Random Structures: from the Discrete to the Continuous

LMS Research School on Probability at the University of Bath

1-5 July 2019

Plenary talks

- Prof. Alison Etheridge
- Prof. Christina Goldschmidt
- Prof. Lorenzo Zambotti

Taught courses

- Scaling limits of random trees - Prof. Nicolas Broutin
- Spatial population genetics - Dr. Sarah Penington
- Stochastic PDE limits - Prof. Hendrik Weber

https://sites.google.com/view/lms-research-school-bath2019

UNIVERSITY OF BATH

LONDON MATHEMATICAL SOCIETY EST. 1865

SAMBa

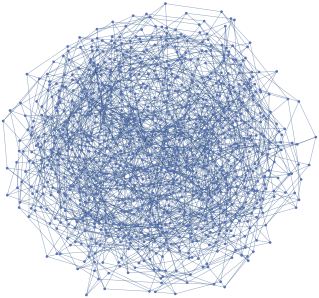
Sparse non-Hermititan random matrices

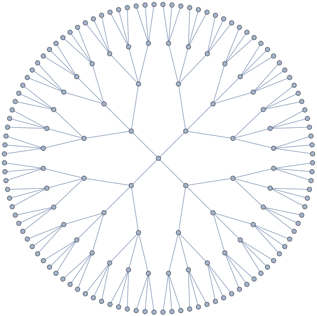
Tim Rogers

Izaak Neri, Fernando Metz,
Pau Aceituno, Henning Schomerus

Department of Mathematical Sciences
University of Bath
http://people.bath.ac.uk/ma3tcr

Random regular graphs



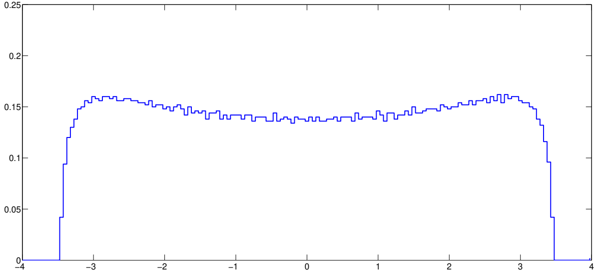


$$A_{ij} = \begin{cases} 1 & i \sim j \\ 0 & \text{else.} \end{cases}$$

$$\varrho(\lambda; A) = \frac{1}{N} \sum_i \delta\left(\lambda - \lambda_i^{(A)}\right).$$

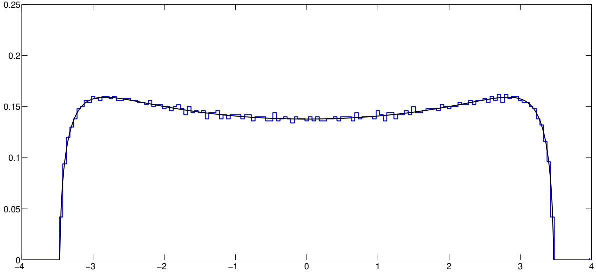
Random regular graphs

Histogram of eigenvalues, $d = 4$, $N = 10,000$.



Random regular graphs

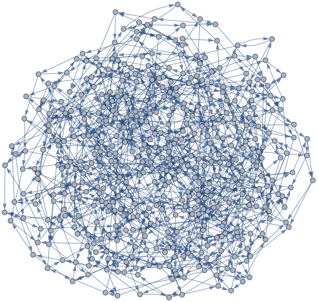
Histogram of eigenvalues, $d = 4$, $N = 10,000$.

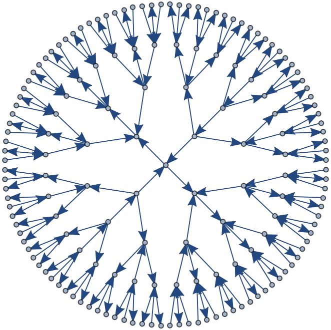


Kesten-McKay Law: In the large N limit,

$$\varrho(\lambda; A) \to \frac{d\sqrt{4(d-1) - \lambda^2}}{2\pi(d^2 - \lambda^2)}$$

Random regular digraphs



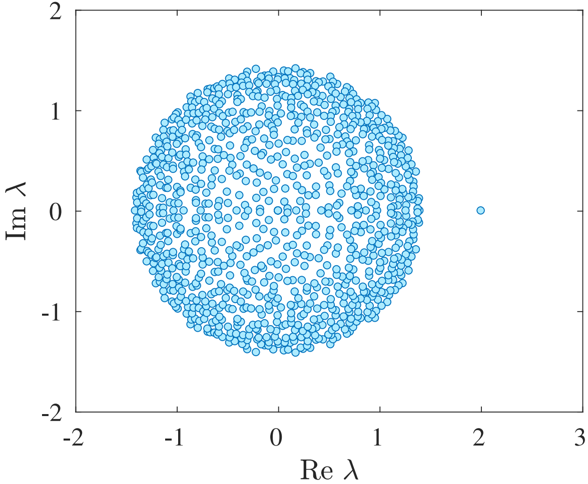


$$A_{ij} = \begin{cases} 1 & i \to j \\ 0 & \text{else.} \end{cases}$$

$$\varrho(\lambda; A) = \frac{1}{N} \sum_i \delta\left(z - \lambda_i^{(A)}\right) .$$

Random regular digraphs

Scatter plot of eigenvalues, $d^+ = d^- = 2$, $N = 1000$.



Spectral regularisation

Introduce the Green's function (resolvent)

$$G_A(z) = \mathrm{Tr}\,(A - zI)^{-1}\,, \qquad z \in \mathbb{C} \setminus \sigma(A)\,.$$

Can check that (in an integral sense)

$$\varrho(\lambda; A) = -\frac{1}{\pi N}\partial_{z^*}\mathrm{Tr}\,G_A(z)\bigg|_{z=\lambda}\,.$$

Problem: only defined outside the spectrum; we would rather work with something that is smooth on the whole of $\mathbb{C}$.

Spectral regularisation

Introduce the $2N \times 2N$ normal block matrix

$$B(z,\eta) = \begin{pmatrix} \eta I & -i(A - zI) \\ -i(A^\dagger - z^* I) & \eta I \end{pmatrix},$$

then

$$B^{-1}(z,\eta) = \begin{pmatrix} \eta X & iX(A - zI) \\ iY(A - zI)^\dagger & \eta Y \end{pmatrix},$$

where $X = (\eta^2 + (A - zI)(A - zI)^\dagger)^{-1}$ and
$Y = (\eta^2 + (A - zI)^\dagger(A - zI))^{-1}$ are the Schur complements.

Sometimes called "Hermitization". [Feinberg & Zee, Nuclear Physics B, 504, 3, 1997].

Spectral regularisation

This is useful because

$$\varrho(\lambda; A) = \lim_{\eta \to 0^+} \varrho^{(\eta)}(\lambda; A)$$

where

$$\rho^{(\eta)}(\lambda; A) = \frac{i}{N\pi} \partial_{z^*} \sum_{j=1}^{N} \left[B(z, \eta)^{-1} \right]_{j+N,j} \Bigg|_{z=\lambda},$$

which is real, positive, and smooth on the whole of $\mathbb{C}$.

Plan of attack: study $\rho^{(\eta)}(\lambda; A)$ for $N \to \infty$, then take $\eta \to 0$.

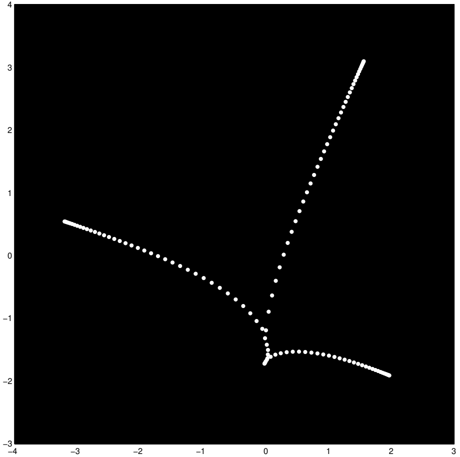
Spectral regularisation

The 'Bull's head' matrix:

$$X = \begin{pmatrix} 0 & 0 & 1 & 0.7 & 0 & \\ 2i & 0 & 0 & 1 & 0.7 & \ddots \\ 0 & 2i & 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

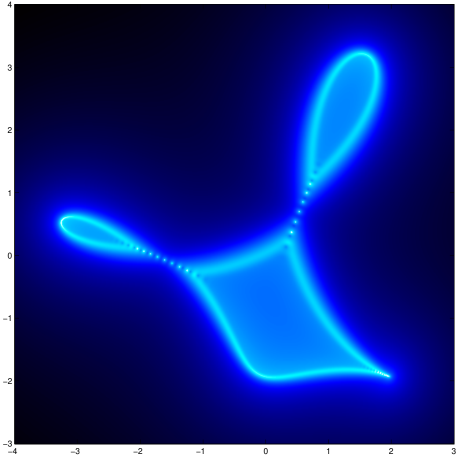
(from *Spectra and Pseudospectra*, Trefethen)

Spectral regularisation



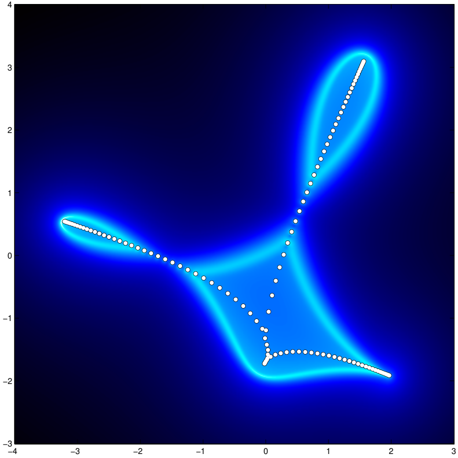
A scatter plot of eigenvalues of the 'Bull's head' of size $N = 120$

Spectral regularisation



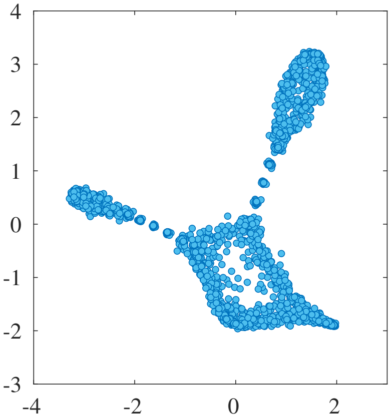
A colourmap of ϱ_η for the 'Bull's head' of size $N = 120$, with $\eta = 0.0001$.

Spectral regularisation



A colourmap of ϱ_η for the 'Bull's head' of size $N = 120$, with $\eta = 0.0001$.

Spectral regularisation



Eigenvalues of random perturbations $X + \eta S$

Spectral regularisation

Theorem: Let P and Q be $N \times N$ matrices of IID standard complex Gaussian random variables, and let $S = PQ^{-1}$. For any $N \times N$ matrix X,

$$\varrho^{(\eta)}(\lambda; A) = \mathbb{E}\,\varrho(\lambda; X + \eta S)\,.$$

Proof is by a matrix version of the Möbius transformation.

[TR, Journal of Mathematical Physics 51 (9), 093304]

Spectral regularisation

Theorem: Let P and Q be $N \times N$ matrices of IID standard complex Gaussian random variables, and let $S = PQ^{-1}$. For any $N \times N$ matrix X,

$$\varrho^{(\eta)}(\lambda; A) = \mathbb{E}\, \varrho(\lambda; X + \eta S)\,.$$

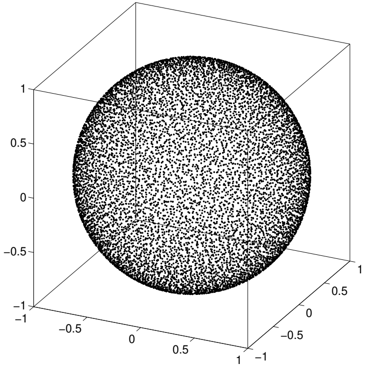
Proof is by a matrix version of the Möbius transformation.

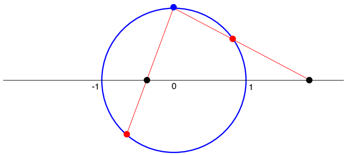
[TR, Journal of Mathematical Physics 51 (9), 093304]

Exchanging the limits $\eta \to 0$ and $N \to \infty$ in the calculation of the spectral density relies on the stability of the spectrum; generally need control over angle between eigenvectors. Very few cases have been proved - we will make this an assumption.

Spectral regularisation

Corollary: $S = PQ^{-1}$ has uniform density of the Reimann sphere





Result is universal in limit $N \to \infty$.
[C Bordenave - Electronic Communications in Probability, 2011]

Recursion relations

Recall we care about the diagonal of the lower left block of $B(z,\eta)^{-1}$, where

$$B(z,\eta) = \begin{pmatrix} \eta I & -i(A - zI) \\ -i(A^\dagger - z^* I) & \eta I \end{pmatrix}.$$

Re-package into 2×2 matrices

$$\mathsf{G}_{jk} = i \begin{pmatrix} [B^{-1}]_{j,k} & [B^{-1}]_{j,k+N} \\ [B^{-1}]_{j+N,k} & [B^{-1}]_{j+N,k+N} \end{pmatrix},$$

so

$$\rho(z) = \lim_{\eta \to 0^+} \lim_{n \to \infty} \frac{1}{N\pi} \sum_{j=1}^{n} \partial_{z^*} \left[\mathsf{G}_j\right]_{21}.$$

Recursion relations

If

$$\mathsf{A}_{jk} = \begin{pmatrix} 0 & A_{jk} \\ A^*_{kj} & 0 \end{pmatrix}, \quad \mathsf{z} = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix},$$

then

$$\mathsf{G}_{jj}(z) = \left(\mathsf{z} - i\eta\mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k,\ell\neq j} \mathsf{A}_{jk}\mathsf{G}^{(j)}_{k\ell}\mathsf{A}_{\ell j} \right)^{-1}.$$

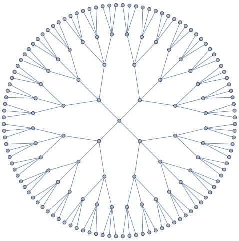
Recursion relations

If

$$\mathsf{A}_{jk} = \begin{pmatrix} 0 & A_{jk} \\ A_{kj}^* & 0 \end{pmatrix}, \quad \mathsf{z} = \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix},$$

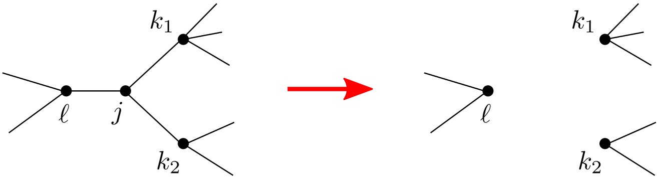
then

$$\mathsf{G}_{jj}(z) = \left(\mathsf{z} - i\eta \mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k,\ell \neq j} \mathsf{A}_{jk} \mathsf{G}_{k\ell}^{(j)} \mathsf{A}_{\ell j} \right)^{-1}.$$



Recursion relations

Exploit local tree structure...

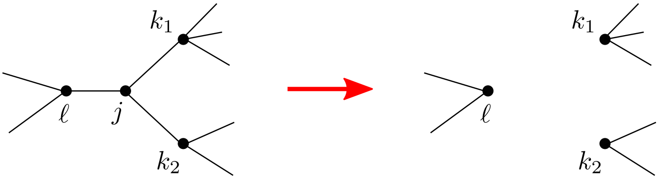


$$
\mathsf{G}_{jj}(z) = \left(\mathsf{z} - i\eta \mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k_1,k_2 \neq j} \mathsf{A}_{jk_1} \mathsf{G}^{(j)}_{k_1 k_2} \mathsf{A}_{k_2 j} \right)^{-1}
$$

$$
\approx \left(\mathsf{z} - i\eta \mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k \sim j} \mathsf{A}_{jk} \mathsf{G}^{(j)}_{kk} \mathsf{A}_{kj} \right)^{-1}
$$

Recursion relations

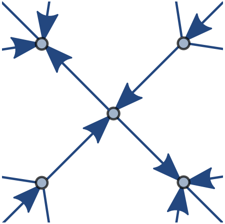
"Cavity method" allows for self-consistent solution



$$\mathsf{G}^{(\ell)}_{jj}(z) = \left(\mathsf{z} - i\eta\mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k_1,k_2 \neq j,\ell} \mathsf{A}_{jk_1}\mathsf{G}^{(j)(\ell)}_{k_1k_2}\mathsf{A}_{k_2j}\right)^{-1}$$

$$\approx \left(\mathsf{z} - i\eta\mathbf{1}_2 - \mathsf{A}_{jj} - \sum_{k\sim j, k\neq \ell} \mathsf{A}_{jk}\mathsf{G}^{(j)}_{kk}\mathsf{A}_{kj}\right)^{-1}$$

Recursion relations



$$\begin{aligned} \mathsf{G}_j &= \mathsf{g} \\ \mathsf{G}_k^{(j)} &= \mathsf{g}_+, \quad \text{if } j \to k, \\ \mathsf{G}_k^{(j)} &= \mathsf{g}_-, \quad \text{if } k \to j. \end{aligned}$$

$$\begin{aligned} \mathsf{g}^{-1} &= -i\eta \mathbf{1}_2 + \mathsf{z} - d\,\sigma_+ \mathsf{g}_+ \sigma_- - d\,\sigma_- \mathsf{g}_- \sigma_+, \\ \mathsf{g}_+^{-1} &= -i\eta \mathbf{1}_2 + \mathsf{z} - d\,\sigma_+ \mathsf{g}_+ \sigma_- - (d-1)\,\sigma_- \mathsf{g}_- \sigma_+, \\ \mathsf{g}_-^{-1} &= -i\eta \mathbf{1}_2 + \mathsf{z} - (d-1)\,\sigma_+ \mathsf{g}_+ \sigma_- - d\,\sigma_- \mathsf{g}_- \sigma_+, \end{aligned}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Recursion relations

Warm-up: $d >> 1$ then $\mathsf{g} \approx \mathsf{g}_+ \approx \mathsf{g}_-$. Send $\eta \to 0$, and

$$\mathsf{g}^{-1} = \mathsf{z} - d\,\sigma_+ \mathsf{g} \sigma_- - d\,\sigma_- \mathsf{g}_- \sigma_+,$$

...or written out in full

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{11}^* \end{pmatrix} = \frac{-d}{d^2|g_{11}|^2 - |z|^2} \begin{pmatrix} g_{11} & z/d \\ z^*/d & g_{11}^* \end{pmatrix}$$

Trivial solution $g_{11} = 0$; non-trivial solution only possible if $|z|^2 < d$, where we get $\rho(z) = 1/\pi d$.

Recursion relations

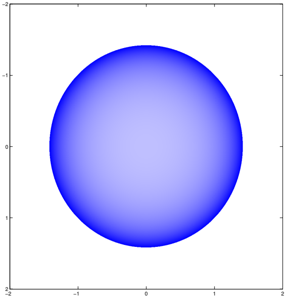
General solution for any d:

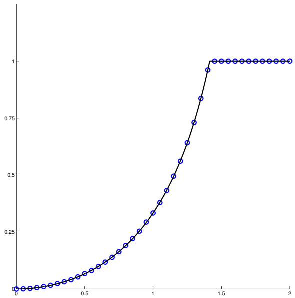
$$\mathrm{g} = \frac{(d-1)}{(d^2 - |z|^2)} \begin{pmatrix} \sqrt{\left(\frac{d}{d-1}\right)(d - |z|^2)} & z \\ z^* & -\sqrt{\left(\frac{d}{d-1}\right)(d - |z|^2)} \end{pmatrix}$$

Spectral density of random d-in d-out graphs:

$$\rho(z) = \frac{(d-1)}{\pi} \left(\frac{d}{d^2 - |z|^2} \right)^2, \quad |z| \leq \sqrt{d}\,,$$

Recursion relations





$$\int_{|z|<r} \rho(z)\mathrm{d}z = \min\left\{(d-1)\frac{r^2}{d^2-r^2}, 1\right\}.$$

Recursion relations

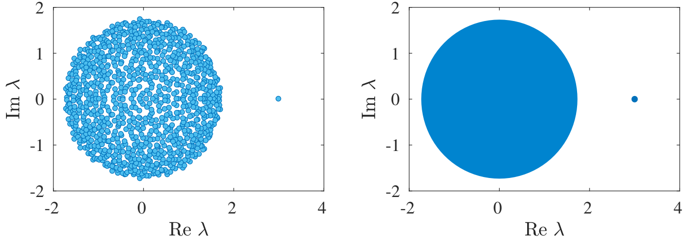
What problem did we actually solve?

- Approximated large random matrix A with infinite random operator $\mathcal{A}$.
- Domain: space of sequences indexed by elements of the free group $\mathcal{F}_2$ on generators $\{\alpha, \beta\}$, consists of all finite length strings of symbols $\{\alpha, \beta\}$, or their inverses $\{\alpha^{-1}, \beta^{-1}\}$, after cancellation of adjacent reciprocal pairs.
- Action: $\mathcal{A}|x\rangle_w = |x\rangle_{w\alpha} + |x\rangle_{w\beta}$

We computed the “Brown's measure” of $\mathcal{A}$.

Recursion relations

Another example: $d^+ = d^- = 3$ and the forward operator on $\mathcal{F}_3$



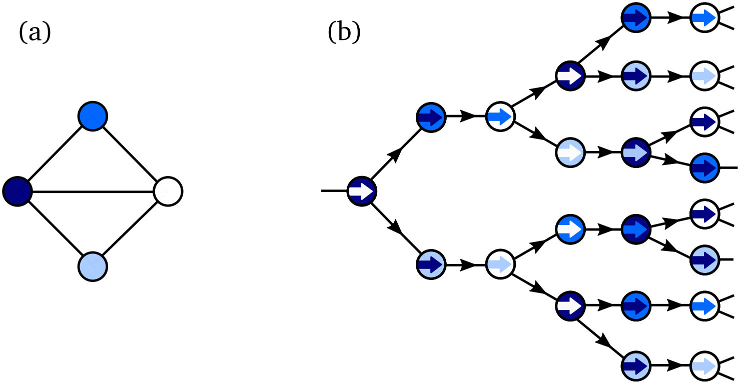
Note outlier eigenvalue at $\sqrt{3}$

...need slightly different theory needed to pick this up.

Recursion relations

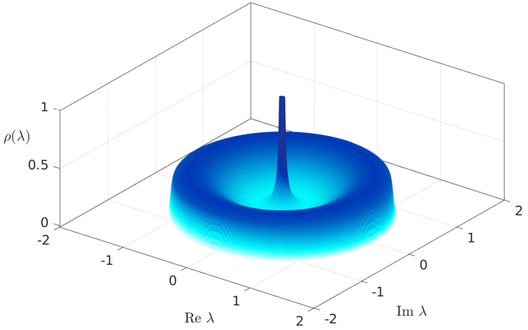
General case: quasi-transitive tree of non-backtracking walks

(a) (b)



Recursion relations

Alternative approach: population dynamics in ensemble average for random digraphs



$$\mathsf{G}' \stackrel{D}{=} \left(\mathsf{z} - i\eta \mathbf{1}_2 - \mathsf{D} - \sum_{\ell=1}^{K'} \mathsf{A}_{i\ell} \mathsf{G}'_\ell \mathsf{A}^\dagger_{j\ell} \right)^{-1} .$$

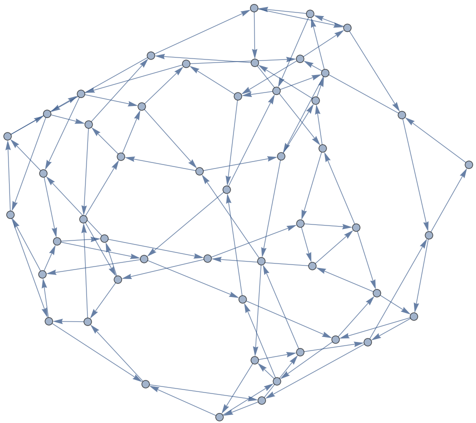
Spectra of Sparse non-Hermitian Random Matrices

Review with Izaak Neri and Fernano Metz [arXiv:1811.10416], including:

- Hertmitization method
- Cavity method for recursion relations
- Theory for spectral outliers
- Eigenvector correlators

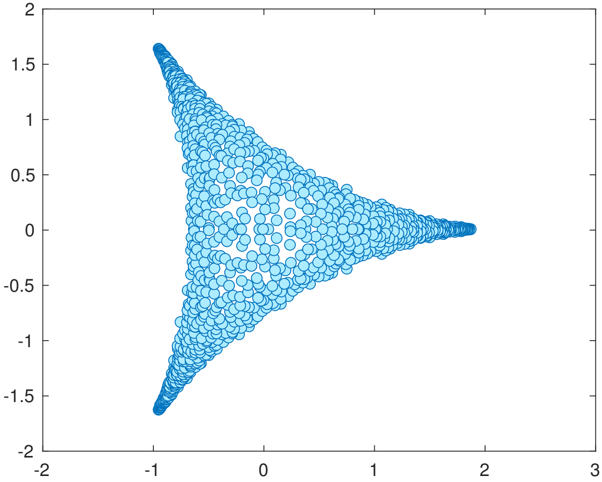
Cyclic motifs

What about digraphs that aren't tree-like?



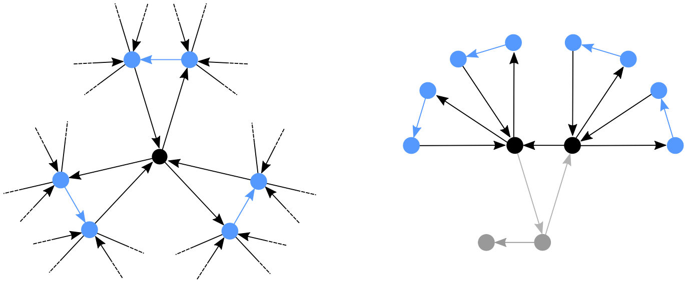
Example: $d^+ = d^- = 2$, composed of 3-cycles

Cyclic motifs



Cyclic motifs

Can still use cavity method, on the graph between cycles:



Group nodes into pairs p, denote by $A^{\star}$ the block matrix describing edges between pairs. Need to solve 4×4 matrix equation

$$G_{pp}^{-1} = A_{pp}^{\star} - \begin{pmatrix} \mathsf{z} & 0 \\ 0 & \mathsf{z} \end{pmatrix} - (d-1) A_{pq}^{\star -} G_{pp} A_{qp}^{\star -} - (d-1) A_{pq}^{\star +} G_{pp} A_{qp}^{\star +}$$

Cyclic motifs

Boundary of spectrum is determined by existence of non-trivial solutions. After some algebra we can parameterise the curve by an angle $\varphi \in [0, 2\pi)$, giving (for general $k \geq 3$)

$$z_b(\varphi) = \frac{1}{t} e^{-i\varphi} + (d-1) t^{k-1} e^{i(k-1)\varphi},$$

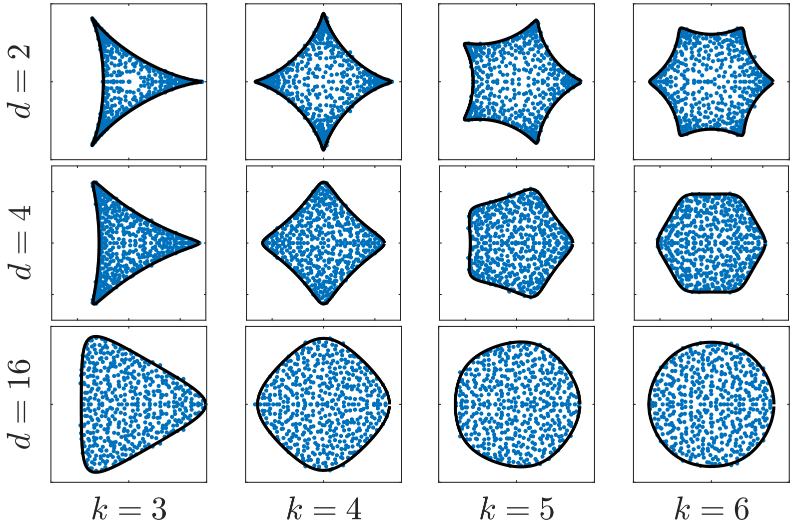
where t is the minimal positive real solution of

$$(d-1)t^{2k} - dt^2 + 1 = 0.$$

Cyclic motifs

Hypotrochoids:

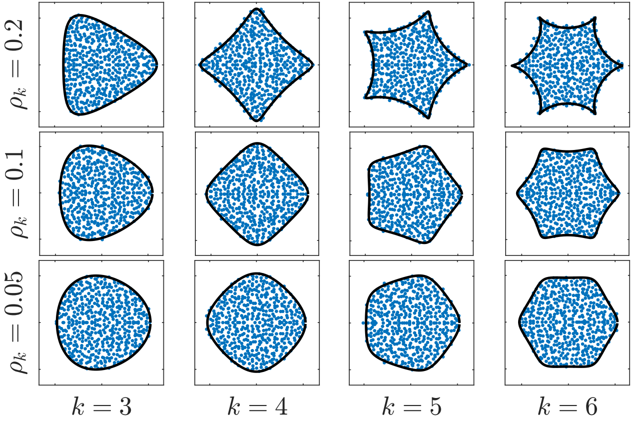
Cyclic motifs



Cyclic motifs

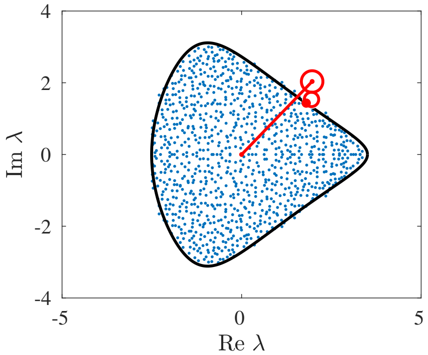
Dense limit: random matrices with correlations $\mathbb{E}\mathrm{Tr}A^k/N = \rho_k$ have spectral boundary

$$z_b(\varphi) = e^{-i\varphi} + \rho_k e^{i(k-1)\varphi}$$



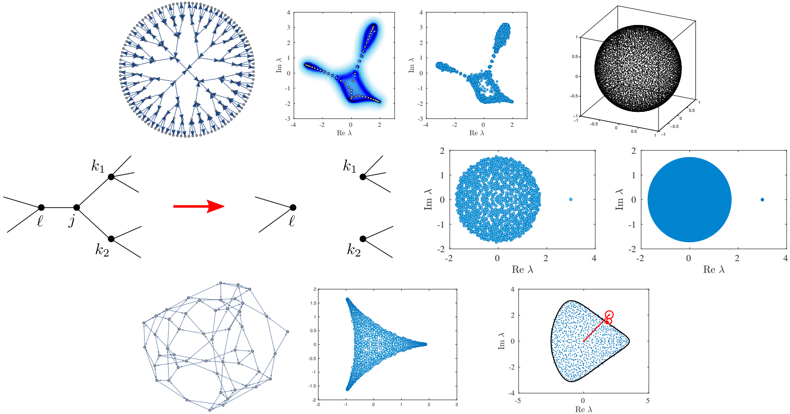
Cyclic motifs

Competing cycles lead to **polytrochoid** spectra:



$$e^{-i\varphi} + d_1 \left(\frac{1}{\sqrt{d_1 + d_2}} \right)^{k_1} e^{i(k_1 - 1)\varphi} + d_2 \left(\frac{1}{\sqrt{d_1 + d_2}} \right)^{k_2} e^{i(k_2 - 1)\varphi}$$

Summary



Preprints available at **http://people.bath.ac.uk/ma3tcr**