On the Centered Maximum of the Sine$_\beta$ process

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Outline

1. An introduction to Sine_β
2. Maxima of log-correlated processes
3. The centered maximum of the Sine_β counting process
4. Ideas from the proof

The Circular β-ensemble

This is an n-point measure on the unit circle (or $[0, 2\pi]$) with density

$$f(\theta_1, \dots, \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta. \qquad (0.1)$$

for any $\beta > 0$.

- This is a generalization of the Circular Orthogonal/Unitary/Symplectic ensembles.
- We can define an associated empirical spectral measure

$$\mu_n = \sum_{k=1}^{n} q_k \delta_{e^{i\theta_k}}, \qquad \text{where} \qquad \sum q_k = 1.$$

- For the Circular β-ensemble we will choose

$$(q_1, \dots, q_n) \sim \text{Dirichlet}(\tfrac{\beta}{2}, \dots, \tfrac{\beta}{2})$$

OPUCs (part I)

For any measure on the unit circle ($\partial\mathbb{D}$), including our choice μ_n, we can associate a family of orthogonal polynomials, $\Phi_0(z), \Phi_1(z), \Phi_2(z), \ldots$.

If the measure μ has finite support it can be written as $\mu = \sum_{k=1}^{n} q_k \delta_{z_k}$ and there exists a bijection

$$(\{z_k\}_{k=1}^{n}, \{q_k\}_{k=1}^{n-1}) \leftrightarrow \{\alpha_k\}_{k=0}^{n-1}$$

where the α_k's give recurrence coefficients that may be used to build the associated OPUCs.

Properties:

- $\alpha_k \in \mathbb{D}$ for $k \leq n-1$ and $|\alpha_{n-1}| = 1$.
- The α_ks are called the Verblunsky coefficients associated to the measure.
- More generally there is a bijection between sequences of Verblunsky coefficients and measures on the unit circle.

OPUCs (part II): The Szegö Recursion

Suppose that $\Phi_0(z), \Phi_1(z), \ldots$ are a family of OPUCs associated to a measure μ on $\partial\mathbb{D}$.

Define: $\Phi_k^*(z) = z^k \overline{\Phi}_k(\frac{1}{z})$.
Then:

$$\begin{aligned} \Phi_{k+1}(z) &= z\Phi_k(z) - \bar{\alpha}_k \Phi_k^*(z) \\ \Phi_{k+1}^*(z) &= \Phi_k^*(z) - \alpha_k z \Phi_k(z) \end{aligned}$$

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = \begin{bmatrix} z & -\bar{\alpha}_k \\ -\alpha_k z & 1 \end{bmatrix} \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix} = T_k \begin{bmatrix} \Phi_k(z) \\ \Phi_k^*(z) \end{bmatrix}$$

Using this notation we can write

$$\begin{bmatrix} \Phi_{k+1}(z) \\ \Phi_{k+1}^*(z) \end{bmatrix} = T_k \cdots T_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Finding a counting function

For a measure μ supported on n points we can use the Szegö recursion to define the function $\Phi_n(z)$ (not an OPUC) which is 0 on the support of μ.

$$\begin{aligned} e^{ix} \in \text{supp } \mu \iff & \Phi_n(e^{ix}) = 0 \\ \iff & e^{ix}\Phi_{n-1}(e^{ix}) = \overline{\alpha}_{n-1}\Phi^*_{n-1}(e^{ix}) \end{aligned}$$

On $\partial\mathbb{D}$ the definition of Φ^*_k becomes $\Phi^*_k(e^{ix}) = e^{ixk}\overline{\Phi_k(e^{ix})}$:

$$e^{ix} \in \text{supp } \mu \iff \arg\overline{\alpha}_{n-1} = 2\arg(\Phi_{n-1}(e^{ix})) - x(n-2).$$

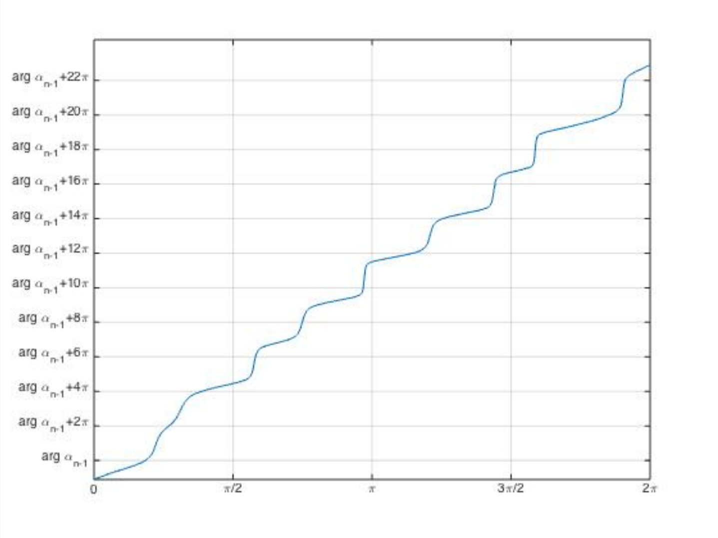
More generally define

$$\omega_k(x) = 2\arg(\Phi_k(e^{ix})) - x(k-1),$$

then...

$$N([0,x]) = \left\lfloor \frac{\omega_{n-1}(x) - \arg\overline{\alpha}_{n-1}}{2\pi} \right\rfloor$$

The counting function from $\omega_{n-1}(x)$ for Circular β



$\omega_{n-1}(x)$ for $n = 12$, $\beta = 4$.

Back to β-ensembles

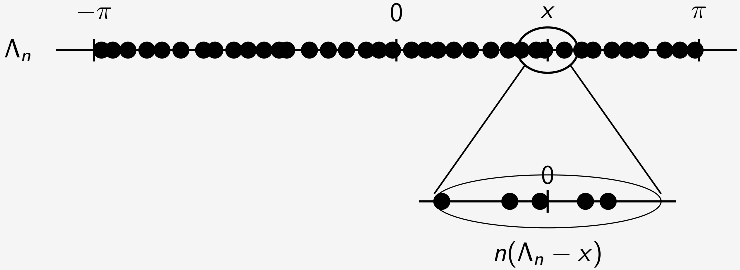
We look at the n-point measure with joint intensity

$$f(\theta_1, ..., \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta.$$

We build a spectral measure μ_n with spectral weights $q_1, ..., q_n$ satisfying $(q_1, ..., q_n) \sim$ Dirichlet $(\frac{\beta}{2}, ..., \frac{\beta}{2})$ then the associated Verblunsky coefficients will be independent with rotationally invariant distribution and

$$|\alpha_k|^2 \sim \begin{cases} \text{Beta } (1, \frac{\beta}{2}(n-k-1)) & k < n-1 \\ 1 & k = n-1 \end{cases}$$

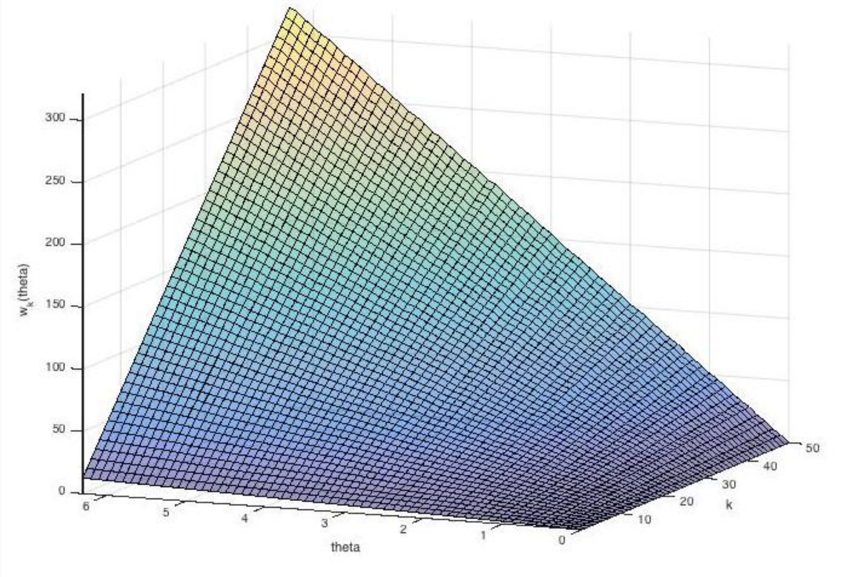
Local limits for Circular β



Rotational invariance means that we will see the same type of structure everywhere in the spectrum (on the circle).

We will focus near 0 which means we need to look at $\omega_{n-1}(x/n)$ in order to see the counting function.

Seeing the local limit structure at a finite level



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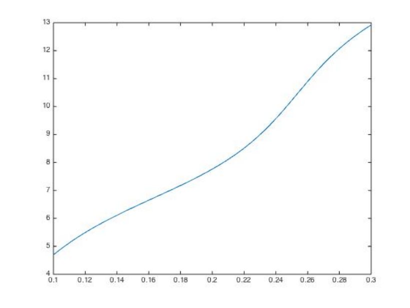
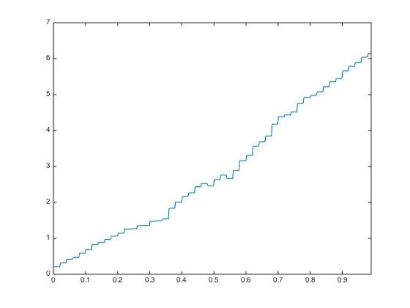


$\omega_{40}(x)$ on $[0.1, 0.3]$ for $\beta = 4$



$\omega_{\lfloor 50t \rfloor}(\frac{5}{50})$ on $[0, .99]$ for $\beta = 4$

The bulk limit

Theorem (Killip-Stoiciu, Valkó-Virág)

Let $\{\cdots < x_{-1} < 0 < x_0 < x_1 < \cdots\}$ *have* β*-circular distribution (in the argument), then*

$$\{..., nx_{-1}, nx_0, nx_1, ...\} \Rightarrow \mathrm{Sine}_\beta \qquad \text{as } n \to \infty.$$

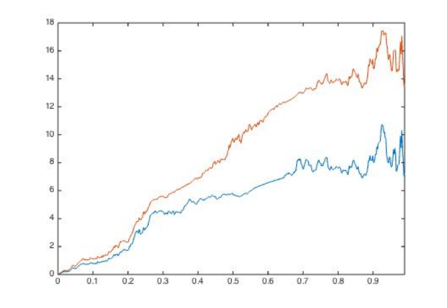
Sine_β *may be characterized by its counting function which has distribution* $N_\beta(\lambda) = \lim_{t\to\infty} \frac{\alpha_\lambda(t)}{2\pi}$ *where*

$$d\alpha_\lambda = \lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + \mathrm{Re}[(e^{-i\alpha_\lambda} - 1) d(B^{(1)} + iB^{(2)})], \qquad \alpha_\lambda(0) = 0.$$

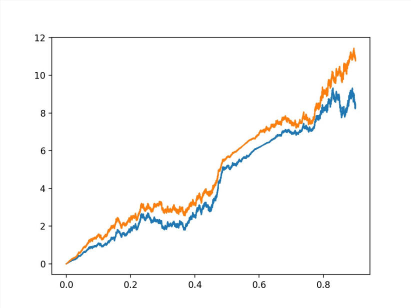
Morally: $\hat{\alpha}_\lambda(t) = \alpha_\lambda(-\frac{4}{\beta} log(1-t)) \approx \omega_{\lfloor nt \rfloor}(\lambda/n)$. Under this time change $\hat{\alpha}_\lambda(0) = 0, t \in [0, 1)$

$$d\hat{\alpha}_\lambda(t) = \lambda dt + \frac{2}{\sqrt{\beta(1-t)}} \mathrm{Re}[(e^{-i\hat{\alpha}_\lambda} - 1) d(B^{(1)} + iB^{(2)})].$$

Moral proof by picture



$\omega_{\lfloor 500t \rfloor}(\frac{10}{500})$ and $\omega_{\lfloor 500t \rfloor}(\frac{14}{500})$ for $\beta = 4$

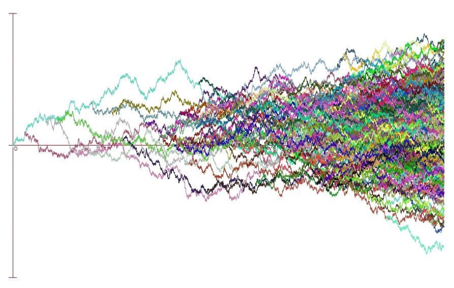


$\hat{\alpha}_{10}(t)$ and $\hat{\alpha}_{14}(t)$ for $\beta = 4$

Natural Questions for Sine_β

- Asymptotic properties of $N_\beta(\lambda)$ as $\lambda \to \infty$
  - Large deviations (H., Valkó)
  - Large gaps (Valkó, Virág)
  - Central limit theorem (Krichevsky, Valkó, Virág)
- Process limits as $\beta \to 0$ (Allez, Dumaz)
- Rigidity of the process (Chhaibi, Najnudel)
- Maximum deviation of the counting function from its norm (H., Paquette)

Log-correlated fields and branching processes



Branching Brownian Motion
(Borrowed from Matt Roberts)

Models with log-correlated structure: Branching Random walk, Branching Brownian motion, log-correlated Gaussian field, characteristic polynomials of random matrices.

A few people who have worked in the area: Derrida-Spohn, Hu-Shi, Aídékon-Shi, Arguin-Zindy

Full results for log-correlated Gaussian fields: Ding-Roy-Zeitouni

Log-correlated fields and Circular β

Conjecture (Fyodorov, Hiary, Keating)

For $\beta = 2$, and K_1, K_2 independent Gumble distributions

$$\sup_{z\in\partial\mathbb{D}} \log|\Phi_n(z)| - (\log n - \frac{3}{4}\log\log n) \to \frac{1}{2}(K_1 + K_2)$$

- 1st term: Arguin, Belius, Bourgade (2017)
- 2nd term: Paquette-Zeitouni (2017)
- tightness of the distribution ($\beta > 0$): Chhaibi, Mandaule, Najnudel (2018)

Recall that we said that $\hat{\alpha}_\lambda(t)$ was morally $2\arg\Phi_{nt}(e^{i\lambda/n}) + t\lambda$. This gives that $2\mathrm{Im}\log\Phi_n(e^{i\lambda/n})$ is comparable to $2\pi N(\lambda) - \lambda$. For Sine_β the analogous question is

$$\sup_{|\lambda|\le x} (N_\beta(\lambda) - N_\beta(-\lambda) - \frac{\lambda}{\pi}) - C_\beta(\log x - \frac{3}{4}\log\log x) \Rightarrow ?$$

The Result

Theorem (H., Paquette)

$$\max_{0\le\lambda\le x} \frac{N(\lambda)-N(-\lambda)-\frac{\lambda}{\pi}}{\log x} \to \frac{2}{\sqrt{\beta}\pi} \quad \textit{in probability as } x\to\infty.$$

- Notice that

$$N(\lambda)-N(-\lambda)-\frac{\lambda}{\pi} = \frac{1}{2\pi}\mathrm{Re}\int_0^\infty (e^{-i\alpha_\lambda(t)}-e^{-i\alpha_{-\lambda}(t)})dZ = \frac{1}{2\pi}M_\lambda(\infty)$$

- For the proof we will focus on the martingale

$$M_\lambda(t) = \mathrm{Re}\int_0^t (e^{-i\alpha_\lambda(t)}-e^{-i\alpha_{-\lambda}(t)})dZ$$

Observation 1

Let $T_\lambda = \frac{4}{\beta}\log\lambda$, then $\alpha_\lambda(T_\lambda + t)$ satisfies the same SDE as $\alpha_1(t)$ with a random initial condition. Indeed for $\tilde{\alpha}_1$ a realization of α_1 with the shifted Brownian motion we get

$$\tilde{\alpha}_1(t) \le \alpha_\lambda(T_\lambda + t) - \lfloor \alpha_\lambda(T_\lambda) \rfloor_{2\pi} \le \tilde{\alpha}_1(t) + 2\pi.$$

Proposition

There exists a C such that

$$P(M_\lambda(\infty) - M_\lambda(T_\lambda) \ge C + r) \le e^{-r/C}.$$

This together with the monotonicity of $N_\beta(\lambda)$ gives

$$\max_{0\le\lambda\le x} \frac{M_\lambda(\infty) - M_\lambda(T_\lambda)}{\log x} \to 0 \qquad \text{in probability.}$$

A conjecture

New problem: $\max_{0\leq\lambda\leq x} \frac{M_\lambda(T_\lambda)}{\log x} \to \frac{4}{\sqrt{\beta}}$ in probability.

Suppose that we replaced the α_λ and $\alpha_{-\lambda}$ in the definition of M_λ by their expectations.

$$G_\lambda(t) = \mathrm{Re} \int_0^t (e^{-i\mathbb{E}\alpha_\lambda(t)} - e^{-i\mathbb{E}\alpha_{-\lambda}(t)})dZ.$$

Then the process $G_\lambda(T_\lambda)$ is a Gaussian process and satisfies the conditions of Ding-Roy-Zeitouni.

$$\max_{0\leq\lambda\leq x} G_\lambda(T_\lambda) - \frac{4}{\sqrt{\beta}}(\log x - \frac{3}{4}\log\log x) \Rightarrow \xi.$$

Conjecture

The same type of limit holds for $M_\lambda(\infty)$.

Observation 2

For a fixed λ the diffusion $\alpha_\lambda - \alpha_{-\lambda}$ satisfies the SDE

$$d(\alpha_\lambda - \alpha_{-\lambda}) = 2\lambda \frac{\beta}{4} e^{-\frac{\beta}{4}t} dt + 2 \sin\left(\frac{\alpha_\lambda - \alpha_{-\lambda}}{2}\right) dB^{(\lambda)}$$

Here the Brownian motion $B^{(\lambda)}$ depends on λ.

Using this we can:

- Tilt the measure
- Study integrals of the form $\int_0^t \sin(\alpha_\lambda - \alpha_{-\lambda}) ds$ which are highly oscillatory for large λ.

Tilting the measure

Let $\xi \in \mathbb{R}$ and consider the measure $Q_{\xi,\lambda}$ such that

$$dX_s = dB^{(\lambda)} - \xi \sin\left(\frac{\alpha_\lambda - \alpha_{-\lambda}}{2}\right) dt$$

is a Brownian motion. The Radon-Nikodym derivative may be explicitly computed as

$$\frac{dQ_{\xi,\lambda}}{d\mathbb{P}} = \mathcal{E}(\xi M_\lambda) = \exp\left(\xi M_{\lambda,t} - \frac{\xi^2}{2}[M_\lambda]_t\right)$$

Under this change of measure we can compute explicitly the processes that M_λ and α_λ become.

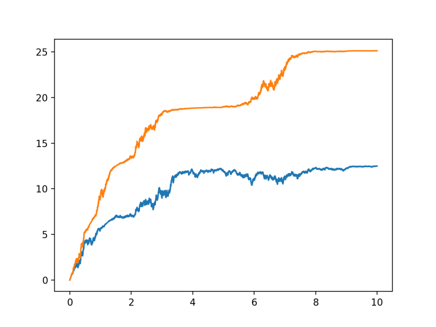
$$M_{\lambda,t} = \int_0^t 2\sin\left(\frac{u}{2}\right) dX + \int_0^t 2\xi \sin^2\left(\frac{u}{2}\right) dt$$
$$du_{\lambda,\xi} = 2\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + 2\xi\sin^2\left(\frac{u}{2}\right) dt + 2\sin\left(\frac{u}{2}\right) dX$$

We call this the "accelerated Sine equation."

The accelerated Sine equation

$$d\alpha_\lambda = 2\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + 2\sin\left(\frac{\alpha_\lambda}{2}\right)dB$$

$$du_{\lambda,\xi} = 2\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}t}dt + 2\xi\sin^2\left(\frac{u}{2}\right)dt + 2\sin\left(\frac{u}{2}\right)dX$$



One realization of α_{10} and $u_{10,2}$ with $\beta = 4$.

Oscillatory integrals

Proposition

There exist R, γ uniform in T, β, λ, a such that

$$\mathbb{E}\left[\sup_{t\le T}\left|\int_0^t e^{iau_{\lambda,\xi}}\,ds\right|\right] \le \frac{R(1+|\xi|)}{|a\lambda|\frac{\beta}{4}e^{-\frac{\beta}{4}T}},$$

$$\mathbb{P}\left[\sup_{t\le T}\left|\int_0^t e^{iau_{\lambda,\xi}}\,ds\right| - \frac{R(1+|\xi|)}{|a\lambda|\frac{\beta}{4}e^{-\frac{\beta}{4}T}} \ge C\right] \le \exp\left(-\gamma\left(Ca\lambda\frac{\beta}{4}e^{-\frac{\beta}{4}T}\right)^2\right).$$

This gives us good control over integrals like $\int_0^t \sin^2(\frac{u}{2})ds$.

How can this be used?

Suppose you wanted a bound on

$$\mathbb{P}(\sup_{t\leq T} M_{\lambda,t} \geq C).$$

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Using the tilting we can compute

$$\mathbb{E}(e^{\xi M_{\lambda,T}}) = \mathbb{E}(\mathcal{E}(\xi M_\lambda)e^{\frac{\xi^2}{2}[M_\lambda]}) = \mathbb{Q}_{\lambda,\xi}(e^{\frac{\xi^2}{2}[M_\lambda]})$$

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Then $[M_\lambda]_t = \int_0^2 4\sin^2(\frac{u}{2})ds$ where u satisfies the 'accelerated Sine equation' with parameters λ, ξ. This can be controlled with the results on oscillatory integrals.

Thank You!