University of BRISTOL

Eigenvector distribution for certain random matrix models in the intermediate regime

joint with

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XVI Brunel-Bielefeld Workshop on Random Matrix Theory
14-15 December 2018, Brunel University London

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Content

- Introduction
- Rosenzweig-Porter model
- Power law random-banded matrices and ultrametric matrices

All the matrix ensembles in this talk are real Hermitian

1) Eugene Bogomolny & M.S., PRE **98**, 032139 (2018)

2) Eugene Bogomolny & M.S., PRE **98**, 042116 (2018)

Motivations for unusual random matrix ensembles

Random regular graphs with on-site energy disorder:

- Anderson Transition between localised and ergodic extended states on random regular trees (Abou-Chacra et al,1973)
- No consensus about extended states on random regular graphs (RRG)
  - Only one ergodic extended phase (Mirlin, Tikhonov, 2018; Biroli,Tarzia, 2018)
  - There is a second transition between ergodic and non-ergodic states with non-trivial fractal dimensions (Kravtsov et al, 2018)

Porter-Thomas distribution for eigenstates:

- The distribution of eigenvectors is universal for all standard invariant ensembles distribution. For real matrices: ($x = \sqrt{N}\Psi_j$)

$$P(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}, \qquad P(y = x^2) = \frac{\mathrm{e}^{-y/2}}{\sqrt{2\pi y}}, \qquad \langle \Psi^2 \rangle = 1$$

- Recent experimental neutron resonance data are in contradiction with this distribution (Koehler et al, 2011 and 2013)

Rosenzweig-Porter model

- Each element is i.i.d. Gaussian variable (up to symmetry)

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ii}^2 \rangle = 1, \quad \langle H_{ij}^2 \rangle_{i \neq j} = \frac{\epsilon^2}{N^\gamma}, \quad 1 \leq i, j \leq N$$

- Rule of thumb for the different regimes

$$S_1(N) = \frac{1}{N} \sum_{i,j=1}^{N} \langle |H_{ij}| \rangle, \qquad S_2(N) = \frac{1}{N} \sum_{i,j=1}^{N} \langle |H_{ij}|^2 \rangle.$$

  - If $\lim_{N \to \infty} S_1(N) < \infty \implies$ eigenvectors are localised and the spectral statistics is Poissonian
  - If $\lim_{N \to \infty} S_2(N) = \infty \implies$ eigenvectors are fully delocalised and the spectral statistics is GOE

- $\gamma > 2 \implies$ localisation
- $\gamma < 1 \implies$ standard GOE

Intermediate region: $1 < \gamma < 2$ (Kravtsov et al, 2015)

Moments of eigenvectors ($q > 1/2$)

$$I_q = \langle \sum_j |\Psi_j|^{2q} \rangle \underset{N \to \infty}{\longrightarrow} C_q \, N^{-(q-1)D_q}$$

where D_q is the fractal dimension

- Localised regime ($\gamma > 2$): $D_q = 0$
- Ergodic regime ($\gamma < 1$): $D_q = 1$
- Intermediate regime ($1 < \gamma < 2$): $D_q = 2 - \gamma$

Recent rigorous proofs (von Soosten & Warzel, 2017)

In this talk: distribution of eigenvectors when $1 < \gamma < 2$ based on

- Breit-Wigner distribution of the variances $\langle |\Psi_j(E)|^2 \rangle$
- Local Gaussian distribution for $\Psi_j(E)$

Follows from rigorous results (Benigni, 2017)

Breit-Wigner distribution of eigenvector variances

$$\Sigma_j^2(E) \equiv \langle |\Psi_j(E)|^2 \rangle \approx \frac{C^2\, \Gamma(E)}{\pi \rho(E) N \left[(E - e_j)^2 + \Gamma^2(E) \right]}$$

Average is over off-diagonal elements, diagonal elements $e_j = H_{jj}$ are fixed.

- The spreading width $\Gamma(E)$ is given by the Fermi golden rule

$$\Gamma(E) = \frac{\pi \epsilon^2}{N^{\gamma-1}} \rho_f(E)$$

- The normalised level density $\rho(E)$ is given by

$$\rho(E) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{E^2}{2} \right)$$

  the density of diagonal elements for $N \to \infty$ and $\gamma > 1$.

- C depends on the normalisation. For the standard normalisation

$$\sum_j |\Psi_j(E_\alpha)|^2 = 1 \quad \text{or} \quad \sum_\alpha |\Psi_j(E_\alpha)|^2 = 1 \longrightarrow C = 1$$

Derivation of the Breit-Wigner formula

Recursive relation for the Green function $G = (E - \mathrm{i}\eta - H)^{-1}$

$$G_{ii}(E - \mathrm{i}\eta) = \Big(E - \mathrm{i}\eta - H_{ii} - \sum_{j,k\neq i} H_{ij} G^{(i)}_{jk}(E - \mathrm{i}\eta) H_{ki}\Big)^{-1}$$

where $G(E)^{(i)}$ is the Green function after removing the row and column i

(Schur complement formula, also Feshbach's projection method)

For large N

$$\sum_{j,k\neq i} H_{ij} G^{(i)}_{jk} H_{ki} \approx \frac{\epsilon^2}{N^\gamma} \sum_{j\neq i} G^{(i)}_{jj} \underset{N\to\infty}{\longrightarrow} \frac{\epsilon^2}{N^\gamma} \int \frac{N\rho(e)\mathrm{d}e}{E - \mathrm{i}\eta - e}$$

The variance $\langle |\Psi_i(E)|^2 \rangle$ follows from

$$\mathrm{Im}\; G_{ii}(E - \mathrm{i}\eta) \underset{\eta\to 0}{\longrightarrow} \pi \langle |\Psi_i(E)|^2 \rangle \rho(E)$$

Full eigenvector distribution

The second ingredient is a local Gaussian distribution of $\Psi_j(E)$ (for fixed e_j)

$$P(\Psi_j(E)) = \frac{1}{\sqrt{2\pi\Sigma_j^2(E)}} \exp\left(-\frac{|\Psi_j(E)|^2}{2\Sigma_j^2(E)}\right)$$

Integrating over the diagonal element e_j gives $[x = \Psi_j(E)]$

$$P(x)_E = \int \frac{\rho(E)}{\sqrt{2\pi\Sigma_j^2(E)}} \exp\left(-\frac{x^2}{2\Sigma_j^2(E)}\right) \mathrm{d}e_j$$

Result for the distribution in a small window around $E = 0$

$$P(x)_{E=0} = \frac{\delta^2}{4\pi\sqrt{a}}\left[K_0(\zeta) + K_1(\zeta)\right]\mathrm{e}^{-\zeta + \frac{\delta^2}{2}}$$

where

$$a = \frac{C^2\epsilon^2}{N^\gamma}, \quad \delta = \Gamma(0) = \frac{\sqrt{\pi}\,\epsilon^2}{\sqrt{2}\,N^{\gamma-1}}, \quad \zeta = \frac{\delta^2}{4a}(x^2 + a).$$

Distribution in the bulk and in the tail

In the bulk, x has values of the order of $\sqrt{a}$. It is convenient to scale

$$y = N^{\gamma/2}\,\Psi_j(E), \quad \langle y^2 \rangle = N^{\gamma-1}, \quad |y| \leq N^{\gamma/2}$$

This corresponds to $C = N^{\gamma/2}$ and $a = \epsilon^2$. As $N \to \infty$

$$P_{\mathrm{bulk}}(y) \approx \frac{\epsilon}{\pi(y^2 + \epsilon^2)}$$

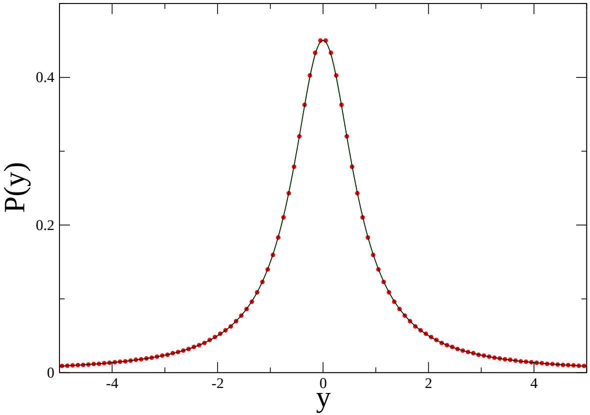
In the tail (small δ, finite ζ) it is convenient to rescale

$$z = N^{1-\gamma/2}\,\Psi_j(E), \quad \langle z^2 \rangle = N^{1-\gamma}, \quad |z| \leq N^{1-\gamma/2}$$

This corresponds to $C = N^{1-\gamma/2}$ and $a = \epsilon^2 N^{2-\gamma}$. Then

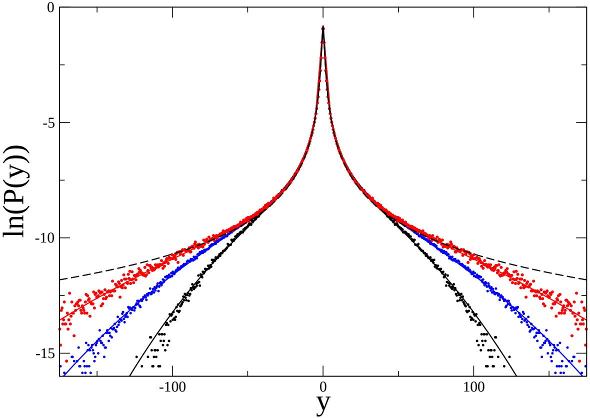
$$P_{\mathrm{tail}}(z) = \frac{2\sqrt{2}\,b^3}{\pi\sqrt{\pi}\,N^{\gamma-1}}(K_0(b^2 z^2) + K_1(b^2 z^2))\mathrm{e}^{-b^2 z^2}, \qquad b = \frac{\sqrt{\pi}\epsilon}{2\sqrt{2}}$$

Distribution of eigenvector components in the bulk



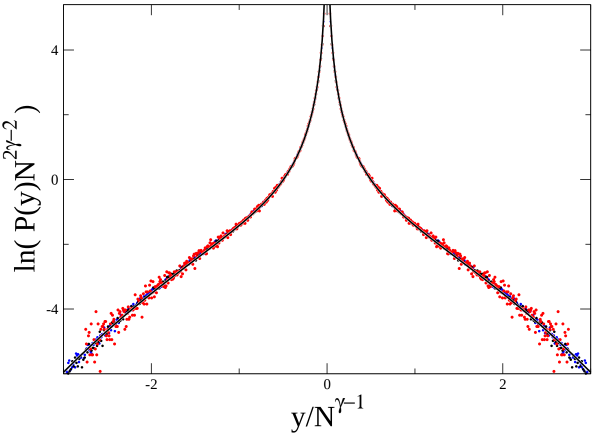
Distribution of $y = N^{\gamma/2}\,\Psi_j(E)$ for the RP model with $\gamma = 1.5$ and $\epsilon = \frac{1}{\sqrt{2}}$ for $N = 4096, 2048, 1024$.

Distribution in logarithmic scale



Distribution of $y = N^{\gamma/2}\,\Psi_j(E)$ for the RP model with $\gamma = 1.5$ and $\epsilon = \frac{1}{\sqrt{2}}$ for $N = 4096$ (red), $N = 2048$ (blue) and $N = 1024$ (black).

Rescaled distribution of eigenvector components in the tail



Distribution of $z = N^{1-\gamma/2}\,\Psi_j(E)$ for the RP model with $\gamma = 1.5$ and $\epsilon = \frac{1}{\sqrt{2}}$ for $N = 4096$ (red), $N = 2048$ (blue) and $N = 1024$ (black).

Moments of the eigenvectors

Results for the centre of the spectrum

$$I_q \equiv \langle \sum_{j=1}^{N} |\Psi_j(E)|^{2q} \rangle = \frac{2^{q-1/2} a^q \Gamma(q+1/2)}{\sqrt{\pi}\delta^{2q-1}} \Psi\Big(\frac{1}{2}, \frac{3}{2} - q; \frac{\delta^2}{2}\Big)$$

where $\Psi(\alpha, \beta; z) =$ is the Tricomi confluent hypergeometric function

In the limit $\delta \to 0$

$$I_{q>\frac{1}{2}} = N^{-(q-1)(2-\gamma)} C_{q>\frac{1}{2}}, \qquad C_{q>\frac{1}{2}} = \frac{\Gamma(q-1/2)\Gamma(q+1/2)}{\pi\, b^{2q-2}\, 2^{q-2}\, \Gamma(q)}$$

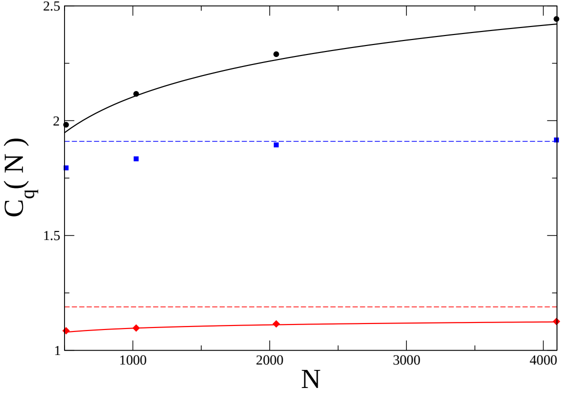
$$I_{q=\frac{1}{2}} = N^{1-\gamma/2} C_{\frac{1}{2}}, \qquad C_{\frac{1}{2}} = \frac{\epsilon}{\pi}\Big[2(\gamma-1)\ln N - \ln\Big(\frac{\pi\epsilon^4}{16}\Big) - \gamma\Big]$$

$$I_{q<\frac{1}{2}} = N^{-\gamma q+1} C_{q<\frac{1}{2}}, \qquad C_{q<\frac{1}{2}} = \frac{\epsilon^{2q}}{\pi}\Gamma(q+1/2)\Gamma(1/2-q)c_{\mathrm{cor}}(q)$$

Corrective factor for $q < 1/2$

$$c_{\mathrm{cor}}(q) = 1 + \frac{\pi^{1-q}\, \epsilon^{2-4q}\, \Gamma(q-1/2)}{2^{1-2q}\, \Gamma(q)\, \Gamma(1/2-q)} N^{-(\gamma-1)(1-2q)}$$

Eigenvector moments for $N = 512, 1024, 2048, 4096$



Eigenvector moments for $q = \frac{1}{8}$ (red), $q = 2$ (blue) and $q = \frac{1}{2}$ (black).

Here $C_{\frac{1}{8}} = 1.19$ with $c_{\text{cor}} = (1 - .44/N^{1/4})$, and $C_2 = 1.19$.

Conclusion for the Rosenzweig-Porter model

- The statistical distribution for eigenvectors of the Rosenzweig-Porter model have been obtained in the regime $1 < \gamma < 2$.
- The derivation is based on two physical assumptions (which are exact for the considered model).
- The first states that the mean square modulus of eigenvectors is given by a Breit-Wigner formula with a spreading width in agreement with the Fermi golden rule.
- The second states that the eigenvectors have a local Gaussian distribution with variance given by the above formula.
- This approach leads to transparent explicit formulas that agree extremely well with numerical calculations.

Power-law random banded matrices and ultrametric matrices

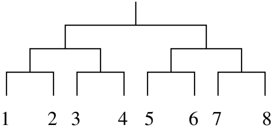
Each matrix element is i.i.d. Gaussian variable (up to symmetry)

$$\langle H_{ij} \rangle = 0, \quad \langle H_{ii}^2 \rangle = 2, \qquad \langle H_{ij}^2 \rangle_{i \neq j} = a^2(i,j)$$

Power-law random banded matrices (Mirlin et al, 1996)
$a(r)$ with $r = |i-j|$ decreases as a power of the distance $a(r) \underset{r\to\infty}{\longrightarrow} \epsilon\, r^{-s}$
A translation-invariant choice to avoid boundary effects is

$$a(r) = \epsilon \left[1 + \left(\frac{N}{\pi} \sin(\frac{\pi r}{N}) \right)^2 \right]^{-s/2} .$$

Ultrametric random matrices (Fyodorov et al, 2009)
$2^n \times 2^n$ matrices with $a(i,j) = \epsilon\, 2^{-s\,\mathrm{dist}(i,j)}$



$\mathrm{dist}(i,j)$ is the ultrametric distance on a binary tree.
For example, $\mathrm{dist}(1,2) = 1$, $\mathrm{dist}(1,3) = 2$ and $\mathrm{dist}(1,5) = 3$.

Intermediate region $\frac{1}{2} < s < 1$

The rule of thumb for the two moments $S_1(N)$ and $S_2(N)$ predicts for both ensembles

- $s > 1 \implies$ eigenvectors are localised and the spectral statistics is Poissonian
- $s < \frac{1}{2} \implies$ eigenvectors are fully delocalised and the spectral statistics is GOE

Intermediate region

$$\frac{1}{2} < s < 1$$

Due to the absence of a small or large parameter standard analytical approaches to random matrices are not applicable.

$\implies$ Numerical investigation of the two ensembles

Main numerical results

- No indication of non-trivial fractal dimensions when $\frac{1}{2} < s < 1$. Distribution of $x = \sqrt{N}\Psi_j$ becomes quickly independent of N
- Eigenvector distribution is extremely well approximated by the generalised hyperbolic distribution (GHD)(symmetric case)

$$P_{\mathrm{GHD}}(x) = \frac{\sqrt{\alpha}}{\sqrt{2\pi}\delta^{\lambda} K_{\lambda}(\alpha\delta)} (x^2 + \delta^2)^{(\lambda-1/2)/2} K_{\lambda-1/2}\left(\alpha\sqrt{x^2+\delta^2}\right)$$

GHD is a variance mixture of the normal distribution with variance distributed according to the generalised inverse Gaussian distribution (GIG) (normal variance-mean mixture)

$$P_{\mathrm{GHD}}(x) = \int_0^{\infty} P_{\mathrm{GIG}}(y) \frac{\mathrm{e}^{-x^2/2y}}{\sqrt{2\pi y}} \, \mathrm{d}y$$

where

$$P_{\mathrm{GIG}}(x) = \frac{\alpha^{\lambda}}{2\delta^{\lambda} K_{\lambda}(\alpha\delta)} x^{\lambda-1} \mathrm{e}^{-\frac{1}{2}(\alpha^2 x + \delta^2 x^{-1})}$$

Parameters and moments

GHD and GIG depend on three parameters α, δ and λ.

The moments are known analytically

$$C_q \equiv \langle x^{2q} \rangle_{\mathrm{GHD}} = C_{\mathrm{GOE}}(q) \langle x^q \rangle_{\mathrm{GIG}}$$

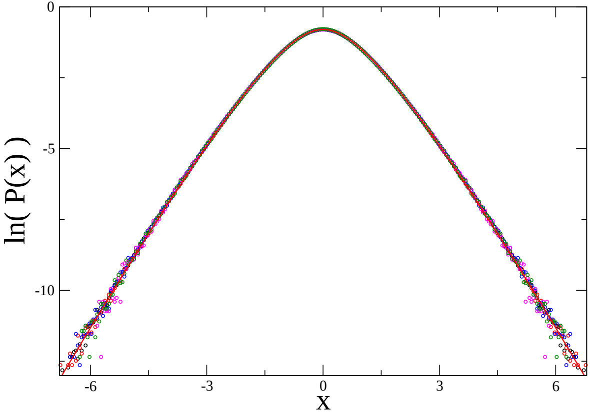
$$C_{\mathrm{GOE}}(q) = \frac{2^q \Gamma(q + \frac{1}{2})}{\sqrt{\pi}}, \qquad \langle x^q \rangle_{\mathrm{GIG}} = \left(\frac{\delta}{\alpha}\right)^q \frac{K_{\lambda+q}(\alpha\delta)}{K_\lambda(\alpha\delta)}$$

The variance of the GHD is fixed to one by the normalisation. We set

$$\alpha = \sqrt{\frac{\xi K_{\lambda+1}(\xi)}{K_\lambda(\xi)}}, \quad \delta = \frac{\xi}{\alpha}, \quad \xi = \alpha\delta$$

With this choice the distributions depend on two parameters: λ and ξ.

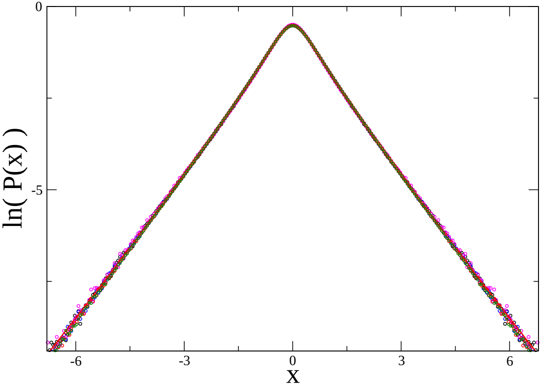
Eigenvector distribution for PLBM with $s = 0.7$ and $\epsilon = 1$



Distribution of $x = \sqrt{N}\Psi_j$ for $N = 8192$ (black), $N = 4096$ (red), $N = 2048$ (blue), $N = 1024$ (green) and $N = 512$ (magenta).

Compared to GHD with $\alpha = 2.6154$, $\lambda = 3.3615$, $\delta = 0.2903$ (red line)

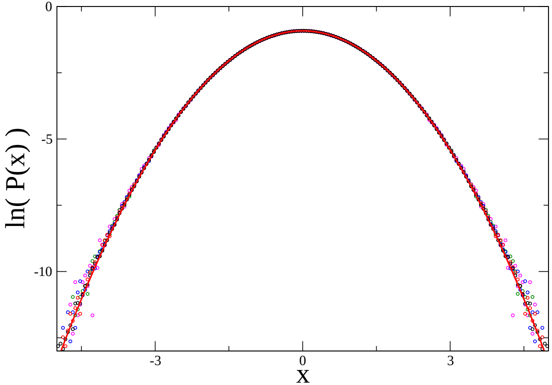
Eigenvector distribution for UMM with $s = 0.7$ and $\epsilon = 1$



Distribution of $x = \sqrt{N}\Psi_j$ for $N = 8192$ (black), $N = 4096$ (red), $N = 2048$ (blue), $N = 1024$ (green) and $N = 512$ (magenta).

Compared to GHD with $\alpha = 1.1673$, $\lambda = 0.3880$, $\delta = 0.4409$ (red line)

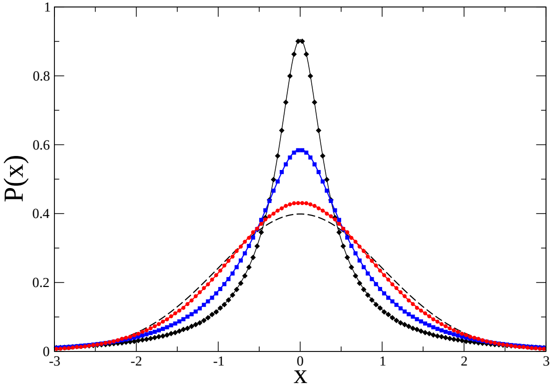
Eigenvector distribution for PLBM with $s = 0.3$ and $\epsilon = 1$ (GOE)



Distribution of $x = \sqrt{N}\Psi_j$ for $N = 8192$ (black), $N = 4096$ (red), $N = 2048$ (blue), $N = 1024$ (green) and $N = 512$ (magenta).

Compared to Gaussian with zero mean and unit variance (red line)

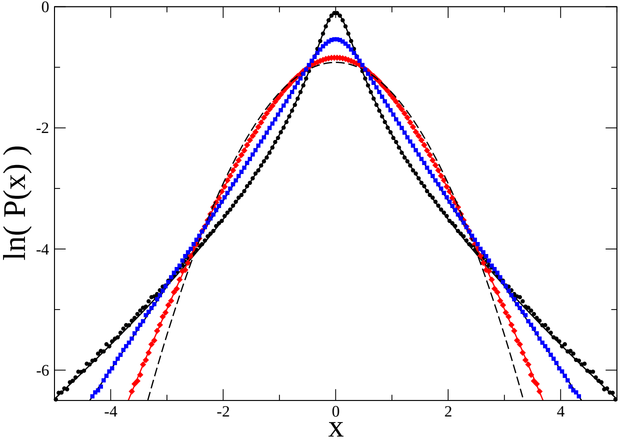
PLBM with $s = 0.7$ and different ϵ (N=2048)



$\epsilon = 0.3$ (black circles), $\epsilon = 0.5$ (blue squares) and $\epsilon = 1.5$ (red diamond)

GHD for $\epsilon = 0.3$: $\alpha = 0.6506$, $\lambda = -0.1067$, $\delta = 0.2805$
GHD for $\epsilon = 0.5$: $\alpha = 1.2754$, $\lambda = 0.5862$, $\delta = 0.3945$
GHD for $\epsilon = 1.5$: $\alpha = 2.9341$, $\lambda = 3.6392$, $\delta = 1.0377$

PLBM with $s = 0.7$ and different ϵ in logarithmic scale ($N = 2048$)



$\epsilon = 0.3$ (black circles), $\epsilon = 0.5$ (blue squares) and $\epsilon = 1.5$ (red diamond)

Local eigenvector variance

- Choose interval $I = [E - \delta E/2, E + \delta E/2]$ with M_I consecutive levels
- Calculate local variance

$$x = \frac{1}{M_I} \sum_{E_\alpha \in I} N|\Psi_j(E_\alpha)|^2$$

- Calculate the distribution $P(x)$ of x for the ensemble
- If $\Psi_j(E_\alpha)$ are independent (GOE) then $P(x)$ is χ^2-distribution with $\nu = M_I$

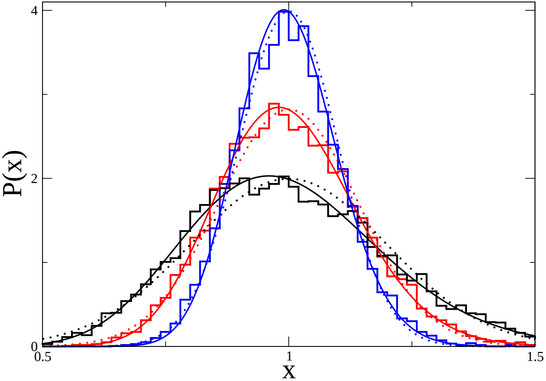
$$P_{\chi^2}(x,\nu) = \frac{\nu^\nu \, x^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)} \mathrm{e}^{-\nu\, x/2}, \qquad \langle x \rangle_{\chi^2} = 1 \ .$$

- Asymptotic formula for $M_I \to \infty$ (central limit theorem)

$$P(x)_{\mathrm{GOE}} \underset{M_I \to \infty}{\longrightarrow} \sqrt{\frac{M_I}{4\pi}} \mathrm{e}^{-M_I (x-1)^2/4}$$

- Deviation from this distribution $\longrightarrow \Psi_j(E_\alpha)$ are not independent

Local eigenvector variance for PLBM with $s = 0.3$ (GOE)

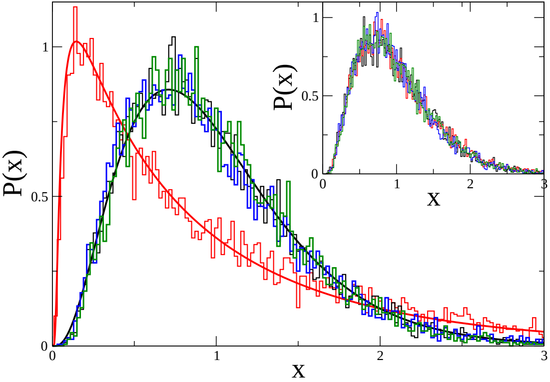


Results for $\epsilon = 1$, $N = 4096$, different M_l:

Staircase lines: $M_l = 200$ (blue), $M_l = 100$ (red), $M_l = 50$ (black)

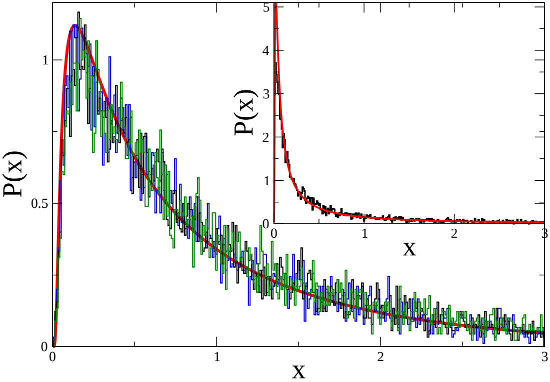
Solid lines: χ^2-distribution with $\nu = M_l$, and their asymptotic form (dotted)

Local eigenvector variance for PLBM with $s = 0.7$



Red staircase: $\epsilon = 0.5$, $N = 4096$, $M_l = 100$
Other staircases: $\epsilon = 1$, $N = 4096$, different M_l
(blue: $M_l = 50$, black: $M_l = 100$, green: $M_l = 200$)
compared to GIG distribution with previous parameters (solid lines)
Insert: $\epsilon = 1$, $M_l = 100$ with $N = 1024, 2048, 4096, 8192$

Local eigenvector variance for UMM with $s = 0.7$



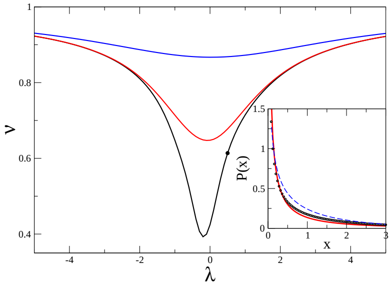
Staircases: $\epsilon = 1$, $N = 4096$, different M_l
(blue: $M_l = 50$, black: $M_l = 100$, green: $M_l = 200$)
compared to GIG distribution with previous parameters (solid red line)
Insert: $\epsilon = 0.5$, $M_l = 100$, $N = 4096$

Comparison with experimental results

- Experimental results for neutron widths were fitted with a χ^2-distribution

$$P_{\chi^2}(x,\nu) = \frac{\nu^{\nu/2}\, x^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)} \mathrm{e}^{-\nu\, x/2}, \qquad \langle x \rangle_{\chi^2} = 1\ .$$

- $^{192}\mathrm{Pt}$: $\nu = 0.57 \pm 0.16$, $^{194}\mathrm{Pt}$: $\nu = 0.47 \pm 0.19$, $^{196}\mathrm{Pt}$: $\nu = 0.60 \pm 0.28$
- The normalised GHD depends on 2 parameters λ and ξ
- We fixed $\xi = 0.02,\ 0.2,\ 2$ (black, red, blue) and fitted ν for different λ



Conclusions for PLBM and UMM

- Power-law banded and ultrametric matrices are representatives of random matrix ensembles with varying strength of interaction
- We numerically investigated the intermediate region $\frac{1}{2} < s < 1$ between the fully delocalised regime ($s < 1/2$) and the localised regime ($s > 1$)
- No non-trivial fractal dimensions were observed. After rescaling by $\sqrt{N}$ the eigenvector distributions become N-independent
- Main result: the eigenvector distributions can be extremely accurately fitted by the **generalised hyperbolic distribution**
- The investigation of the PLBM and UMM in the intermediate regime is of importance as they constitute a new class of random matrices potentially important for different applications
- One possible application is the explanation of deviations of recent experimental data of neutron widths from the Porter-Thomas distribution