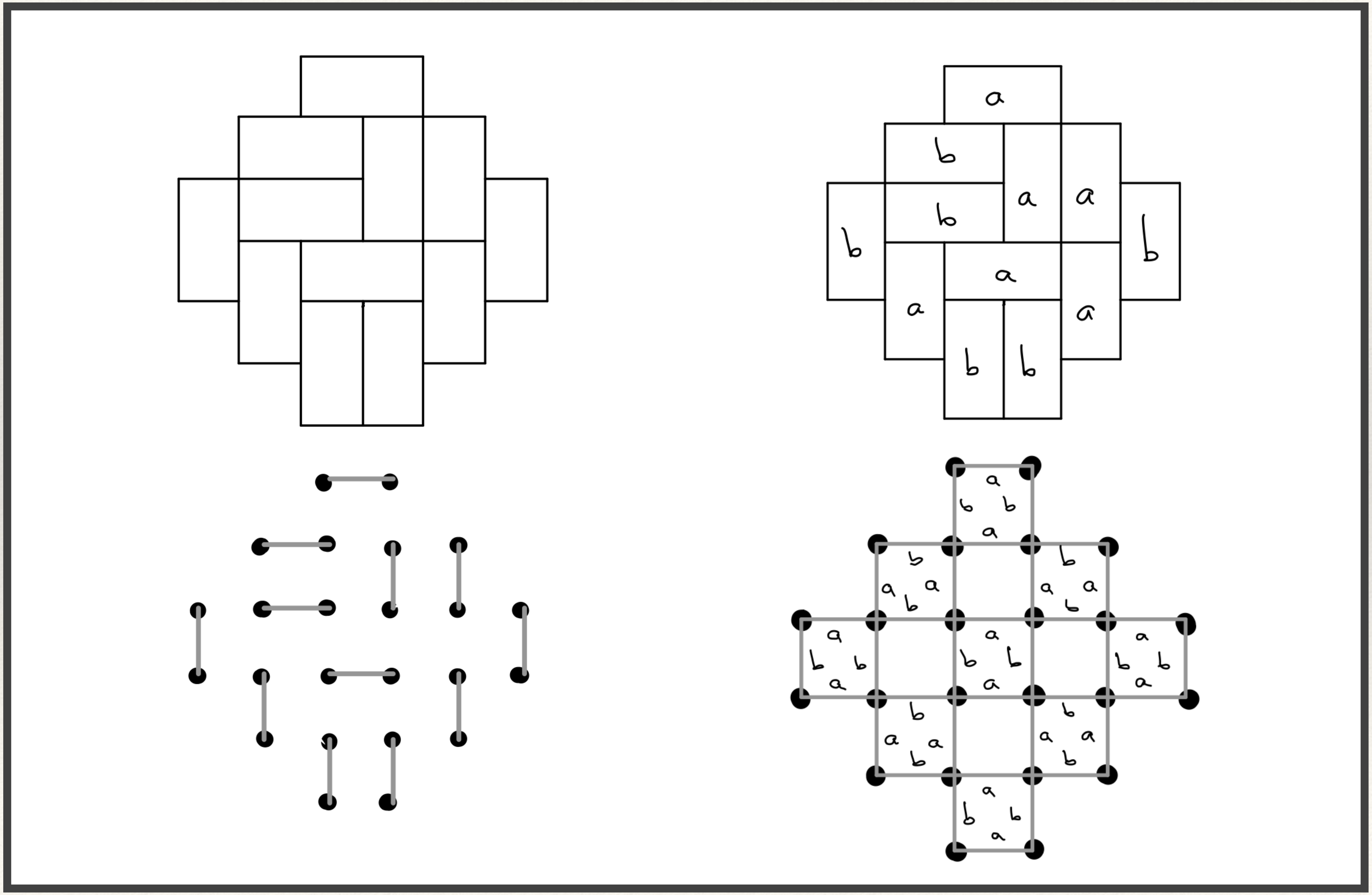
Chhita-Young '14
Chhita-Johansson ′16
Beffara-Chhita-Johansson '16
D-Kuijlaars ′17



Domino tilings of the Aztec Diamond



Domino tilings of the Aztec Diamond

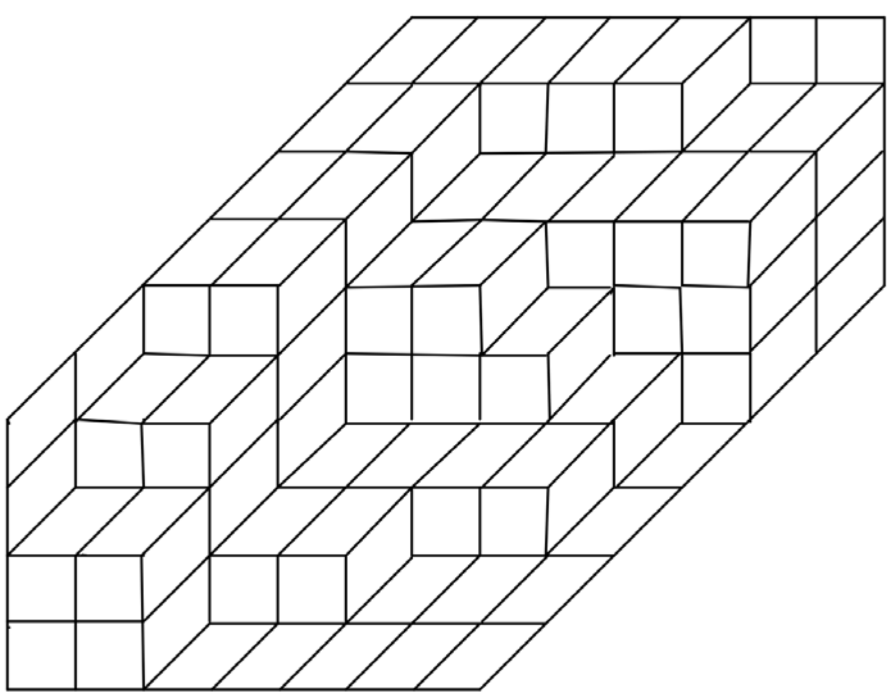


Understanding doubly periodic weightings

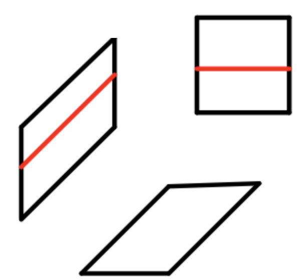
- The first goal of the project started in **D-Kuijlaars '17** was to find an understanding of the 2-periodic Aztec diamond, using orthogonal polynomials.
- More specifically, of our main results is an explicit double integral formula for the correlation kernel for a finite 2 periodic Aztec Diamond, for which classical steepest descent techniques can be applied.
- We believe that the approach we introduced can be successfully applied to more general type of periodic weightings discussed in **Kenyon-Okounkov-Sheffield '06**
- The approach also gives a new perspective on more classical weightings. In **Charlier-D-Kuijlaars-Lenells** results we analyze a model with lozenge tilings of the hexagon that seems hard to analyze with standard machinery.
- Extensions of Schur measures? **Berggren-D**

Non-Intersecting paths

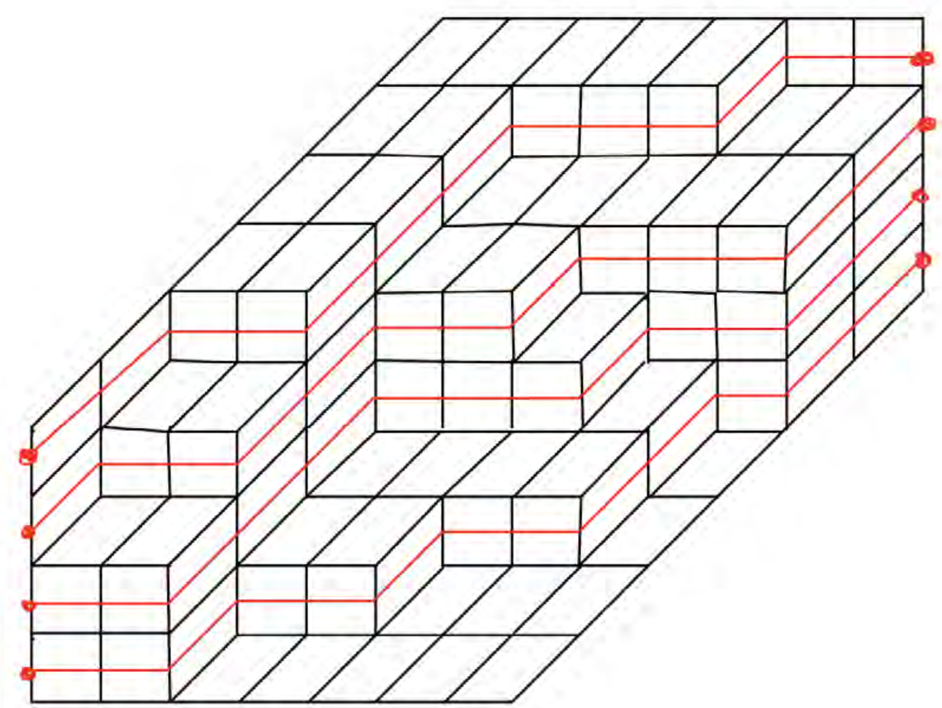
Start with a tiling….



….draw red lines on the lozenges as…

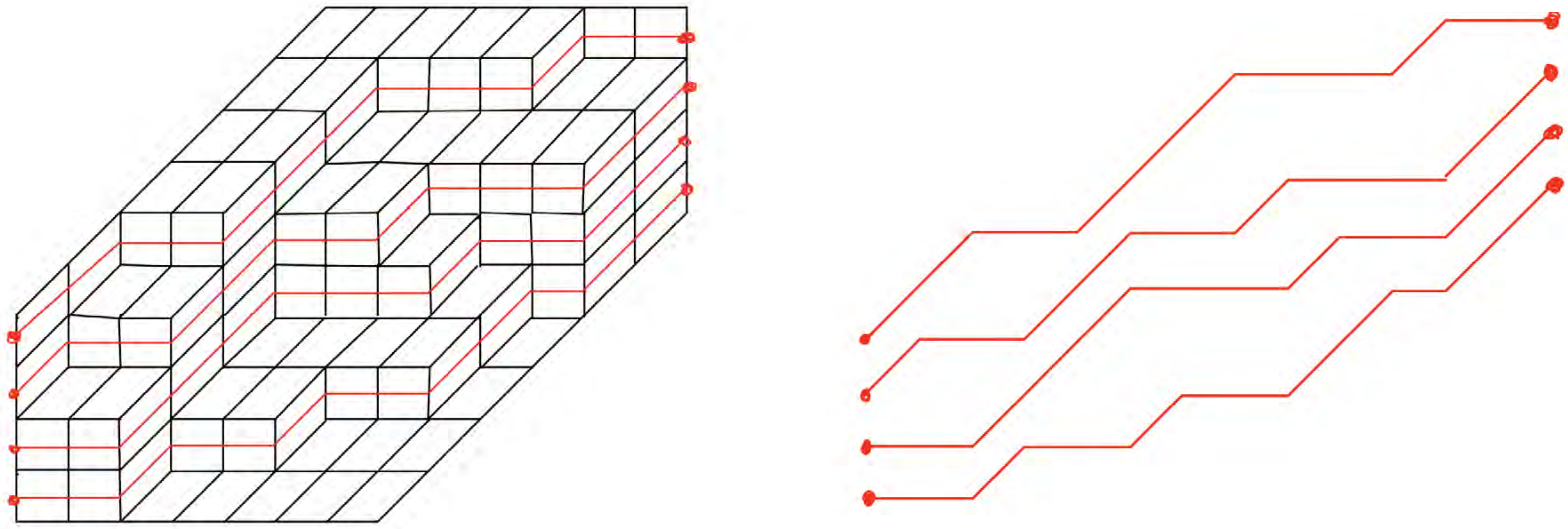


….and non-intersecting up-right paths appear.



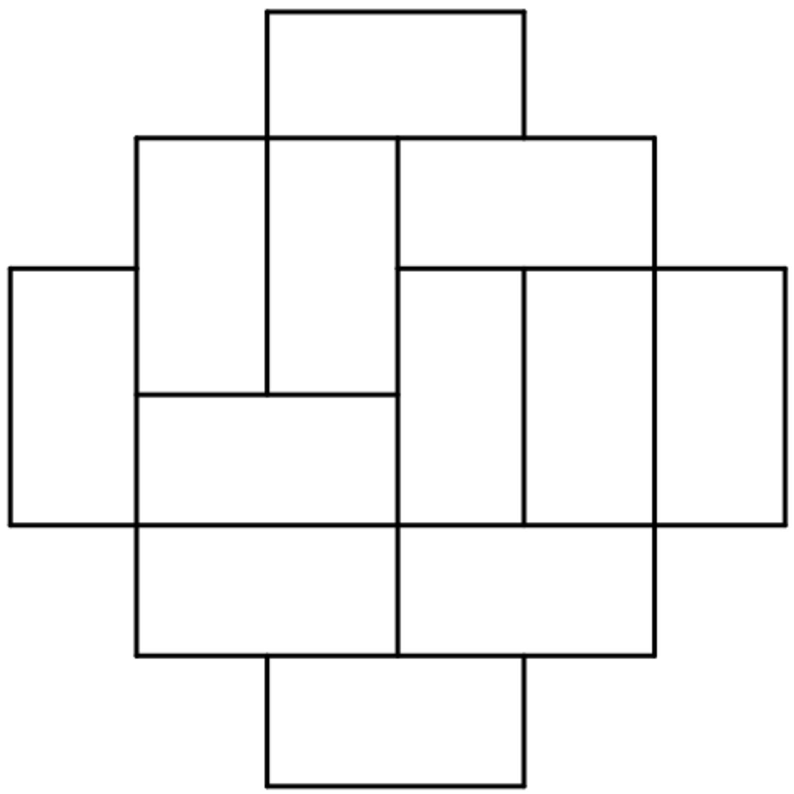
Non-Intersecting paths

The two pictures are in fact equivalent.....

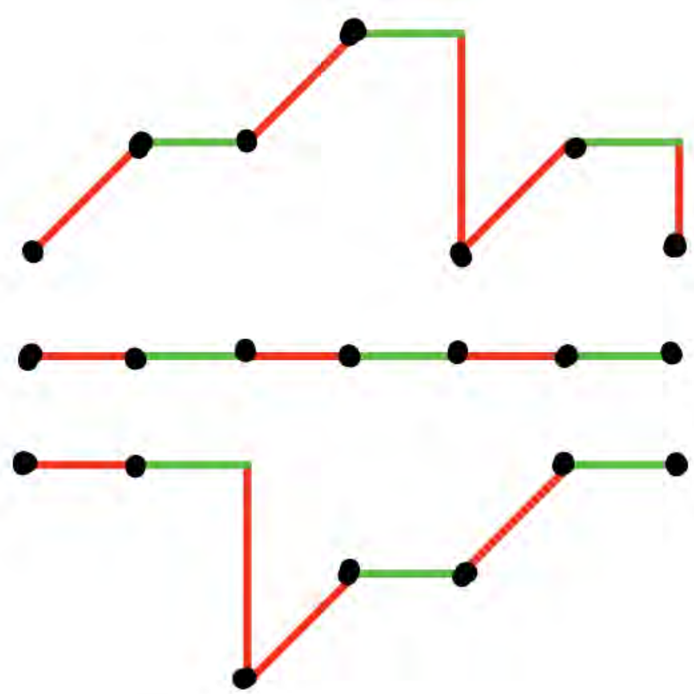


Non-Intersecting paths

A slightly more complicated collection of paths can be found for the Aztec diamond.....



.... leading to paths that end at the same points as they started, and are up-right for odd steps and go down on the even steps



Products of determinants

- We define a probability measure on the tilings by defining a probability measure on the non-intersecting paths (LGV-Theorem)

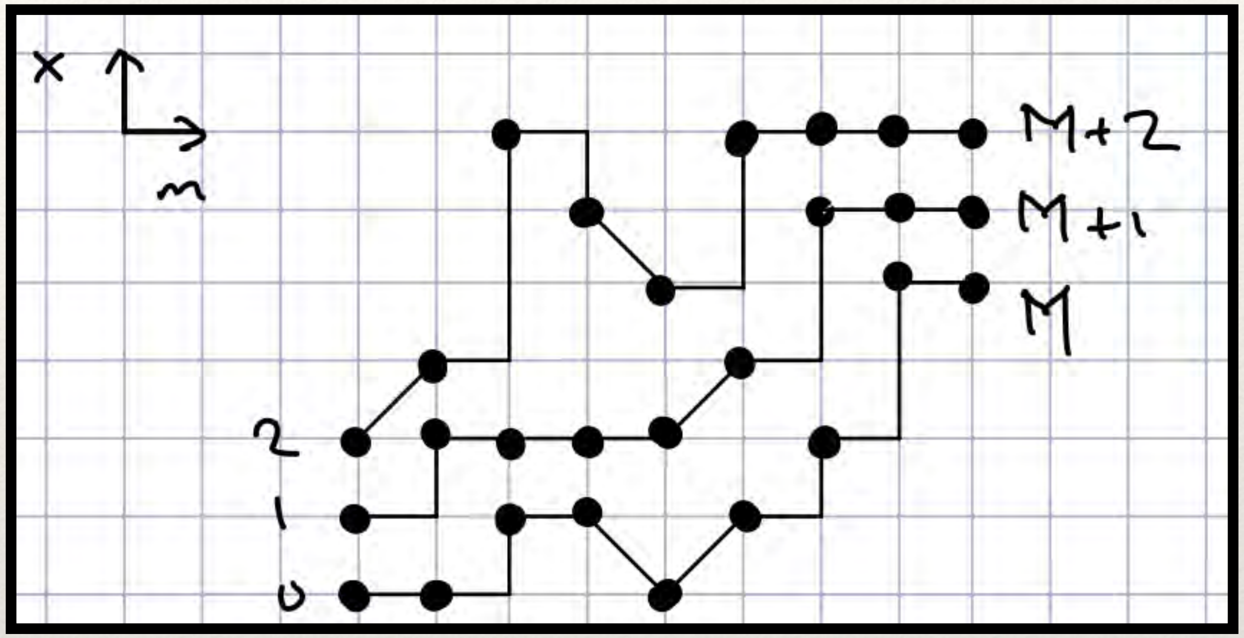
- Denote the position of the *j*-th path after step m by x_j^m

- Consider the probability measure defined by

$$\sim \prod_{m=1}^{N} \det\left(T_m(x_j^{m-1}, x_k^m)\right)_{j,k=1}^{n}$$

where for $j = 1,\ldots,n$ we have as initial and endpoints:

$$x_j^0 = j - 1 \qquad\qquad x_j^N = M + j - 1$$



n = number of paths
N = number of steps
M = the shift at endpoints
$T_m(x, y)$ = Transition probability at step m to jump from x to y

Toeplitz matrices

- The first class of models we discuss are when the transition matrices are Toeplitz matrices

$$T_m(x, y) = \hat{\phi}_m(y - x) = \frac{1}{2\pi i}\oint \phi_m(z)\frac{dz}{z^{y-x+1}}$$

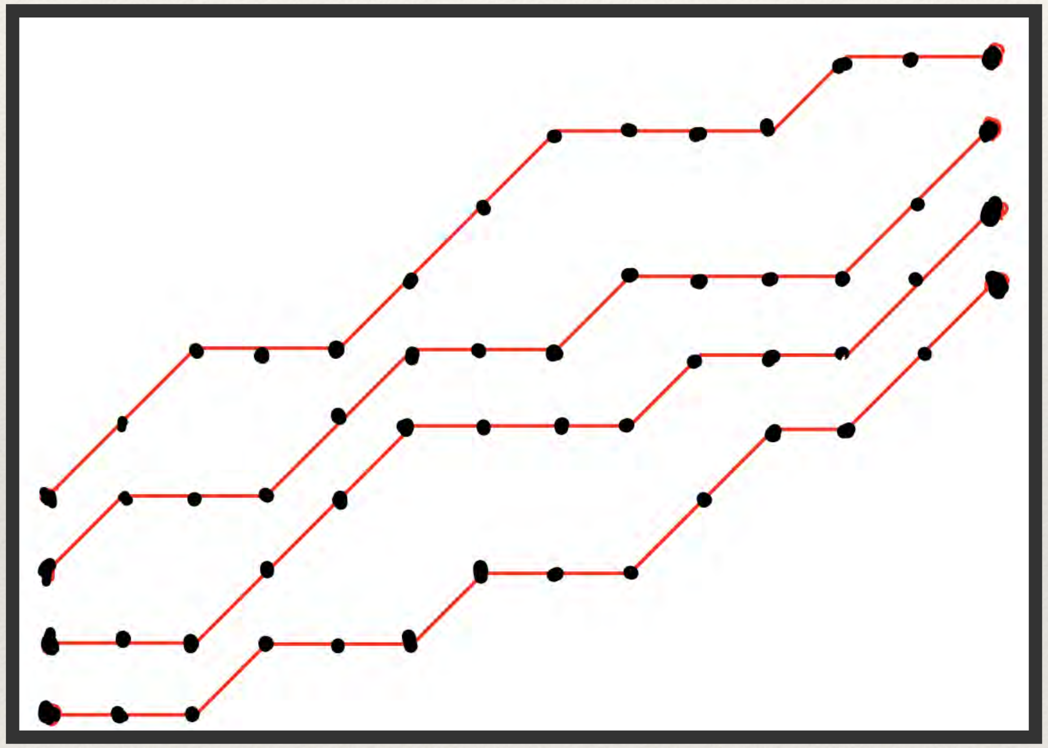
- That is, the step probability from x to y depends only on the size y-x.
- Examples:

$$\phi_m(z) = 1 + a_m z \qquad \phi_m(z) = \frac{1}{1 - a_m z}$$

$$\phi_m(z) = 1 + \frac{a_m}{z} \qquad \phi_m(z) = \frac{1}{1 - \frac{a_m}{z}}$$

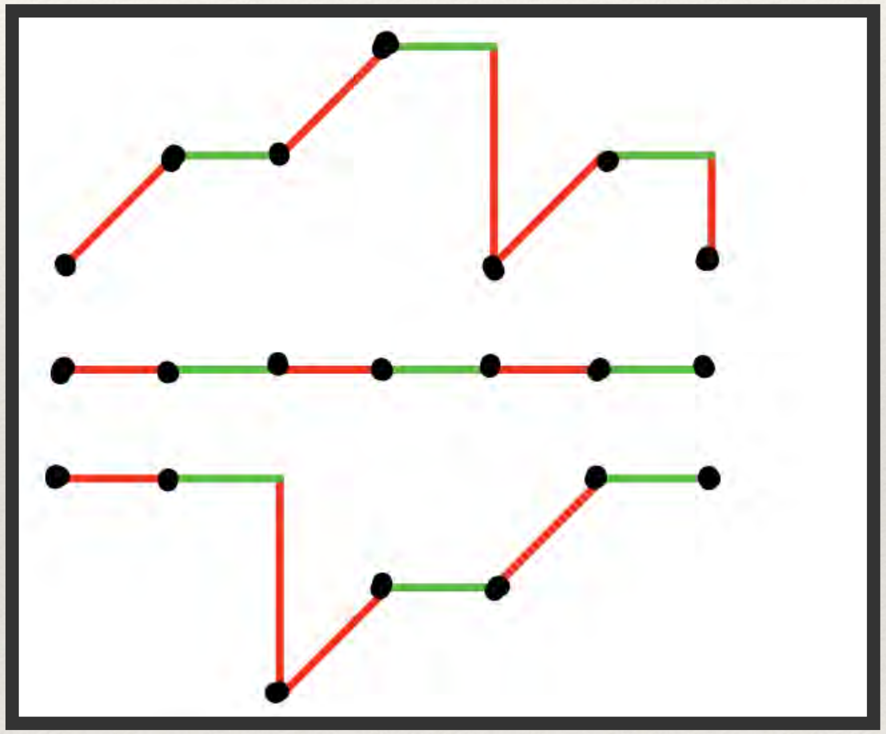
Examples

Uniform lozenge tilings of the hexagon



$$\phi_m(z) = 1 + z$$

Uniform domino tilings of the Aztec diamond



$$\phi_m(z) = \begin{cases} 1 + qz, & m \text{ odd} \\ (1 - \frac{q}{z})^{-1}, & m \text{ even} \end{cases}$$

....and take the limit $q \uparrow 1$

Orthogonal polynomials

- In **D-Kuijlaars '17** we used a biorthogonalization procedure using orthogonal polynomials in the complex plane to describe the k-point correlations.
- Let $p_k(z)$ be the monic polynomial of degree k such that

$$\oint_\gamma p_k(z)\; z^j \; \frac{\prod_{m=1}^N \phi_m(z)dz}{z^{M+n}} = 0, \qquad j = 0,1,\ldots,k-1$$

- Orthogonality relations is with respect to contour in the complex plane and non-hermitian. **The existence is not guaranteed!**

The idea of biorthogonalization is a standard trick for determinantal point processes. However, there are many ways to do it. The way we choose here is very different from the more common one, that would lead to Discrete Orthogonal Polynomials. **Baik-Deift-Kriechenbauer-McLaughlin** The relation between the two is not obvious.

Determinantal point process

- By the Eynard-Mehta Theorem the process is determinantal.

$$\mathbb{P}\left(\text{ points at } (m_1, x_1), \ldots, (m_k, x_k)\right) = \det\left(K(m_j, x_j, m_\ell, x_\ell\right)_{j,\ell=1}^{n}$$

- In **D-Kuijlaars '17** we computed the correlation kernel in terms of the OPs

$$K(m,x,m',y) = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$+\frac{c_n}{(2\pi i)^2}\oint_\gamma\oint_\gamma \prod_{\ell=m'+1}^{N} \phi_\ell(w)\ \frac{p_n(z)p_{n-1}(w)-p_n(w)p_{n-1}(z)}{z-w}\ \prod_{\ell=1}^{m}\phi_\ell(w)\ \frac{w^y}{z^{x+1}w^{M+n}}dzdw$$

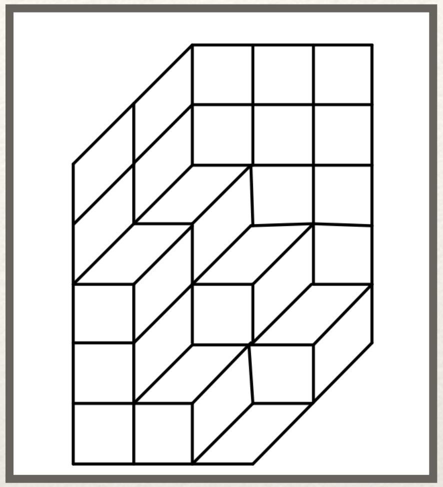
Strategy for asymptotic analysis

- To study the asymptotic behavior $n, N \to \infty$ we
  - First find the asymptotic behavior of the Orthogonal Polynomials. In particular for the Christoffel-Darboux kernel
  - Insert the asymptotics into the double integral formula and perform a steepest descent analysis.
- The asymptotic for the orthogonal polynomials can be done by a Riemann-Hilbert analysis.
- In certain special cases, like uniform lozenge tilings of the hexagon and domino tilings of the Aztec diamond, the orthogonal polynomials are "classical."
- **Schur processes**: when only $n \to \infty$ then the asymptotics of the polynomials is easy.

Jacobi polynomials

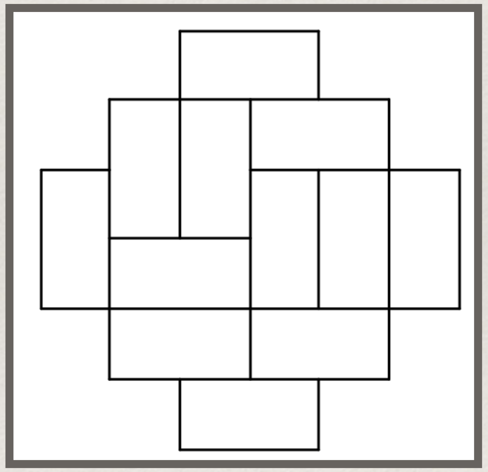
- In case of **uniform lozenge tilings of a hexagon** we obtain the ”orthogonality measure”

$$\frac{(1+z)^N}{z^M}dz$$



- In case of **domino tilings of the Aztec diamond** we obtain the ”orthogonality measure”

$$\left(\frac{1+qz}{1-qz}\right)^N dz$$



- In both cases, this means that the orthogonal polynomials are in fact **Jacobi polynomials** where one of the parameter is negative. In the Aztec diamond the choice is even degenerate and the Christoffel-Darboux kernel is explicit and we retrieve the Krawtchouk kernel from **Johansson ’03**

More complicated models

- In **Charlier-D-Kuijlaars-Lenells** we look at

$$\phi_m(z) = \begin{cases} 1+z, & m \text{ even} \\ 1+\alpha z, & m \text{ odd} \end{cases}$$

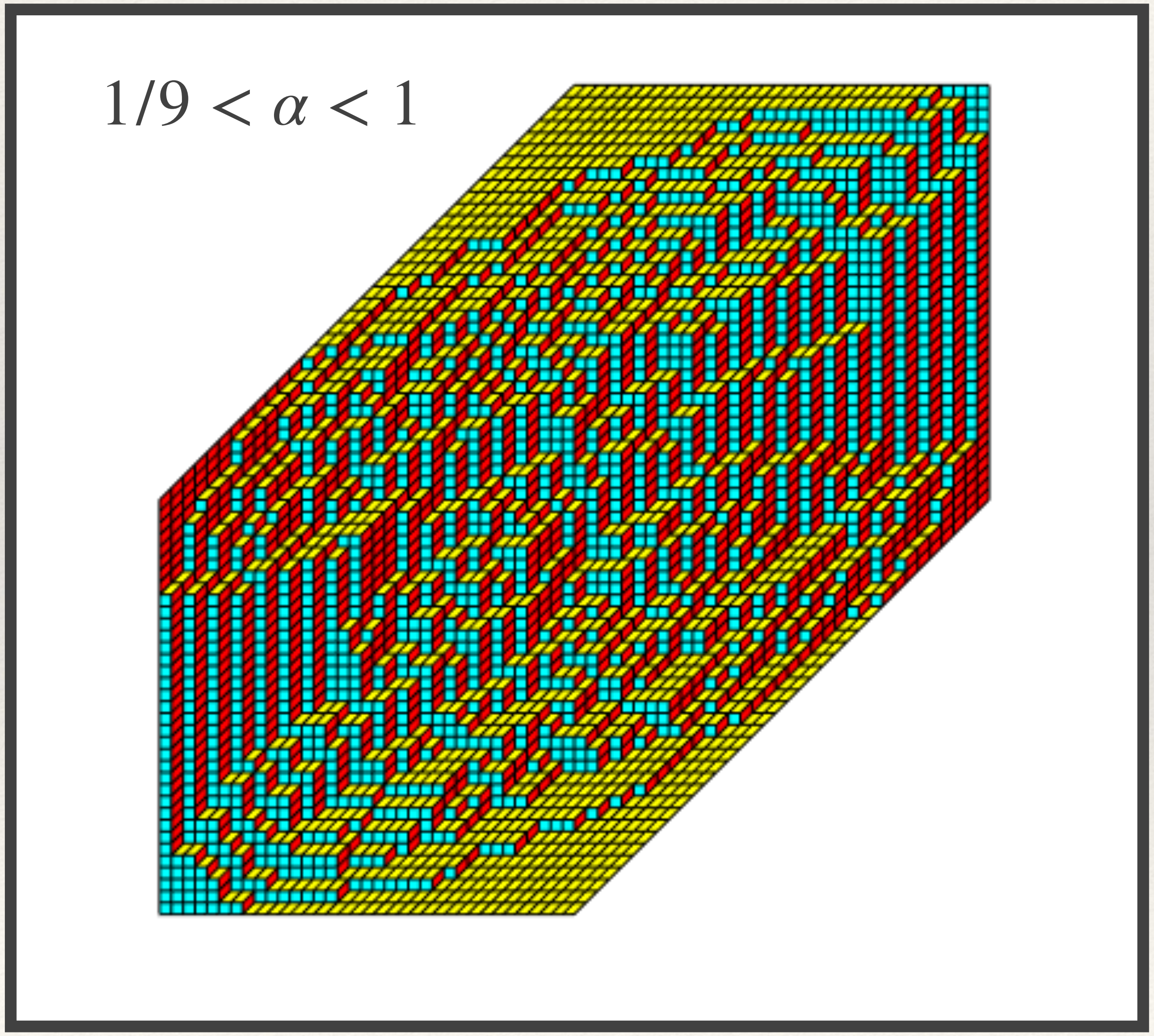
- Meaning that the orthogonality weight is given by

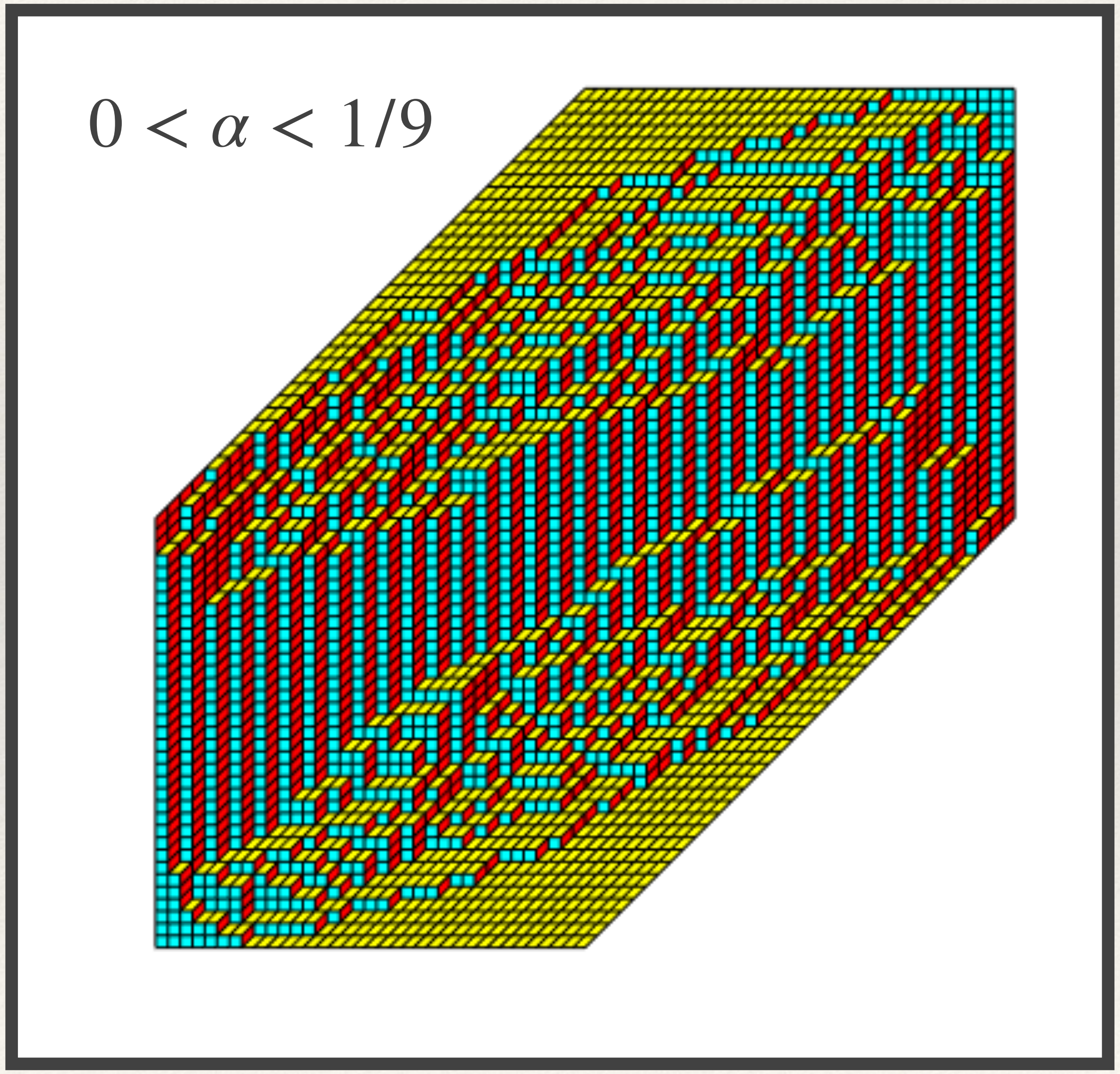
$$\frac{(1+z)^N(1+\alpha)^N}{z^N}dz$$

- By steepest descent analysis on the **Riemann-Hilbert problem** for the polynomials we find the asymptotic behavior of these polynomials. By inserting that in the double integral formula and then performing a classical steepest descent analysis we can compute the thermodynamical limit.

More complicated models

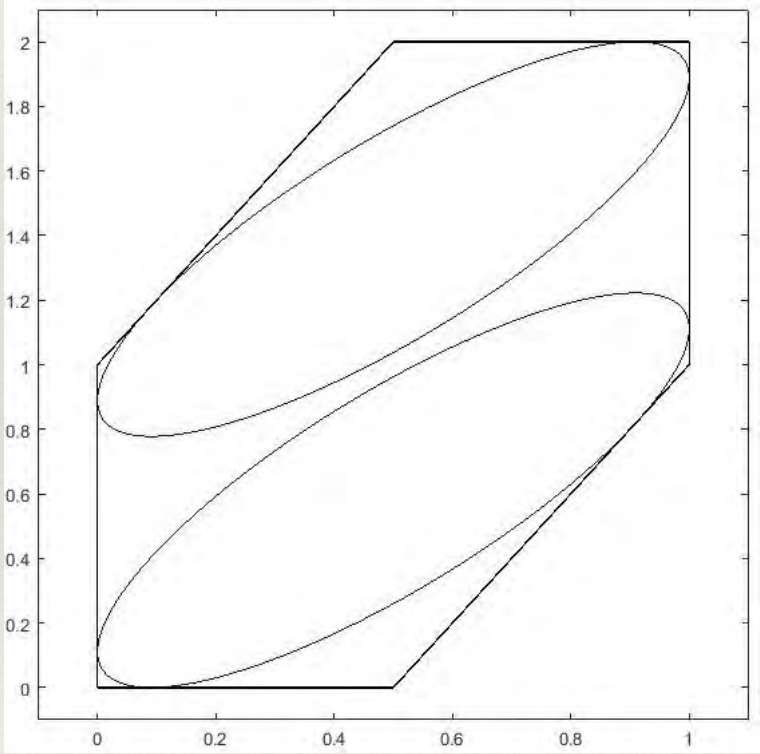
- There are two different type of shapes of the limiting disordered region domain. One

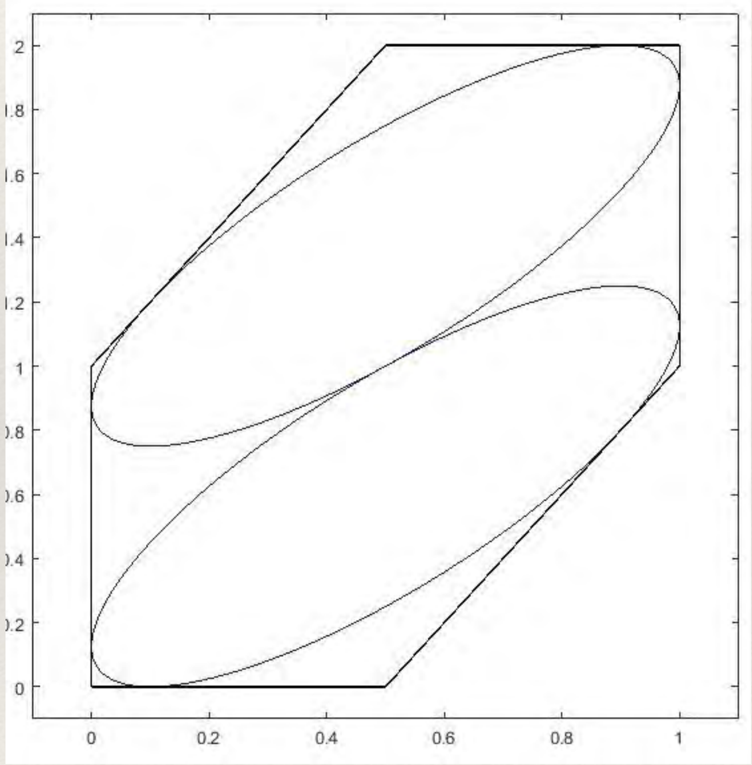


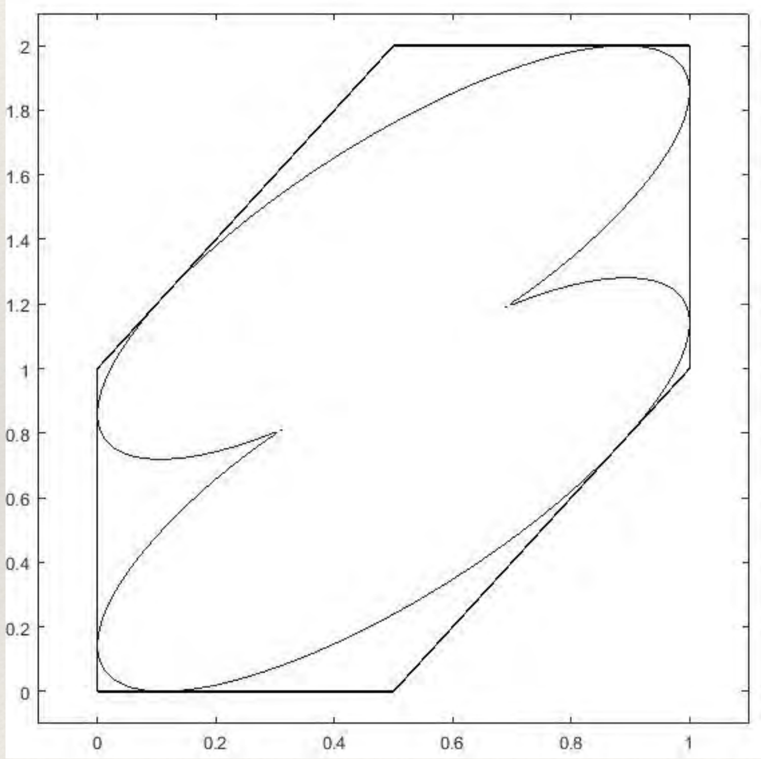


More complicated models

- There are two different type of shapes of the limiting disordered region domain. For large α we have two separate disordered region. When α is close to 1, then there is only one. When the two regions meet meet they form a tacnode and we retrieve the tacnode process.







- Our rigorous analysis follows straightforward principle and combines Riemann-Hilbert techniques with steepest descent integrals. Gaussian free field fluctuations are also within reach.

Schur process

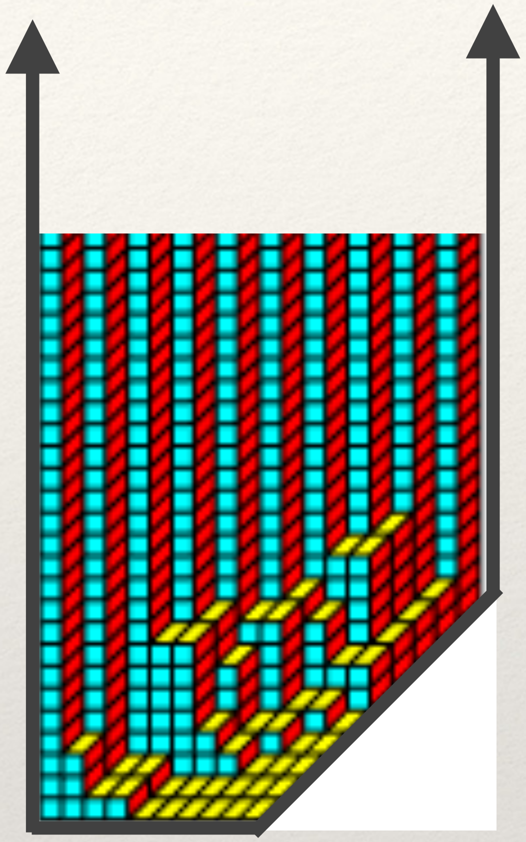
- Instead of keeping the number of paths n finite, we can also define the process for $\mathbf{n} \to \infty$. This recovers the **Schur processes** of **Okounkov-Reshetikhin**

- We assume that the orthogonality weight has a Wiener-Hopf type factorization

$$\prod_{m=1}^{N} \phi_m(z) = \phi_+(z)\phi_-(z)$$

where

  - $\phi_+^{\pm 1}(z)$ is analytic for $|z| < 1$ and continuous for $|z| \leq 1$
  - $\phi_-^{\pm 1}(z)$ is analytic for $|z| > 1$ and continuous for $|z| \geq 1$
  - $\phi_-(z) \sim z^M$ as $z \to \infty$



Schur process

- The bottom part of the paths converge to a DPP with kernel

$$K_{bottom}(m,x;m',y) = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$-\frac{1}{(2\pi i)^2}\iint_{|z|<|w|} \prod_{\ell=m'+1}^{N} \phi_\ell(z)\phi_-^{-1}(w)\phi_+^{-1}(z)\prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}(z-w)}dzdw$$

- The top part of the paths converge to a DPP with kernel

$$K_{top}(m,x;m',y) = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\iint_{|w|<|z|} \prod_{\ell=m'+1}^{N} \phi_\ell(z)\phi_+^{-1}(w)\phi_-^{-1}(z)\prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}(z-w)}dzdw$$

Block Toeplitz transition matrices

- The original motivation for **D-Kuijlaars '17** was to analyze the 2-periodic Aztec diamond (see also **Chhita-Young '14**, **Chhita-Johansson '14**, **Beffara-Chhita-Johansson '15)**

- The 2-periodic Aztec diamond is only one example of a larger class of models, with a periodic weighting as discussed in **Kenyon-Okounkov-Sheffield '06**

- In a more general setup, we consider the case where the transition matrices are **block Toeplitz matrices with blocks of size** $p \times p$

$$T_m(px + r, py + s) = \left(\hat{\phi}_m(y - x)\right)_{r,s} = \frac{1}{2\pi i} \oint \left(\phi_m(z)\right)_{r+1,s+1} \frac{dz}{z^{y-x+1}},$$

$$r, s = 1, \ldots, p \qquad x, y \in \mathbb{N}$$

Block Toeplitz transition matrices

- In the case $p = 2$ the following matrix symbols are canonical:

$$\phi_m(z) = \begin{pmatrix} a_m & b_m z \\ c_m & d_m \end{pmatrix}$$

"Bernoulli up"

$$\phi_m(z) = \frac{1}{1 - qz} \begin{pmatrix} a_m & b_m z \\ c_m & d_m \end{pmatrix}$$

"Geometric up"

$$\phi_m(z) = \begin{pmatrix} a_m & b_m \\ c_m/z & d_m \end{pmatrix}$$

"Bernoulli up"

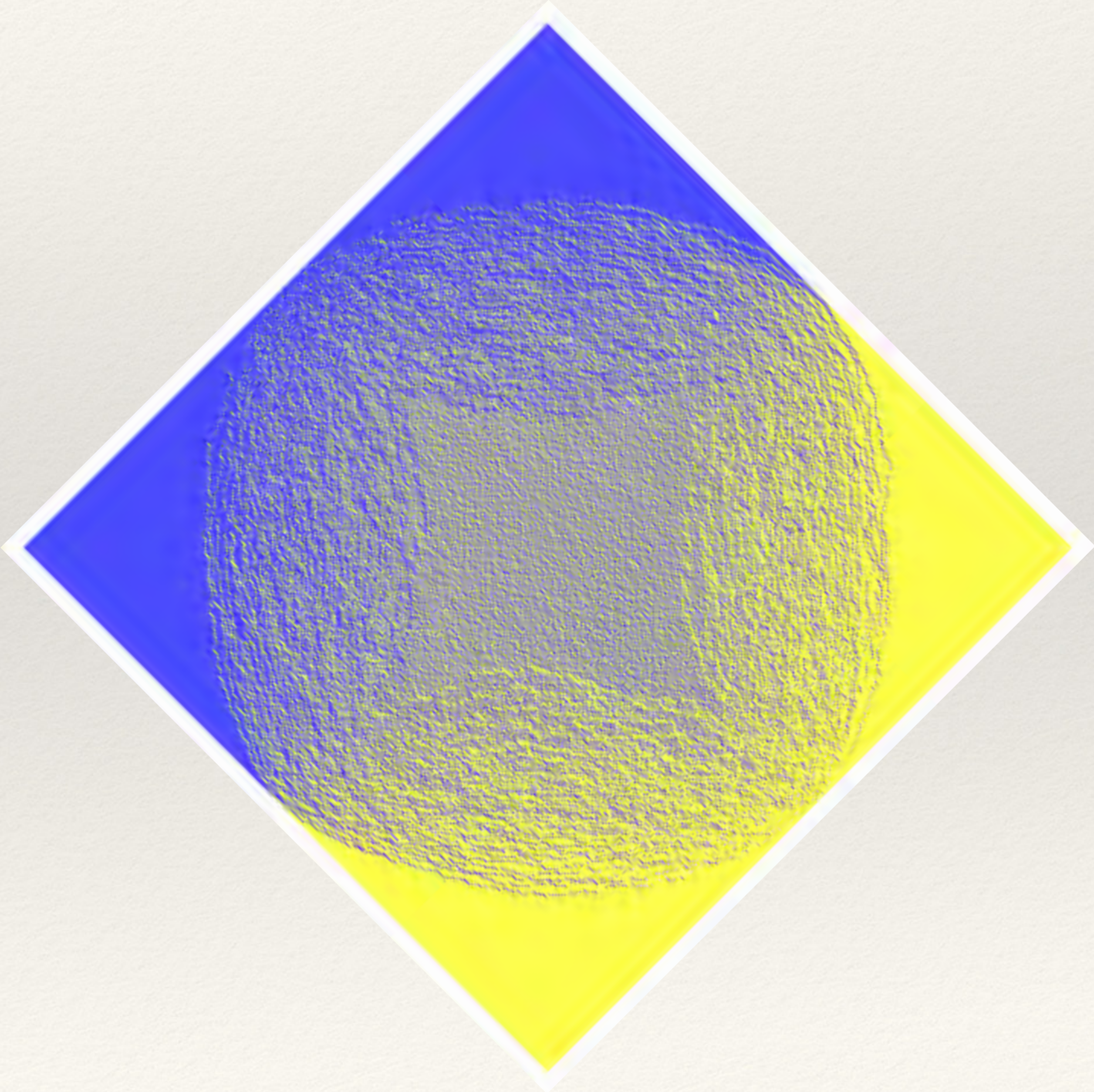
$$\phi_m(z) = \frac{1}{1 - q/z} \begin{pmatrix} a_m & b_m \\ c_m/z & d_m \end{pmatrix}$$

"Geometric down"

2 periodic Aztec diamond

- The 2-periodic Aztec diamond has the weight

$$\phi_m(z) = \begin{cases} \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \begin{pmatrix} 1 & az \\ a & 1 \end{pmatrix}, & m \text{ even}, \\ \frac{1}{1-a^2/z} \begin{pmatrix} 1 & a \\ a/z & 1 \end{pmatrix}, & m \text{ odd}. \end{cases}$$



- Here $\alpha > 1$
- where we also need $a \uparrow 1$

Matrix Orthogonal Polynomials

- In **D-Kuijlaars '17** we used **matrix orthogonal polynomials** in the complex plane to describe the k-point correlations.
- Let $p_k(z) = I_p z^k + \dots$ be the monic polynomial of degree k such that

$$\oint_\gamma p_k(z)\; z^j\; \frac{\prod_{m=1}^N \phi_m(z) dz}{z^{M+n}} = 0, \qquad j = 0,1,\dots,k-1$$

- Orthogonality relations is with respect to contour in the complex plane and non-hermitian.
- **The weight is matrix valued. Order in the product is important!**

Correlation kernel

- In **D-Kuijlaars '17** we show that the correlation kernel is given in block form by

$$\left[K(m, px+j; m', py+i)\right]_{i,j=0}^{p-1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_\gamma \prod_{\ell=m'+1}^{m} \phi_\ell(z) z^{y-x}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\oint_\gamma\oint_\gamma \prod_{\ell=m'+1}^{N} \phi_\ell(w)\ R(z,w)\ \prod_{\ell=1}^{m}\phi_\ell(z)\frac{w^y}{z^{x+1}w^{M+n}}dzdw$$

  where $R_n(z,w)$ is the Christoffel-Darboux kernel for the matrix orthogonal polynomials

- Due to non-commutativity, the order in the product is important!

2 periodic Aztec diamond

- For the 2-periodic the Riemann-Hilbert problem can be solved explicitly. This was done in **D-Kuijlaars '17** and reproved in a different way in **Berggren-D**

$$\left[\mathbb{K}_N(2m+r,n;2m'+s,n')\right]_{r,s=0}^{1} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma_{0,1}} A^{m-m'}(z) z^{(m'+n')/2-(m+n)/2}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\oint_{\gamma_{0,1}}\frac{dz}{z}\oint_{\gamma_1}\frac{dw}{z-w}A^{N-m'}(w)F(w)A^{-N+m}(z)\frac{z^{N/2}(z-1)^N}{w^{N/2}(w-1)^N}\frac{w^{(m'+n')/2}}{z^{(m+n)/2}}$$

where

$$A(z)=\frac{1}{z-1}\begin{pmatrix} 2\alpha z & \alpha(z+1)\\ \beta z(z+1) & 2\beta z\end{pmatrix}$$

and

$$F(z)=\frac{1}{2}I_2+\frac{1}{2\sqrt{z(z+\alpha^2)(z+\beta^2)}}\begin{pmatrix}(\alpha-\beta)z & \alpha(z+1)\\ \beta z(z+1) & -(\alpha-\beta)z\end{pmatrix},$$

2 periodic Aztec diamond

- In **D-Kuijlaars '17** we analyzed this double integral formula asymptotically
- An important role in the analysis is defined by the spectral curve

$$\det\left(A(z)-\lambda\right)=0 \qquad\qquad A(z)=\frac{1}{z-1}\begin{pmatrix}2\alpha z & \alpha(z+1)\\ \beta z(z+1) & 2\beta z\end{pmatrix}$$

  which is an important input for finding the saddle point in the steepest descent analysis.
- An important feature of the spectral curve is that it leads to a Rieman-surface with genus 1. The presence of a gas phase seems intrinsic to a non-zero genus.

Matrix analogue of the Schur process

- What about more general models than the 2-periodic Aztec diamond?
- In Berggren-D we follow the approach of Schur processes and found a general statement for the kernel in case of infinite systems of paths.
- The main idea in **Berggren-D** is that the matrix orthogonal polynomials have an easy to describe limit as $n \to \infty$, in case a matrix-valued Wiener-Hopf factorization is available.

Matrix analogue of the Schur process

- We assume that the orthogonality weight has a matrix Wiener-Hopf type factorization

$$\prod_{m=1}^{N} \phi_m(z) = \phi_+(z)\phi_-(z) = \widetilde{\phi}_-(z)\widetilde{\phi}_+(z)$$

where

  - $\phi_+^{\pm 1}(z), \widetilde{\phi}_+^{\pm 1}(z)$ are analytic for $|z| < 1$ and continuous for $|z| \leq 1$
  - $\phi_-^{\pm 1}(z), \widetilde{\phi}_-^{\pm 1}(z)$ are analytic for $|z| > 1$ and continuous for $|z| \geq 1$
  - $\phi_-(z), \widetilde{\phi}_-(z) \sim z^M I_p$ as $z \to \infty$

- In **Beggren-D (upcoming)** we prove the following statement that is the analogue of the correlation kernels for the Schur process.

Matrix Analogue of the Schur process

- The bottom part of the paths converge to a DPP with kernel

$$\left[K_{bottom}(m, px+r; m', py+s)\right]_{r,s=1}^{p} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma}\prod_{\ell=m'+1}^{m}\phi_{\ell}(z)z^{y-x}\frac{dz}{z}$$

$$-\frac{1}{(2\pi i)^2}\iint_{|z|<|w|}\prod_{\ell=m'+1}^{N}\phi_{\ell}(z)\phi_{-}^{-1}(w)\phi_{+}^{-1}(z)\prod_{\ell=1}^{m}\phi_{\ell}(z)\frac{w^{y}}{z^{x+1}(z-w)}dzdw$$

- The top part of the paths converge to a DPP with kernel

$$\left[K_{top}(m, xp+r; m', yp+s)\right]_{r,s=1}^{p} = -\frac{\chi_{m>m'}}{2\pi i}\oint_{\gamma}\prod_{\ell=m'+1}^{m}\phi_{\ell}(z)z^{y-x}\frac{dz}{z}$$

$$+\frac{1}{(2\pi i)^2}\iint_{|w|<|z|}\prod_{\ell=m'+1}^{N}\phi_{\ell}(z)\widetilde{\phi}_{+}^{-1}(w)\widetilde{\phi}_{-}^{-1}(z)\prod_{\ell=1}^{m}\phi_{\ell}(z)\frac{w^{y}}{z^{x+1}(z-w)}dzdw$$

Matrix Wiener-Hopf factorization

- These results are of course only meaningful if we can find a Matrix Wiener-Hopf factorization.
- The existence of such is a classical problem and many results are known. Existence results apply to the typical cases that we are interested in
- Still, existence is not enough. We want an explicit form of the factorization that is useful for an asymptotic study.
- So far we have been able to do several cases:
  - 2 periodic Aztec diamond
  - Higher periodic Aztec diamonds
  - 2 periodic tilings of the infinite hexagons
- As a result, all of these example can be analyzed asymptotically (work in progress).