Rotating trapped fermions in $2d$ and the complex Ginibre ensemble

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Refs: B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

Plan:

- A brief introduction to ultracold trapped fermions

- An exact mapping in 1-d:

  positions of free fermions in a 1-d harmonic trap (ground state)

  $\equiv$ eigenvalues of Gaussian Unitary Ensemble (GUE)

- An exact mapping in 2-d:

  positions of free fermions in a 2-d rotating harmonic trap (ground state)

  $\equiv$ eigenvalues of Complex Ginibre Ensemble (GinUE)

- Exact results for two specific observables:
  - Number variance
  - Rényi entanglement entropy

Ultracold Trapped Fermions

Ultracold atoms

- Recent great progress in the experimental manipulation of cold atoms

  $\Rightarrow$ to investigate the interplay between quantum and statistical behaviors in many-body systems over a wide range of temperatures

  - atoms in a confining laser trap
  - different spatial dimensions
  - interactions between atoms $\Rightarrow$ tunable
  - temperature T and the number of atoms N $\Rightarrow$ tunable

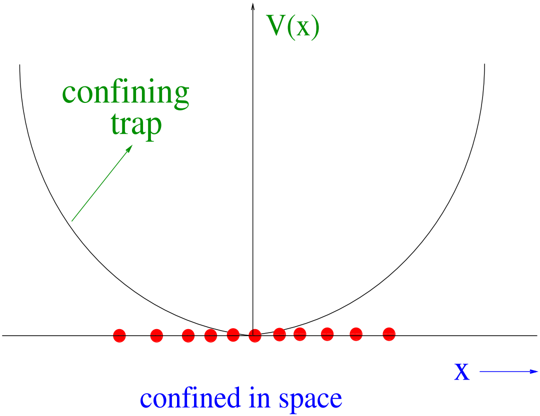
- Interesting quantum many-body effects even in the absence of interactions

  Bosons: Bose-Einstein condensation

  Fermions: Pauli exclusion principle $\Rightarrow$ nontrivial spatial correlations

Ultracold atoms in a confining potential

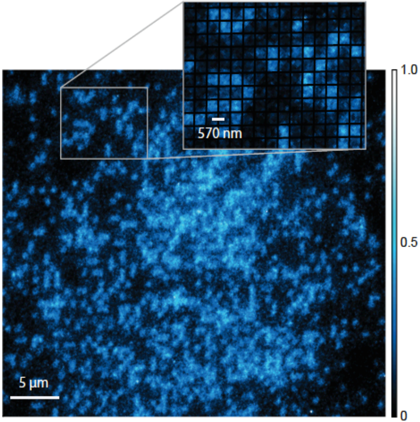
A common feature of these experiments ⇒ presence of a confining potential that traps the particles within a limited spatial region



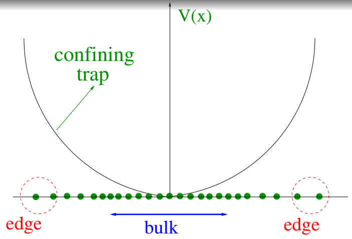
Quantum gas microscope

Parsons et. al. PRL (2015)
(Greiner group, Harvard)

$^6\mathrm{Li}$ atoms $\longrightarrow$ fermions



Free fermions in a trap → edge physics



- bulk: traditional many-body physics (translationally invariant system)
- edges: ?

"The uniform electron gas, the traditional starting point for density-based many-body theories of inhomogeneous systems, is inappropriate near electronic edges. " W. Kohn & A. E. Mattsson, PRL, 81, 3487 (1998)

Edge physics in 1-d can be fully described exploiting a connection to random matrix theory (GUE) ⇒ universal edge properties

Grabsch, Gautie, Grela, Lacroix, Marino, Cunden, Dean, Le Doussal, O'Connell, Schehr, Texier, Vivo,.....,+S.M. (2014 - -)

Recent review: Dean, Le Doussal, S.M., Schehr, arXiv: 1810.12583

$T = 0$ free fermions in a 1-d harmonic trap

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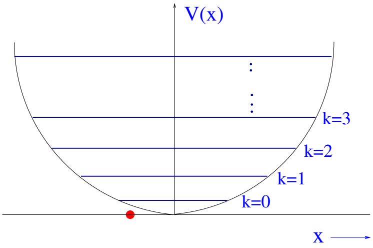
Gaussian Unitary Ensemble (GUE)

A single Fermion in a harmonic trap



A single quantum particle in a harmonic potential

Hamiltonian: $\hat{h} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$

Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + \frac{1}{2} m\omega^2 x^2 \varphi_k(x) = \epsilon_k \varphi_k(x)$$

with $\varphi_k(x \to \pm\infty) = 0$

single particle eigenfunctions: $\varphi_k(x) = \left[\frac{\alpha}{\sqrt{\pi}\, 2^k\, k!}\right]^{1/2} e^{-\alpha^2 x^2/2} H_k(\alpha\, x)$

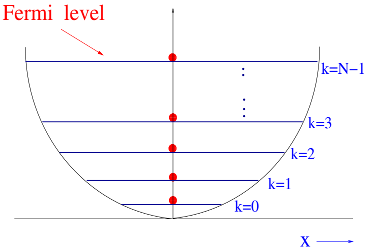
with energy levels: $\epsilon_k = (k + 1/2)\, \hbar\, \omega \quad k = 0, 1, 2, 3 \ldots$

$\alpha = \sqrt{m\omega/\hbar} \to$ inverse of the width of the ground state wave packet

$H_k(x) \to$ Hermite polynomials

For example, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, etc.

N spinless free fermions in a harmonic trap: T=0



many-body Hamiltonian

$$\hat{H}_N = \sum_{i=1}^{N} \hat{h}_i = \sum_{i=1}^{N} \left[\frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 \right]$$

Many-body Ground state wavefunction $\rightarrow \hat{H}_N \, \Psi_0 = E_0 \, \Psi_0$

$\Psi_0(x_1, x_2, \ldots, x_N) \rightarrow$ antisymmetric (Pauli exclusion principle)

can be expressed as a Slater determinant:

$$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \qquad 0 \le i \le (N-1), \quad 1 \le j \le N$$

Ground state energy: $E_0 = \sum_{k=0}^{N-1} (k + 1/2) \hbar\,\omega = \frac{N^2}{2} \, \hbar\,\omega$

Slater determinant

The Slater determinant can be explicitly evaluated

$$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \qquad 0 \le i \le (N-1), \;\; 1 \le j \le N$$

$$\Psi_0(\{x_i\}) \propto e^{-\frac{\alpha^2}{2} \sum_{i=1}^N x_i^2} \det_{1 \le i, j \le N} [H_i(\alpha\, x_j)]$$

$$\propto e^{-\frac{\alpha^2}{2} \sum_{i=1}^N x_i^2} \prod_{j<k} (x_j - x_k)$$

$$\Longrightarrow \qquad \boxed{|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N}\, e^{-\alpha^2 \sum_{i=1}^N x_i^2} \prod_{j<k} (x_j - x_k)^2}$$

$\Rightarrow$ quantum prob. density

$\Rightarrow$ encodes quantum fluctuations

1-d trapped fermions at T=0 $\equiv$ GUE eigenvalues

- Fermions: squared many-body wave function at $T=0$

$$|\Psi_0(\{x_i\})|^2 = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \alpha^2\, x_i^2\right] \prod_{j<k} (x_j - x_k)^2 \text{ where } \alpha = \sqrt{m\omega/\hbar}$$

- GUE eigenvalues $\{\lambda_i\}$: joint probability distribution

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} \lambda_i^2\right] \prod_{j<k} |\lambda_j - \lambda_k|^2$$

$\Rightarrow$ The positions of free fermions in a harmonic trap at $T=0$ behave statistically as the eigenvalues of a GUE random matrix

$$\boxed{(\alpha\, x_1,\, \alpha\, x_2,\, \ldots, \alpha\, x_N) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)}$$

Properties of $T = 0$ fermions using GUE

GUE eigenvalues $\rightarrow$ determinantal process

m-point correlation function:

$$R_m(x_1, x_2, \ldots, x_m) = \frac{N!}{(N-m)!} \int dx_{m+1} \ldots dx_N \, |\Psi_0(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_N)|^2$$
$$= \det_{1 \le i,j \le m} [K_N(x_i, x_j)]$$

The Kernel:

$$\boxed{K_N(x, x') = \sum_{k=0}^{N-1} \varphi_k(x) \varphi_k(x')} \implies \text{central object}$$

In particular, the average density:

$$\boxed{\rho_N(x) = \frac{1}{N} K_N(x, x) = \frac{1}{N} \sum_{k=0}^{N-1} |\varphi_k(x)|^2}$$

1-d trapped fermions at T=0 $\equiv$ GUE eigenvalues

The equivalence

$$\boxed{(\alpha\, x_1,\, \alpha\, x_2,\, \ldots, \alpha\, x_N) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_N)}$$

has been heavily exploited in recent works to predict various properties of 1-d trapped fermions, using the knowledge of GUE

- Average density of fermions $\rightarrow$ Wigner semicircle law
- Correlation functions: bulk (Sine kernel) and edge (Airy kernel)
- The scaled distribution of the position of the rightmost trapped fermion

  $x_{\rm max}$ at $T = 0 \rightarrow$ Tracy-Widom GUE
- Number variance of fermions in an interval $[-l, l]$
- Entanglement entropy of fermions in $[-l, l]$ with the rest of the system

Recent review: Dean, Le Doussal, S.M., Schehr, arXiv:1810.12583

$T = 0$ **free fermions in a** 2**-d trap**

$\&$

Random Matrix Ensemble ?

Complex Ginibre Ensemble (GinUE)

A natural random matrix ensemble with complex eigenvalues in 2-d

$\Longrightarrow$ complex Ginibre ensemble

$[X_{i,j}] \rightarrow$ complex $(N \times N)$ matrix with independent $\mathcal{N}(0, 1/2)$ entries

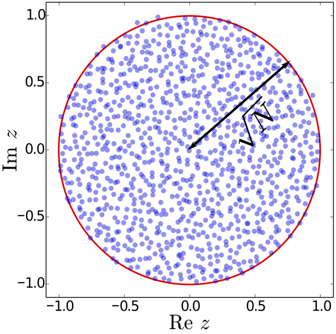
All N eigenvalues $\{z_1, z_2, \ldots, z_N\} \Longrightarrow$ complex

The joint distribution of eigenvalues:

$$P_{\mathrm{Gin}}(z_1, z_2, \ldots, z_N) = \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} |z_i|^2\right] \prod_{j<k} |z_j - z_k|^2$$

The eigenvalues $\{z_1, z_2, \ldots, z_N\} \Longrightarrow$ determinantal process

Complex Ginibre Eigenvalues



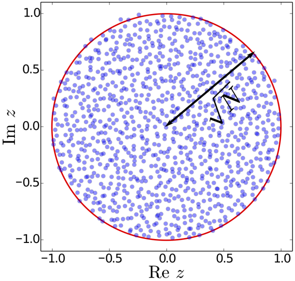
Average density of eigenvalues, for large N, converges to a uniform distribution over a disk of radius $\sqrt{N}$

$$\rho_N(z) \to \frac{1}{\pi}\,\mathbf{1}_{|z|<\sqrt{N}}$$

bulk: $|z| \sim O(1) << \sqrt{N}$

edge: $|z| - \sqrt{N} \sim O(1)$

GinUE eigenvalues



GinUE eigenvalues $\{z_1, z_2, \ldots, z_N\}$

$\Longrightarrow$ determinantal process

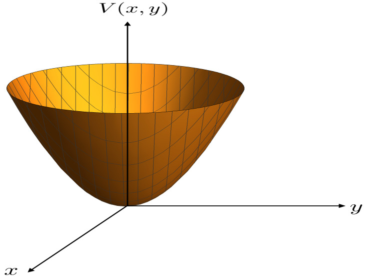
Both bulk and edge properties $\Rightarrow$ well studied

Q: Is there a 2-d trapped Fermion system whose ground state wave function (modulus square) would correspond to the joint distribution of the GinUE eigenvalues?

$$|\Psi_0(z_1, z_2, \ldots, z_N)|^2 \equiv P_{\mathrm{Gin}}(z_1, z_2, \ldots, z_N)$$

$$= \frac{1}{Z_N} \exp\left[-\sum_{i=1}^{N} |z_i|^2\right] \prod_{j<k} |z_j - z_k|^2$$

Fermions in a 2-d harmonic trap: a natural candidate



many-body Hamiltonian

$$\hat{H}_N = \sum_{i=1}^{N} \hat{h}_i = \sum_{i=1}^{N} \left[\frac{\hat{\mathbf{p}}_i^2}{2m} + V(\hat{\mathbf{x}}_i) \right]$$

where $V(x,y) = \frac{1}{2}\, m\, \omega^2 \,(x^2+y^2)$

let $\alpha = \sqrt{m\omega/\hbar}$

Single particle eigenfunctions:

$$\varphi_{n_1,n_2}(x,y) \propto H_{n_1}(\alpha\, x)\, H_{n_2}(\alpha\, y)\, e^{-\frac{\alpha^2}{2}(x^2+y^2)}$$

with associated eigenvalue:

$$\epsilon_{n_1,n_2} = (n_1 + n_2 + 1)\,\hbar\omega \qquad n_1 = 0,1,2,\ldots \text{ and } n_2 = 0,1,2,\ldots$$

Many-body ground state wavefunction $\Rightarrow$ Slater determinant of N lowest energy eigenstates:

$$\Psi_0(\{\mathbf{x}_i\}) = \frac{1}{\sqrt{N!}}\, \det[\varphi_i(\mathbf{x}_j)]$$

Free fermions in a 2-d harmonic trap

$|\Psi_0(\{\mathbf{x}_i\})|^2 \Longrightarrow$ 2-d determinantal process

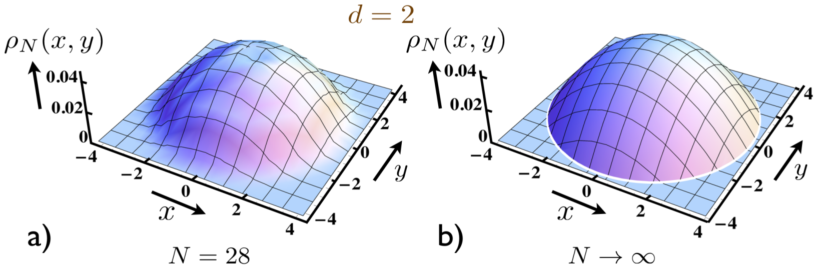
However, it does not correspond to GinUE

$$|\Psi_0(\{\mathbf{x}_i\})|^2 \neq P_{\rm Gin}(z_1, z_2, \ldots, z_N) \qquad \text{where } z_k = x_k + i\, y_k$$

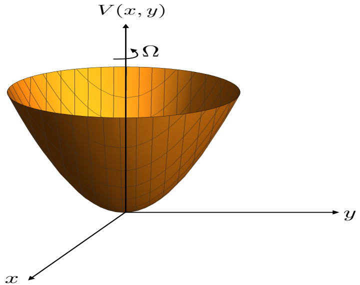
This 2-d determinantal process has been studied analytically both for the bulk and the edge properties

Dean, Le Doussal, S.M & Schehr, EPL, 112, 60001 (2015)

For example, the average density $\rho_N(r) \propto (r_{\rm edge}^2 - r^2)$ where $r_{\rm edge} \sim N^{1/4}$



Free fermions in a 2-d rotating trap



In the rotating frame: $\hat{H}_N = \sum_{i=1}^{N} \hat{h}_i$

$$\hat{h} = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2}\, m\,\omega^2\, \hat{\mathbf{r}}^2 - \Omega\, \hat{L}_z$$

$\hat{L}_z = i\,\hbar\,(y\,\partial_x - x\,\partial_y) \longrightarrow$ angular momentum

$\Omega \leq \omega$ to keep the fermions confined

Single particle hamiltonian can also be expressed as:

$$\hat{h} = \frac{1}{2m}(\hat{\mathbf{p}} - m\,\omega\,\hat{\mathbf{z}} \times \hat{\mathbf{r}})^2 + (\omega - \Omega)\,\hat{L}_z$$

where $\hat{\mathbf{z}} \to$ unit vector along z

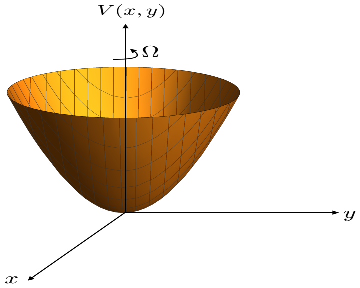
Two interpolating limits:

- $\Omega = 0 \longrightarrow 2$-d harmonic trap (non-rotating)
- $\Omega = \omega \longrightarrow$ free fermions in a uniform perpendicular magnetic field

$\mathbf{B} = -B\,\hat{\mathbf{z}}$ with $B = 2m\,\omega\,c/e \implies$ Landau problem

Vector potential: $\mathbf{A} = \frac{1}{2}[B\,y, -B\,x, 0] \to$ symmetric gauge

Single particle spectrum: exactly solvable



In the rotating frame:

$$\hat{h} = \frac{\hat{\mathbf{p}}^2}{2m} + \frac{1}{2}\, m\,\omega^2\, \hat{\mathbf{r}}^2 - \Omega\, \hat{L}_z$$

$$\hat{h}\,\varphi(x,y) = \epsilon\,\varphi(x,y)$$

$z = \alpha\,(x + i\,y)$ and $\bar{z} = \alpha\,(x - i\,y)$

where $\alpha = \sqrt{m\omega/\hbar}$

Eigenfunctions and eigenvalues of $\hat{h}$ are labelled by (n_1, n_2):

$$\varphi_{n_1,n_2}(z,\bar{z}) \propto e^{\frac{1}{2}\,z\bar{z}}\ \partial_{\bar{z}}^{n_1}\ \partial_{z}^{n_2}\ e^{-z\bar{z}}$$

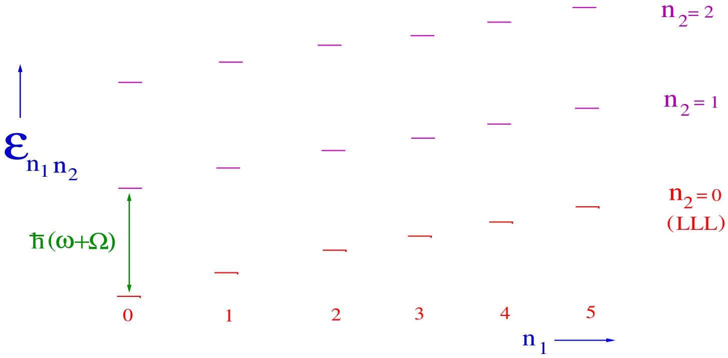
$$\epsilon_{n_1,n_2} = \hbar\,[\omega + (\omega - \Omega)\,n_1 + (\omega + \Omega)\,n_2]$$

with $n_1 = 0, 1, 2, \ldots$ and $n_2 = 0, 1, 2, \ldots$

$n_2 \Longrightarrow$ Landau levels ($n_2 = 0 \Longrightarrow$ Lowest Landau Level (LLL))

Ho & Ciobanu, '00; Aftalion, Blanc, Dalibard, '05

Single particle spectrum



Single particle energy levels: $\epsilon_{n_1,n_2} = \hbar\left[\omega + (\omega - \Omega)\, n_1 + (\omega + \Omega)\, n_2\right]$

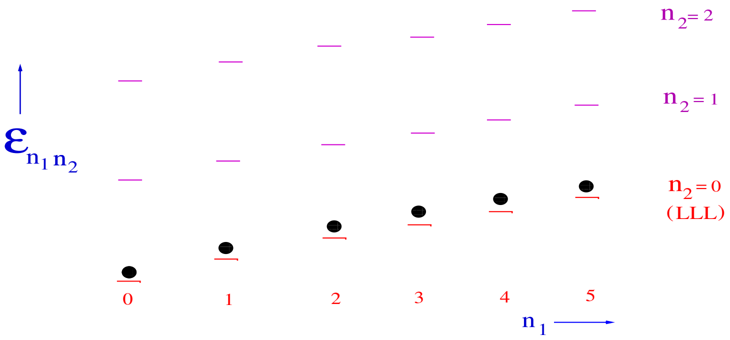
where $\Omega \leq \omega \implies$ needed for confinement

In the Landau limit $\Omega = \omega$:

LLL $\implies$ infinitely degenerate (labelled by $n_1 = 0, 1, 2, \ldots$)

Setting $\Omega < \omega \implies$ lifts the degeneracy of the LLL

Ground state of the N-Fermi system



Ground state $\rightarrow$ N lowest energy levels each filled with 1 Fermion

N occupied levels $\in$ LLL ($n_2 = 0$) sector if: $\epsilon_{n_1=N-1,\, n_2=0} < \epsilon_{n_1=0,\, n_2=1}$

highest occupied level in $n_2 = 0$ < lowest energy of the $n_2 = 1$ sector

$$\Longrightarrow \boxed{\omega\left(1 - \frac{2}{N}\right) < \Omega < \omega}$$

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Rapidly Rotating Bose-Einstein Condensates in and near the Lowest Landau Level

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Ground state wavefunction and GinUE

If the rotating frequency is in the range: $\omega\left(1-\frac{2}{N}\right) < \Omega < \omega$

Ground state corresponds exactly to filling the first N levels of LLL

labelled by $n_2 = 0$ and $n_1 = 0, 1, \ldots, N-1$

These occupied single particle levels have simple eigenfunctions:

$$\varphi(z) \propto z^{n_1}\, e^{-\frac{1}{2}\,|z|^2} \qquad n_1 = 0, 1, 2, \ldots, N-1$$

with associated eigenvalues: $\epsilon_{n_1,0} = \hbar(\omega + (\omega - \Omega)\, n_1)$

The N-body ground state wavefunction $\Longrightarrow$ Slater determinant of these N lowest LLL levels

$$\Psi_0(z_1, z_2, \ldots, z_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(z_j)] \qquad 0 \le i \le N-1, \;\; 1 \le j \le N$$

$$\Psi_0(\{z_i\}) \propto e^{-\frac{1}{2}\sum_{i=1}^N |z_i|^2} \det_{1\le i, j\le N}[z_j^{i-1}] \;\; \propto e^{-\frac{1}{2}\sum_{i=1}^N |z_i|^2} \prod_{j<k} (z_j - z_k)$$

Consequently: $|\Psi_0(\{z_i\})|^2 \propto e^{-\sum_{i=1}^N |z_i|^2} \prod_{j<k} |z_j - z_k|^2$

$$= P_{\rm Gin}(z_1, z_2, \ldots, z_N)$$

Rotating $2d$ harmonic trap and GinUE

The equivalence:

$$\boxed{|\Psi_0(\{z_i\})|^2 = P_{\rm Gin}(z_1, z_2, \ldots, z_N) = \frac{1}{Z_N}\, e^{-\sum_{i=1}^N |z_i|^2} \prod_{j<k} |z_j - z_k|^2}$$

A 2-d harmonic trap rotating with a frequency $\omega(1-\frac{2}{N}) < \Omega < \omega$ in its ground state provides

$\Longrightarrow$ an experimental realisation of GinUE

B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

Exact results from this correspondence

GinUE eigenvalues $\{z_1, z_2, \ldots, z_N\} \Longrightarrow$ determinantal process with Kernel

$$K_N(z, z') = \frac{1}{\pi}\, e^{-\frac{1}{2}(|z|^2+|z'|^2)} \sum_{k=0}^{N-1} \frac{(z\bar{z}')^k}{k!}$$

Correlation functions in the bulk and the edges are known

Forrester & Honner, '99, . . .

$\Longrightarrow$ provide exact position correlations (bulk and edge) of the Fermions in the ground state of the 2-d rotating trap

Exact results for two specific observables:

- Number Variance (Full Counting Statistics)
- Rényi Entanglement Entropy (not a natural observable in RMT)

B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

Number Variance

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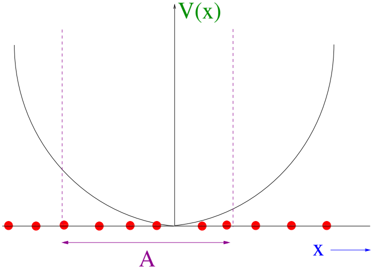
Rényi Entanglement Entropy

N free fermions in a confining potential at $T = 0$



N free fermions at $T = 0$

in a confining trap $V(x)$ (any dim.)

ground state: $|\Psi_0\rangle \to$ pure state

density matrix: $\hat{\rho} = |\Psi_0\rangle\langle\Psi_0|$

subsystem: $A \longrightarrow$ interval (domain)

- Number Variance: $\mathrm{Var}(N_A) = \langle\Psi_0|\hat{N}_A^2|\Psi_0\rangle - \left[\langle\Psi_0|\hat{N}_A|\Psi_0\rangle\right]^2$

  where $\hat{N}_A = \int_A dx\, \hat{n}(x) \to$ no. of fermions in A

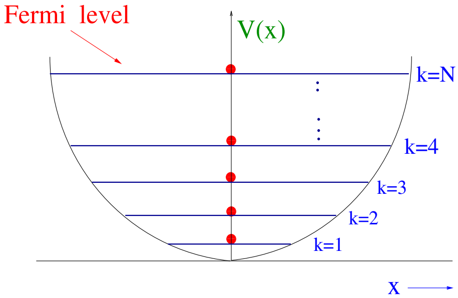
- Rényi Entanglement Entropy: $S_q = \frac{1}{1-q} \ln \mathrm{Tr}\, \hat{\rho}_A^q$

  where $\hat{\rho}_A = \mathrm{Tr}_{\bar{A}}\,[\hat{\rho}] \to$ reduced density matrix of A

As $q \to 1$, $\quad S_q \to S_{VN} = -\mathrm{Tr}\,[\hat{\rho}_A \ln \hat{\rho}_A] \longrightarrow$ von Neumann entropy

Calabrese, Mintchev & Vicari '11, Eisler '13, Eisler & Peschel '14, Calabrese, le Doussal & S.M '15 .....

Ground state for non-interacting fermions



Ground state many-body wavefunction $\rightarrow$ $(N \times N)$ Slater determinant

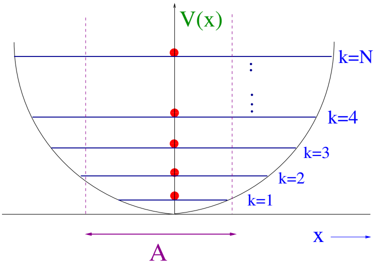
$$\Psi_0(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\varphi_i(x_j)] \text{ with } 1 \leq i, j \leq N$$

where $\varphi_k(x) \longrightarrow$ single particle eigenfunction

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_k}{dx^2} + V(x)\,\varphi_k(x) = \epsilon_k \varphi_k(x)$$

$\epsilon_k \longrightarrow$ single particle energy eigenvalues

Overlap matrix



$(N \times N)$ overlap matrix associated to an arbitrary interval A

$$\hat{\mathcal{A}}_{m,n} = \int_A \varphi_m^*(x)\, \varphi_n(x)\, dx$$

where $1 \le m,\ n \le N$

with N real eigenvalues $0 \le a_n \le 1$

Number Variance :

$$\mathrm{Var}(N_A) = \mathrm{Tr}\, \left[\hat{\mathcal{A}}\, (I - \hat{\mathcal{A}})\right] = \sum_{n=1}^{N} a_n\, (1 - a_n)$$

Rényi Entropy :

$$S_q = \frac{1}{1-q}\, \mathrm{Tr}\, \ln \left[\hat{\mathcal{A}}^q + (I - \hat{\mathcal{A}})^q\right] = \frac{1}{1-q} \sum_{n=1}^{N} \ln\, [(a_n)^q + (1 - a_n)^q]$$

Peschel '99, Vidal et. al. '03, Klich '06, ...Rodriguez & Sierra '09,...

Overlap matrix and Kernel

- Trapped non-interacting fermions $\longrightarrow$ determinantal process with kernel

$$K(x,y) = \sum_{i=1}^{N} \varphi_k^*(x)\, \varphi_k(y)$$

- Restricted kernel: $K_A(x,y) \equiv I_A(x)\, K(x,y)\, I_A(y)$

$$I_A(x) = \begin{cases} 1 \ \text{if} & x \in A \\ 0 \ \text{if} & x \notin A \end{cases}$$ projects onto A

- $K_A(x,y) \longrightarrow$ integral operator: $\int K_A(x,y)\, \psi_\lambda(y)\, dy = a_\lambda\, \psi_\lambda(x)$

spectrum $\{a_\lambda\}$ of $K_A(x,y) \Longleftrightarrow$ spectrum $\{a_i\}$ of $\hat{\mathcal{A}}$

- Consequently:

Number Variance: $\mathrm{Var}(N_A) = \mathrm{Tr}\left[\hat{\mathcal{A}}\,(I - \hat{\mathcal{A}})\right] = \mathrm{Tr}\left[K_A\,(I - K_A)\right]$

Rényi Entropy :

$$S_q = \frac{1}{1-q}\,\mathrm{Tr}\,\ln\left[\hat{\mathcal{A}}^q + (I - \hat{\mathcal{A}})^q\right] = \frac{1}{1-q}\,\mathrm{Tr}\,\ln\left[{K_A}^q + (I - K_A)^q\right]$$

Number Variance and Rényi Entropy

For a translationally invariant free Fermi gas (no edge), there exists a universal relation (irrespective of the subsystem A):

$$\boxed{S_q = \frac{\pi^2}{6}\left(1+\frac{1}{q}\right)\mathrm{Var}(N_A)}$$

Klich & Levitov '09, Song et. al. '11-'12, ..., Calabrese, Le Doussal & S.M. '15

Q: What happens to this proportionality for a finite N trapped Fermi gas (with an edge) **?**

For 1-d trapped fermions (GUE), this proportionality breaks down near the edge, although Rényi entropy is difficult to compute near the edge

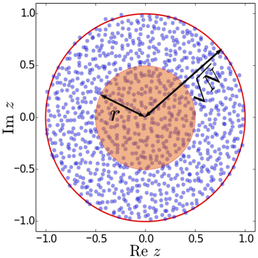
Marino, S.M., Schehr, Vivo '14 Calabrese, Le Doussal & S.M. '15

For 2-d free fermions in a rotating trap (GinUE)

$\Longrightarrow$ number variance and Rényi entropy can be computed exactly (bulk and edge)

B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

2-d rotating trap and GinUE



subsystem: A

$\Longrightarrow$ circular disk of radius r

Overlap matrix: $\hat{A}_{m,n} = \int_{|z|\le r} d^2z\, \varphi^*_{m-1}(z)\, \varphi_{n-1}(z)$

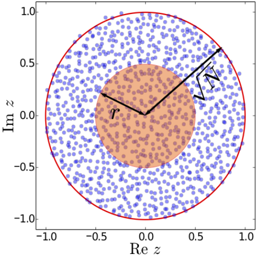
where the eigenfunction $\varphi_n(z) = \frac{z^n}{\sqrt{\pi\, n!}}\, e^{-|z|^2/2}$

Circular symmetry of the subsystem A implies:

$$\hat{A}_{m,n} = a_n(r)\,\delta_{m,n} \quad \text{with} \quad a_n(r) = \frac{\gamma(n,r^2)}{\Gamma(n)} \qquad \gamma(n,x) = \int_0^x dt\, e^{-t}\, t^{n-1}$$

$\Longrightarrow$ Overlap matrix is fully diagonal with eigenvalues $a_n(r) = \frac{\gamma(n,r^2)}{\Gamma(n)}$

Exact Number Variance & Rényi Entropy



Overlap matrix:

$\hat{\mathcal{A}}_{m,n} = a_n(r)\, \delta_{m,n}$

with eigenvalues: $a_n(r) = \frac{\gamma(n,r^2)}{\Gamma(n)}$

Consequently :

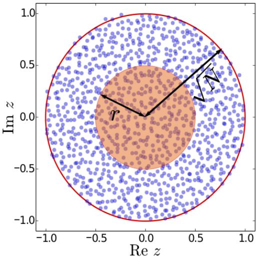
Number Variance:

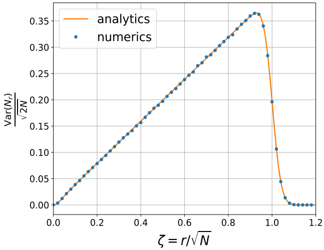
$$\mathrm{Var}(N_r) = V(N,r) = \mathrm{Tr}\left[\hat{\mathcal{A}}\,(I - \hat{\mathcal{A}})\right] = \sum_{n=1}^{N} a_n(1-a_n)$$

Rényi Entropy:

$$S_q(N,r) = \frac{1}{1-q}\,\mathrm{Tr}\,\ln\left[\hat{\mathcal{A}}^q + (I-\hat{\mathcal{A}})^q\right] = \frac{1}{1-q}\sum_{n=1}^{N} \ln\left[(a_n)^q + (1-a_n)^q\right]$$

Large N asymptotics: Number Variance





$$\mathrm{Var}(N_r) \simeq \mathcal{K}_2^b(r) \qquad r \sim O(1) \quad \text{(deep bulk)}$$

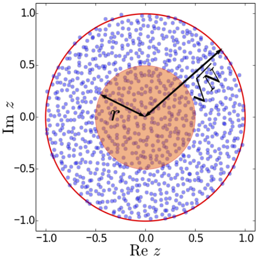
$$\simeq \frac{1}{\sqrt{\pi}}\, r \qquad 0 < \frac{r}{\sqrt{N}} < 1 \quad \text{(extended bulk)}$$

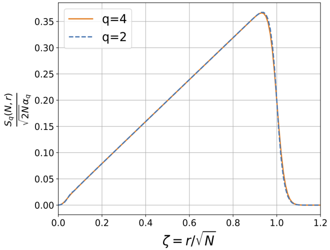
$$\simeq \sqrt{2N}\, \mathcal{K}_2^e\left(\sqrt{2}\,(r - \sqrt{N})\right) \qquad |\sqrt{N} - r| \sim O(1) \quad \text{(edge)}$$

where the bulk and edge scaling functions are:

$$\mathcal{K}_2^b(r) = \sum_{k=1}^{\infty} \frac{\Gamma(k, r^2)\gamma(k, r^2)}{\Gamma^2(k)} \quad \text{and} \quad \mathcal{K}_2^e(s) = \frac{1}{4} \int_s^{\infty} dx\, \mathrm{erfc}(x)\, \mathrm{erfc}(-x)$$

Large N asymptotics: Rényi Entropy





$$S_q(N,r) \simeq \mathcal{S}_q^b(r) \qquad r \sim O(1) \quad \text{(deep bulk)}$$

$$\simeq \frac{\alpha_q}{\sqrt{\pi}}\, r \qquad 0 < \frac{r}{\sqrt{N}} < 1 \quad \text{(extended bulk)}$$

$$\simeq \sqrt{2N}\,\mathcal{S}_q^e\left(\sqrt{2}\,(r-\sqrt{N})\right) \qquad |\sqrt{N}-r| \sim O(1) \quad \text{(edge)}$$

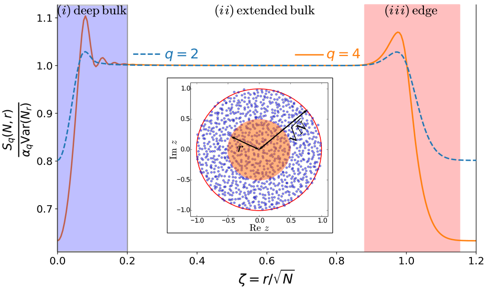
where the bulk and edge scaling functions are:

$$\mathcal{S}_q^b(r) = \frac{1}{1-q}\sum_{k=1}^{\infty} \ln\left[\left(\frac{\gamma(k,r^2)}{\Gamma(k)}\right)^q + \left(\frac{\Gamma(k,r^2)}{\Gamma(k)}\right)^q\right] \quad \text{and}$$

$$\mathcal{S}_2^e(s) = \frac{1}{1-q}\int_s^{\infty} dx\, \ln\left[\frac{1}{2^q}\mathrm{erfc}(x)^q + \frac{1}{2^q}\mathrm{erfc}(-x)^q\right]$$

the prefactor: $\alpha_q = \sqrt{2\pi}\,\mathcal{S}_2^e(-\infty)$

Number Variance and Rényi Entropy



Only in the extended bulk $0(1) << r << \sqrt{N}$

$$S_q(N,r) \simeq \frac{\alpha_q}{\sqrt{\pi}}\, r \quad \text{and} \quad \mathrm{Var}(N_r) \simeq \frac{1}{\sqrt{\pi}}\, r$$

$\Longrightarrow$ the ratio $\frac{S_q(N,r)}{\mathrm{Var}(N_r)} = \alpha_q$ is constant (ind. of r)

The proportionality breaks down in both (i) deep bulk $r \sim O(1)$
(ii) at the edge $|\sqrt{N} - r| \sim O(1)$

B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

Summary and Conclusion

- An exact mapping between the positions of free fermions in a 2-d rotating trap (in the frequency range $\omega\,(1-\frac{2}{N}) < \Omega < \omega$) in its ground state $\Longleftrightarrow$ eigenvalues of the Complex Ginibre Ensemble

  $\Longrightarrow$ an experimental realisation of GinUE

- Known results from GinUE

  $\Longrightarrow$ exact correlation functions of the Fermion positions (bulk and edge)

- Exact results for:
  - Number variance
  - Rényi entanglement entropy

- Proportionality between Rényi entropy and number variance holds only in the extended bulk, but fails in (i) deep bulk and (ii) at the edge

- The higher cumulants of N_r (Full counting statistics) $\Longrightarrow$ exact results

B. Lacroix-A-Chez-Toine, S.M. & G. Schehr, arXiv: 1809.05835

" An exactly solvable 2-d quantum system for Rényi entanglement entropy for arbitrary N "

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