Universality of the third-order pushed-to-pulled transition

Pierpaolo Vivo

joint work with

Fabio D Cunden, Paolo Facchi and Marilena Ligabo'

- Largest Eigenvalue of Gaussian ensembles
- Our result for symmetric walls
- Electrostatic interpretation and generalizations
- Sketch of the proof (symmetric case)
- Interesting questions

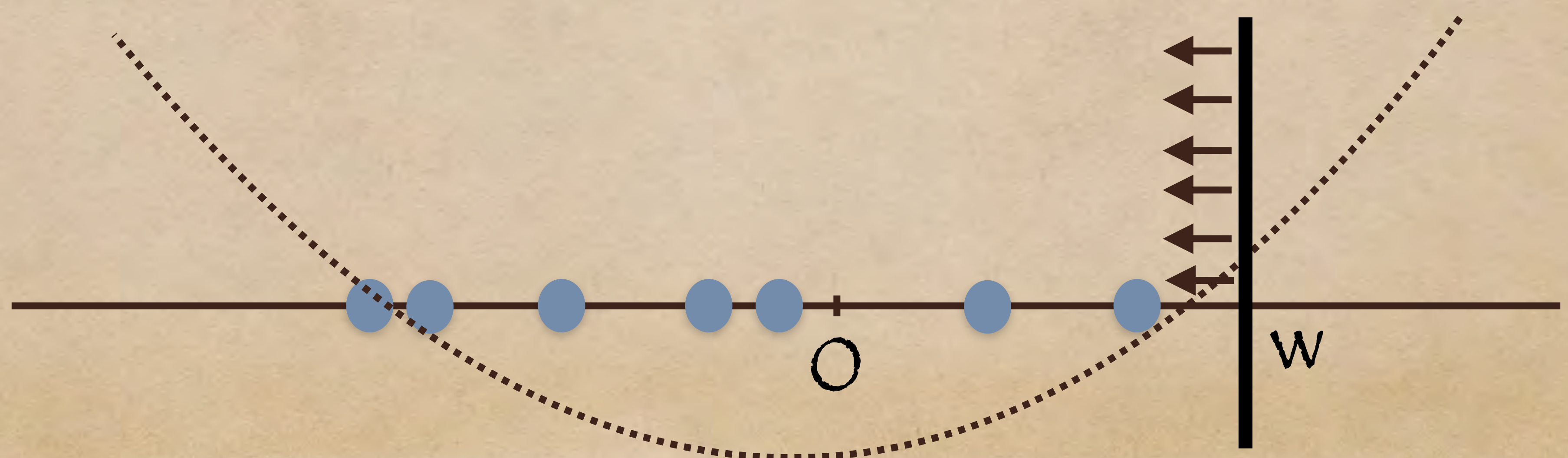
Largest eigenvalue of Gaussian ensembles

$$F_N(w) = \mathrm{Prob}[x_{\max} \leq w]$$

$$F_N(w) = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

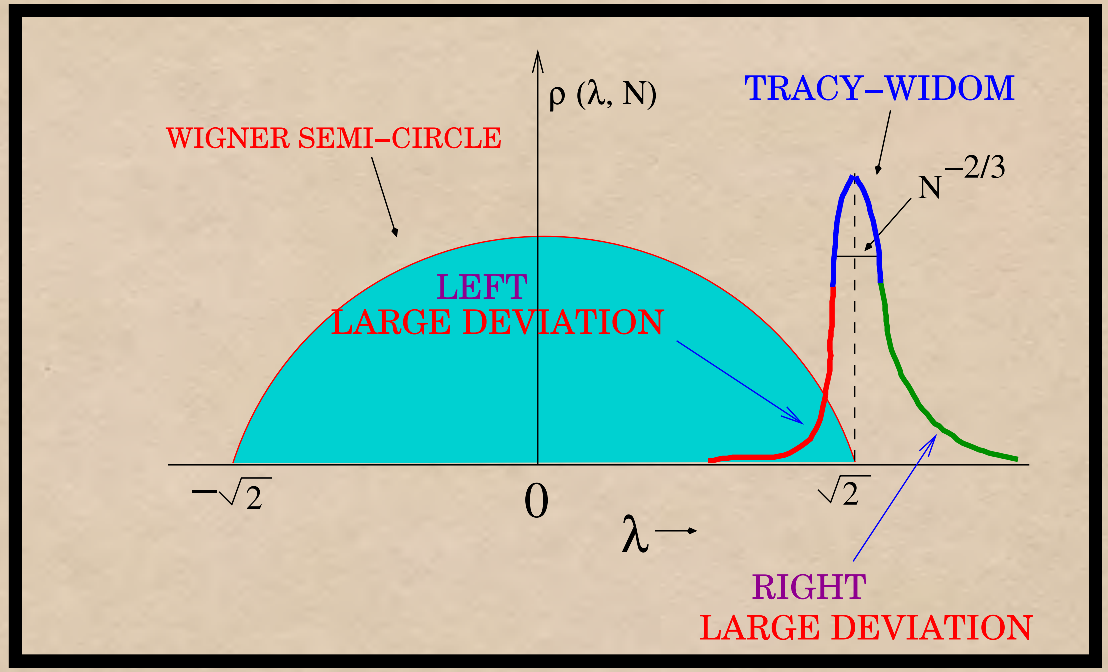
$$Z_N(w) = \int_{-\infty}^{w} \mathrm{d}x_1 \cdots \int_{-\infty}^{w} \mathrm{d}x_N \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N} x_i^2 - \sum_{i \neq j} \ln|x_i - x_j|\right)\right]$$

Canonical partition function



Largest eigenvalue of Gaussian ensembles

taken from
Majumdar & Schehr
JSTAT 2014



[Tracy & Widom, 1994-1996]

[Dean & Majumdar, 2006-2008]

$$F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right], & w < \sqrt{2} \text{ and } |w-\sqrt{2}| \sim O(1) \\ \mathscr{F}_\beta\left(\sqrt{2}N^{2/3}(w-\sqrt{2})\right), & |w-\sqrt{2}| \sim O(N^{-2/3}) \\ 1-\exp[-\beta N \Phi_+(w)], & w > \sqrt{2} \text{ and } |w-\sqrt{2}| \sim O(1) \end{cases}$$

[Majumdar & Vergassola, 2009]

Largest eigenvalue of Gaussian ensembles

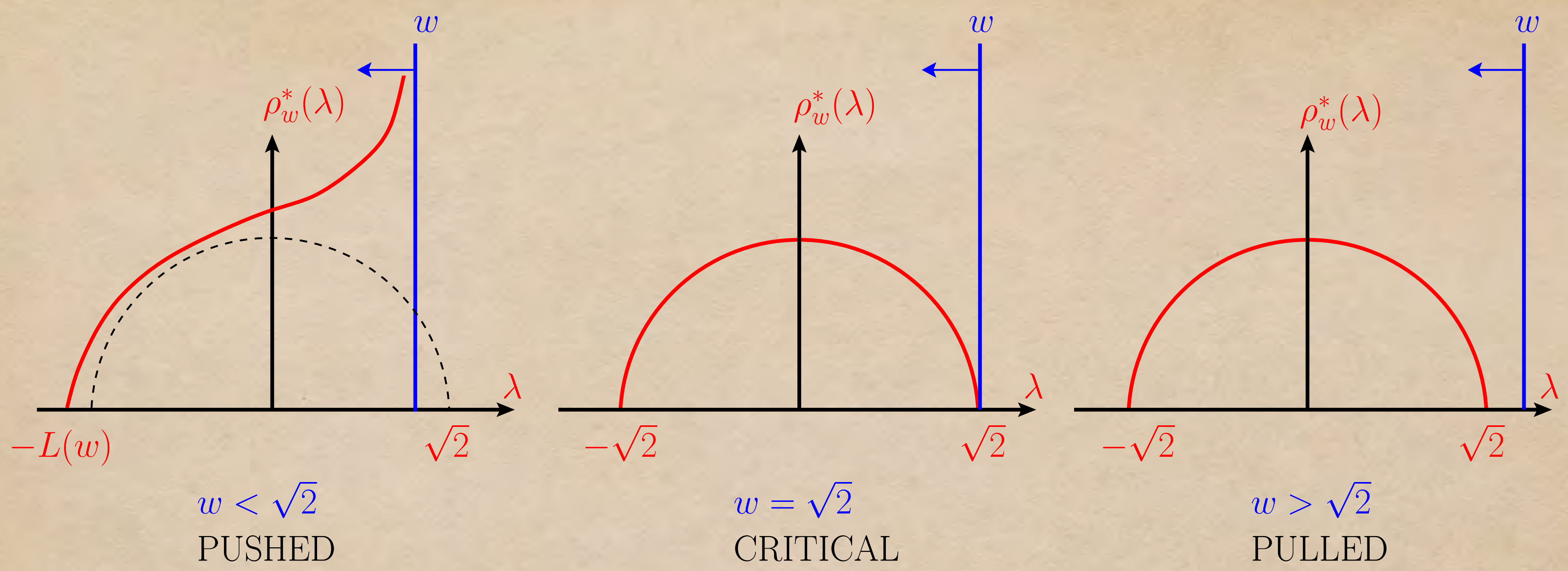
$$\lim_{N\to\infty} -\frac{1}{N^2}\ln F_N(w) = \begin{cases} \Phi_-(w), & w < \sqrt{2} \\ 0 & w > \sqrt{2} \end{cases}$$

Free energy of a Coulomb gas with a hard wall at 'w'

$$F_N(w) = \text{Prob}[x_{\max} \le w]$$

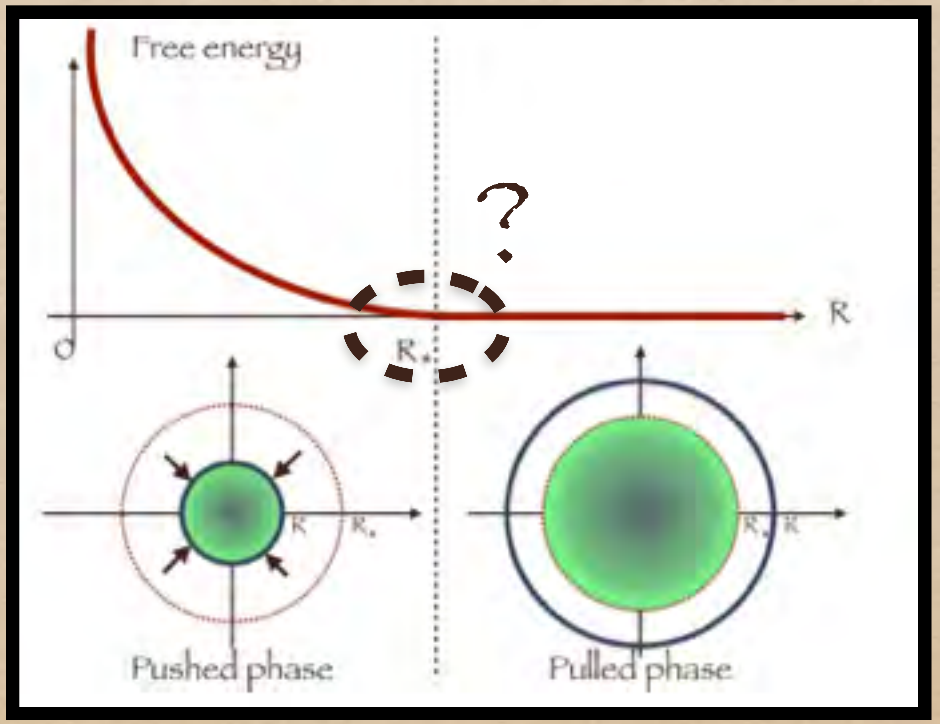
$$F_N(w) = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

$$Z_N(w) = \int_{-\infty}^{w} \mathrm{d}x_1 \cdots \int_{-\infty}^{w} \mathrm{d}x_N \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N} x_i^2 - \sum_{i\neq j} \ln|x_i - x_j|\right)\right]$$



$$\Phi_-(w) \sim \frac{1}{6\sqrt{2}}(\sqrt{2}-w)^3, \qquad w \to \sqrt{2}$$

<u>3rd order phase transition</u>



Ubiquity of the 3rd order "pushed-to-pulled" transition

- U(N) lattice QCD [Gross & Witten, Wadia 1980]
- Maximum height of non-intersecting Brownian walkers [Forrester, Majumdar, Schehr & Comtet 2011]
- Extreme eigenvalues for Wishart [PV, Majumdar & Bohigas 2007; Deift, Its and Krasovsky 2006] and Jacobi ensembles [Ramli, Katzav & Castillo 2012]
- Conductance and other observables in mesoscopic cavities [PV, Majumdar & Bohigas 2008-10; Grabsch & Texier 2015-16]
- Entanglement entropy for bipartite random pure states [Nadal, Majumdar & Vergassola 2010]
- Multiple Input Multiple Output (MIMO) channels [Karadimitrakis et al 2013]
- Minima of random landscapes [Fyodorov & Nadal 2012]
- Random Tilings [Colomo & Pronko 2013]
- Height in 1d KPZ growth models at late times [Le Doussal, Majumdar & Schehr 2016]
- Rightmost particle of 1-d Coulomb gas [Dhar, Kundu, Majumdar, Sabhapandit & Schehr 2017]
- Truncated linear statistics [Grabsch-Majumdar-Texier 2018]

•••

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Top eigenvalue of a random matrix: large deviations and third order phase transition

Satya N Majumdar and Grégory Schehr

Université Paris-Sud, LPTMS, CNRS (UMR 8626), F-91405 Orsay Cedex, France

$$\rho(x) \sim \sqrt{R_\star - x}$$ → 3rd order

"off-critical" case

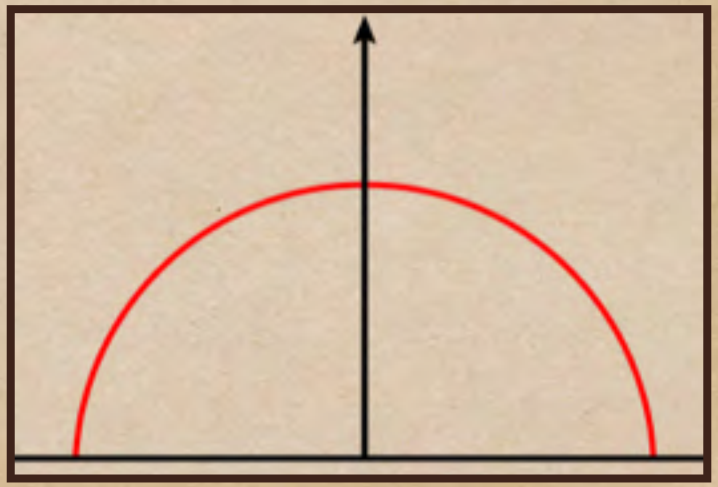
Our setting

$$p_N(x_1,\dots,x_N) = \frac{1}{Z_N} e^{-\beta E_N(x_1,\dots,x_N)}$$

$$E_N(x_1,\dots,x_N) = -\frac{1}{2}\sum_{i\neq j} \log|x_i - x_j| + N\sum_k V(x_k), \quad x_i \in \mathbb{R}.$$

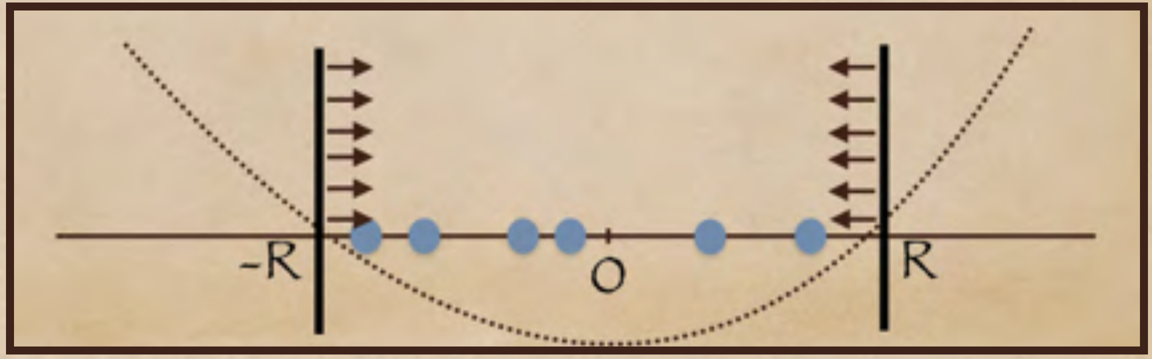
Assumption 1 *$V(x)$ is $C^3(\mathbb{R})$, symmetric $V(x) = V(-x)$, strictly convex and satisfies* $\liminf_{|x|\to\infty} \frac{V(x)}{\log|x|} > 1$.

“one-cut” and “off-critical”



Symmetric walls

$$\text{Prob}(\max |x_i| < R)$$?



Related works

- $$\lim_{N\to\infty} \Pr\left(\max |x_i| \leq R_\star + \frac{t}{\sqrt{2}N^{2/3}}\right) = \mathscr{F}_2^2(t)$$ for GUE

[Dean - Le Doussal - Majumdar - Schehr 2017]

- d-dimensional Coulomb (jellium) model

[Cunden -Facchi - Ligabo' - PV 2017]

$$\text{Prob}(\max|x_i| < R) \approx \exp\left(-\beta N^2 F(R)\right)$$

Equilibrium density of the fluid constrained between two walls

In general

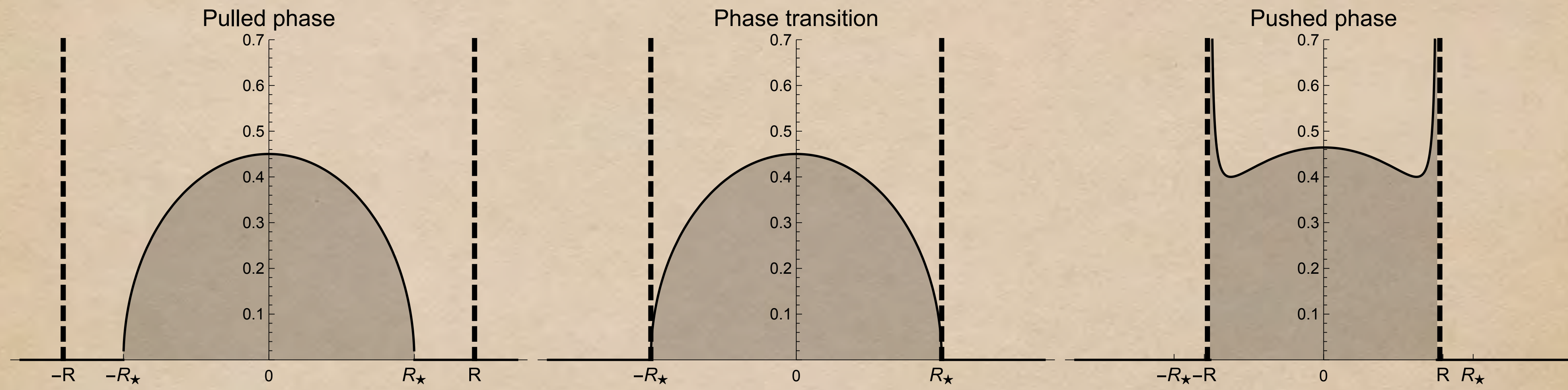
$$F(R) = \mathscr{E}[\rho_R] - \mathscr{E}[\rho_{R_\star}]$$

$$\mathscr{E}[\rho] = -\frac{1}{2}\iint \log|x-y|\mathrm{d}\rho(x)\mathrm{d}\rho(y) + \int V(x)\mathrm{d}\rho(x)$$

['continuum' version of the 'granular' energy

$$E_N(x_1,\ldots,x_N) = -\frac{1}{2}\sum_{i\neq j}\log|x_i - x_j| + N\sum_k V(x_k)$$]

Density of particles constrained between symmetric walls



$$
\mathrm{d}\rho_R(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{P_R(x)}{\sqrt{R^2-x^2}}\mathbb{1}_{|x|<R}\,\mathrm{d}x & \textit{if } R < R_\star \textit{ (pushed phase)} \\ \dfrac{1}{\pi}Q(x)\sqrt{R_\star^2-x^2}\,\mathbb{1}_{|x|\le R_\star}\,\mathrm{d}x & \textit{if } R \ge R_\star \textit{ (pulled phase)}\,, \end{cases}
$$

$$
P_R(x) = 1 - \fint_{-R}^{R} \frac{1}{\pi}\frac{\sqrt{R^2-t^2}\,V'(t)}{x-t}\,\mathrm{d}t
$$

[Tricomi 1957]

Our result

$$\text{Prob}(\max|x_i| < R) \approx \exp\left(-\beta N^2 F(R)\right)$$

$$F(R) = \frac{1}{2}\int_{\min(R,R_\star)}^{R_\star} \mathrm{d}r \frac{P_r(r)^2}{r}$$

$$V(x) \to \rho_R(x) \to P_R(x) \to F(R)$$

Our result

$$F(R) = \frac{1}{2}\int_{\min(R,R_\star)}^{R_\star} \mathrm{d}r\,\frac{P_r(r)^2}{r}$$

$$\lim_{R\uparrow R_\star} F(R) = \frac{1}{2}\int_{R_\star}^{R_\star} \frac{P_r(r)^2}{r}\mathrm{d}r = 0\,,$$

$$\lim_{R\uparrow R_\star} F'(R) = -\frac{1}{2}\frac{P_{R_\star}(R_\star)^2}{R_\star} = 0\,,$$

$$\lim_{R\uparrow R_\star} F''(R) = -\frac{2P_{R_\star}(R_\star)P'_{R_\star}(R_\star)R_\star - P_{R_\star}(R_\star)^2}{2R_\star^2} = 0\,.$$

but $\lim_{R\uparrow R_\star} F'''(R) = -\frac{P'_{R_\star}(R_\star)^2}{R_\star} < 0\,.$ off-criticality

→ fully in line with M-S criterion!

Check: Gaussian ensembles

$$\rho_R(x) = \begin{cases} \dfrac{1}{\pi} \dfrac{2+R^2-2x^2}{2\sqrt{R^2-x^2}} \mathbb{1}_{|x|<R} & \text{if } R < R_\star \text{ (pushed phase)} \\ \dfrac{1}{\pi}\sqrt{2-x^2}\,\mathbb{1}_{|x|\le R_\star} & \text{if } R \ge R_\star \text{ (pulled phase)}\ , \end{cases}$$

$$F_{G\beta E}(R) = \begin{cases} \frac{1}{32}(8R^2 - R^4 - 16\log R - 12 + 8\log 2) & \text{if } R < R_\star = \sqrt{2} \\ 0 & \text{if } R \ge R_\star = \sqrt{2} \end{cases}$$

[Dean & Majumdar, PRE 2008]

$P_R(x)$

$$F_{G\beta E}(R) = \frac{1}{2}\int_R^{\sqrt{2}} \frac{(2-r^2)^2}{4r}\,\mathrm{d}r = \frac{1}{32}(8R^2 - R^4 - 16\log R - 12 + 8\log 2) \qquad R < \sqrt{2}$$

✓

Electrostatic interpretation and generalizations

Electrostatic interpretation

The increase in free energy must match the work done in compressing the fluid

$$F(R) = -W_{R_\star \to R} \quad \text{with} \quad W_{R_\star \to R} = \int_{\mathrm{vol}_i}^{\mathrm{vol}_f} P\,\mathrm{d}V = 2\int_{R_\star}^{R} p(r)\mathrm{d}r$$

Electrostatic pressure: normal force per unit length

surface charge x electric field generated by all other charges

$$p(r) = \frac{\pi^2}{2}\left|\mathrm{Res}_{z=r}\rho_r^2(z)\right|$$

Not limited to spherical symmetry!

The electrostatic interpretation allows us to recover more general results

Example 1: largest eigenvalue of Gaussian ensembles

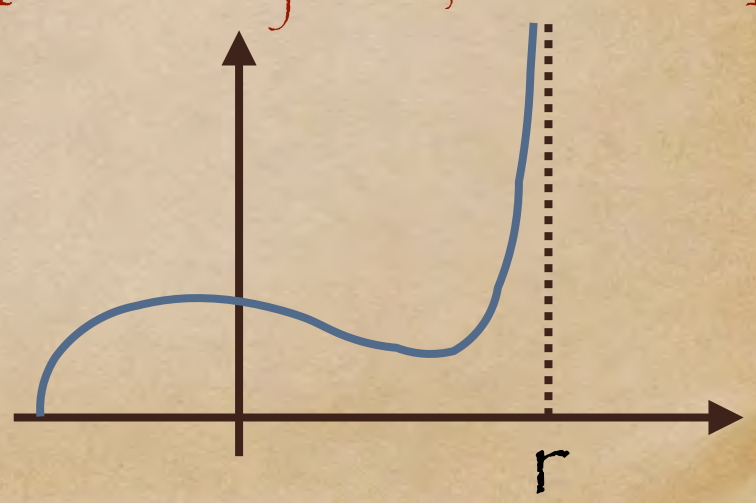
$$F_N(w) = \mathrm{Prob}[x_{\max} \leq w] \approx \mathrm{e}^{-\beta N^2 \Phi_-(w)}$$

$$\Phi_-(w) \stackrel{?}{=} -\int_{\sqrt{2}}^{w} p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}\left|\mathrm{Res}_{z=r}\rho_r^2(z)\right|$$

$$\rho_r(z) = \frac{1}{2\pi}\sqrt{\frac{z-L(r)}{r-z}}(r-L(r)-2z) \qquad L(r) = -\frac{2\sqrt{r^2+6}-r}{3}$$

[Dean & Majumdar, 2006-2008]



The electrostatic interpretation allows us to recover more general results

Example 1: largest eigenvalue of Gaussian ensembles

$$F_N(w) = \mathrm{Prob}[x_{\max} \leq w] \approx \mathrm{e}^{-\beta N^2 \Phi_-(w)}$$

$$\Phi_-(w) \stackrel{?}{=} -\int_{\sqrt{2}}^{w} p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}|\mathrm{Res}_{z=r}\rho_r^2(z)| = \frac{1}{27}(r^3 + (6+r^2)\sqrt{r^2+6} - 18r)$$

$$\rho_r(z) = \frac{1}{2\pi}\sqrt{\frac{z-L(r)}{r-z}}(r-L(r)-2z) \qquad L(r) = -\frac{2\sqrt{r^2+6}-r}{3}$$

The electrostatic interpretation allows us to recover more general results

Examp

$$\Phi_-(w) = \frac{1}{108}\left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln 18 - 2\ln\left(w + \sqrt{w^2+6}\right)\right)\right],$$
$$w < \sqrt{2}.$$

F

$$\Phi_-(w) \stackrel{?}{=} -\int_{\sqrt{2}}^{w} p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}\left|\mathrm{Res}_{z=r}\rho_r^2(z)\right| = \frac{1}{27}\left(r^3 + (6+r^2)\sqrt{r^2+6} - 18r\right)$$

$$\rho_r(z) = \frac{1}{2\pi}\sqrt{\frac{z-L(r)}{r-z}}(r - L(r) - 2z) \qquad L(r) = -\frac{2\sqrt{r^2+6}-r}{3}$$

The electrostatic interpretation allows us to recover more general results

Example 2: largest eigenvalue of Wishart ensembles (c=1)

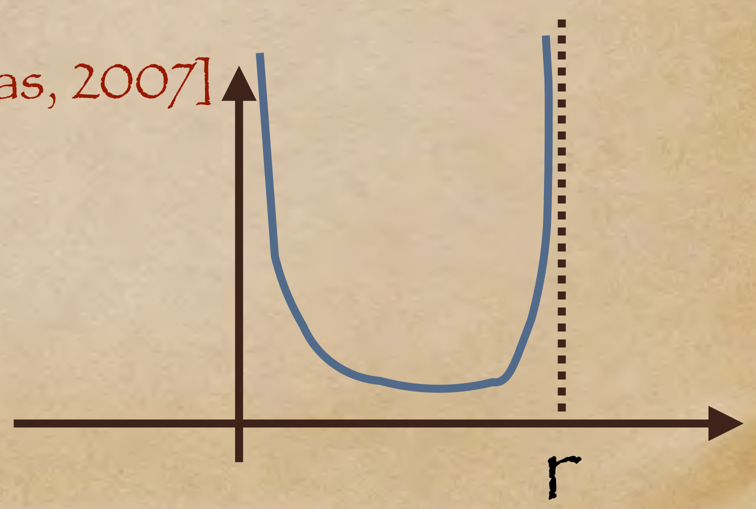
$$F_N(w) = \mathrm{Prob}[x_{\max}^{(\mathrm{W})} \leq w] \approx \mathrm{e}^{-\beta N^2 \Phi_-^{(\mathrm{W})}(w)}$$

$$\Phi_-^{(\mathrm{W})}(w) \stackrel{?}{=} -\int_4^w p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}|\mathrm{Res}_{z=r}\rho_r^2(z)|$$

$$\rho_r(z) = \frac{1}{2\pi}\frac{r/2+2-z}{\sqrt{z(r-z)}}$$

[PV, Majumdar & Bohigas, 2007]



The electrostatic interpretation allows us to recover more general results

Example 2: largest eigenvalue of Wishart ensembles (c=1)

$$F_N(w) = \mathrm{Prob}[x_{\max}^{(\mathrm{W})} \leq w] \approx \mathrm{e}^{-\beta N^2 \Phi_-^{(\mathrm{W})}(w)}$$

$$\Phi_-^{(\mathrm{W})}(w) \stackrel{?}{=} -\int_4^w p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}|\mathrm{Res}_{z=r}\rho_r^2(z)| = \frac{r^2 - 8r + 16}{32r}$$

$$\rho_r(z) = \frac{1}{2\pi}\frac{r/2 + 2 - z}{\sqrt{z(r-z)}}$$

The electrostatic interpretation allows us to recover more general results

Example 2: larg

$$\Phi_-(x;1) = \begin{cases} 2\log 2 - \log(4-x) - \frac{x}{4} - \frac{x^2}{32} & x \geqslant 0 \\ 0 & x \leqslant 0 \end{cases}$$

$$F_N(w) = \mathrm{P}$$

$$\Phi_-^{(\mathrm{W})}(w) \stackrel{?}{=} -\int_4^w p(r)\mathrm{d}r$$

$$p(r) = \frac{\pi^2}{2}\left|\mathrm{Res}_{z=r}\rho_r^2(z)\right| = \frac{r^2-8r+16}{32r}$$

$$\rho_r(z) = \frac{1}{2\pi}\frac{r/2+2-z}{\sqrt{z(r-z)}}$$

Sketch of the proof (symmetric case)

$$F(R) = \frac{1}{2}\int_{\min(R,R_\star)}^{R_\star} \mathrm{d}r \frac{P_r(r)^2}{r}$$

A remarkable formula

$$-\log|x-y| = \log 2 + \sum_{n\geq 1} \frac{2}{n} T_n(x) T_n(y) \qquad x, y \in [-1,1],\ x \neq y$$

Chebyshev Polynomials

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} \mathrm{d}x = \delta_{nm} h_n \quad \text{with} \quad h_n = \begin{cases} \pi & \text{if } n=m=0 \\ \pi/2 & \text{if } n=m\geq 1 \end{cases}$$

used already by Fyodorov, Khoruzhenko, Simm [Ann Prob 2016]
and Garoufalidis and Popescu [Ann Henri Poincare' 2013] -
unpublished notes by Uffe Haagerup

$$V(Ru) = \sum_{n\geq 0} c_n(R) T_n(u) \,, \qquad P_R(Ru) = \sum_{n\geq 0} a_n(R) T_n(u) \,,$$

where

$$a_n(R) = \frac{1}{h_n}\int_{-1}^{1} \frac{P_R(Ru)T_n(u)}{\sqrt{1-u^2}}\mathrm{d}u \,, \qquad c_n(R) = \frac{1}{h_n}\int_{-1}^{1} \frac{V(Ru)T_n(u)}{\sqrt{1-u^2}}\mathrm{d}u \,.$$

At electrostatic equilibrium (pushed phase)

$$-\int_{-R}^{R} \log|x-y|\rho_R(y)\mathrm{d}y + V(x) = \mu(R) \qquad |x| \leq R$$

$$= \log 2 + \sum_{n\geq 1} \frac{2}{n} T_n(x) T_n(y)$$

$$\frac{1}{\pi}\frac{P_R(x)}{\sqrt{R^2-x^2}}$$

At electrostatic equilibrium (pushed phase)

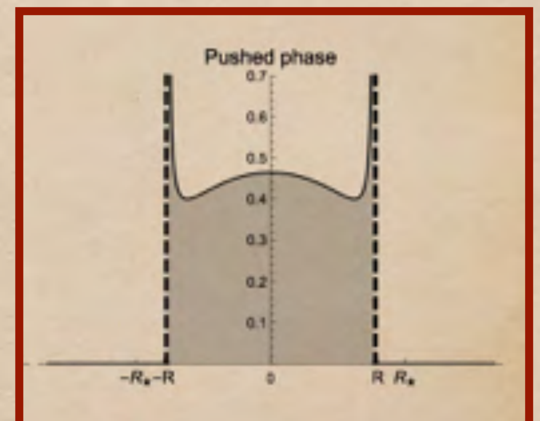
$$-\int_{-R}^{R} \log|x-y|\rho_R(y)\mathrm{d}y + V(x) = \mu(R) \qquad |x| \leq R$$

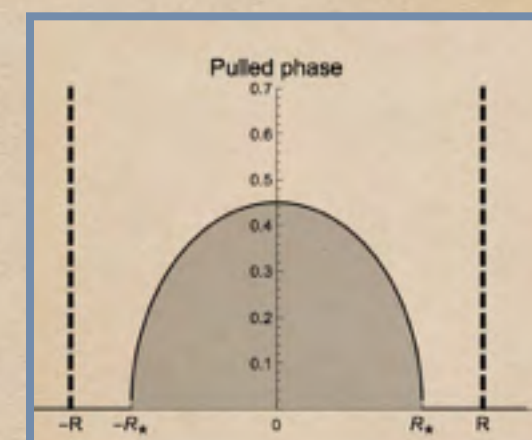
$$-\log\frac{R}{2} + a_0(R)c_0(R) + \sum_{n\geq 1}\underbrace{\left(\frac{1}{n}a_n(R) + c_n(R)\right)}_{\approx 0} T_n\left(\frac{x}{R}\right) = \mu(R) \quad \text{if } |x| \leq R$$

Now, expand the excess free energy in Chebyshev

$$F(R) = \mathscr{E}[\rho_R] - \mathscr{E}[\rho_{R_\star}]$$

$$\mathscr{E}[\rho] = -\frac{1}{2}\iint \log|x-y| \mathrm{d}\rho(x)\mathrm{d}\rho(y) + \int V(x)\mathrm{d}\rho(x)$$





$$\mathrm{d}\rho_R(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{P_R(x)}{\sqrt{R^2-x^2}}\mathbb{1}_{|x|<R}\,\mathrm{d}x & \textit{if } R < R_\star \textit{ (pushed phase)} \\ \dfrac{1}{\pi}Q(x)\sqrt{R_\star^2-x^2}\,\mathbb{1}_{|x|\le R_\star}\,\mathrm{d}x & \textit{if } R \ge R_\star \textit{ (pulled phase)}, \end{cases}$$

Now, expand the excess free energy in Chebyshev

$$F(R) = \mathscr{E}[\rho_R] - \mathscr{E}[\rho_{R_\star}]$$

$$\mathscr{E}[\rho] = -\frac{1}{2}\iint \log|x-y|\mathrm{d}\rho(x)\mathrm{d}\rho(y) + \int V(x)\mathrm{d}\rho(x)$$

$$\Downarrow$$

$$F(R) = \frac{1}{2}\left(-\log\frac{R}{2} + 2c_0(R) - \sum_{n\geq 1}\frac{nc_n^2(R)}{2}\right) \stackrel{?}{=} \frac{1}{2}\int_{\min(R,R_\star)}^{R_\star} \mathrm{d}r \frac{P_r(r)^2}{r}$$

$$\Downarrow$$

$$F'(R) = -\frac{1}{2R}\left(1 - 2Rc_0'(R) + R\sum_{n\geq 1} nc_n(R)c_n'(R)\right) \stackrel{?}{=} -\frac{P_R(R)^2}{2R}$$

How to prove this?

$$F'(R) = -\frac{1}{2R}\left(1 - 2Rc_0'(R) + R\sum_{n\geq 1} nc_n(R)c_n'(R)\right) \stackrel{?}{=} -\frac{P_R(R)^2}{2R}$$

$$P_R(Ru) = \sum_{n\geq 0} a_n(R)T_n(u) \longrightarrow P_R(R) = \sum_{n\geq 0} a_n(R)T_n(1) = \sum_{n\geq 0} a_n(R) = 1 - \sum_{n\geq 1} nc_n(R)$$

so the identity to prove is

$$1 - 2Rc_0'(R) + R\sum_{n\geq 1} nc_n(R)c_n'(R) \stackrel{?}{=} \left(1 - \sum_{n\geq 1} nc_n(R)\right)^2$$

which can be established using

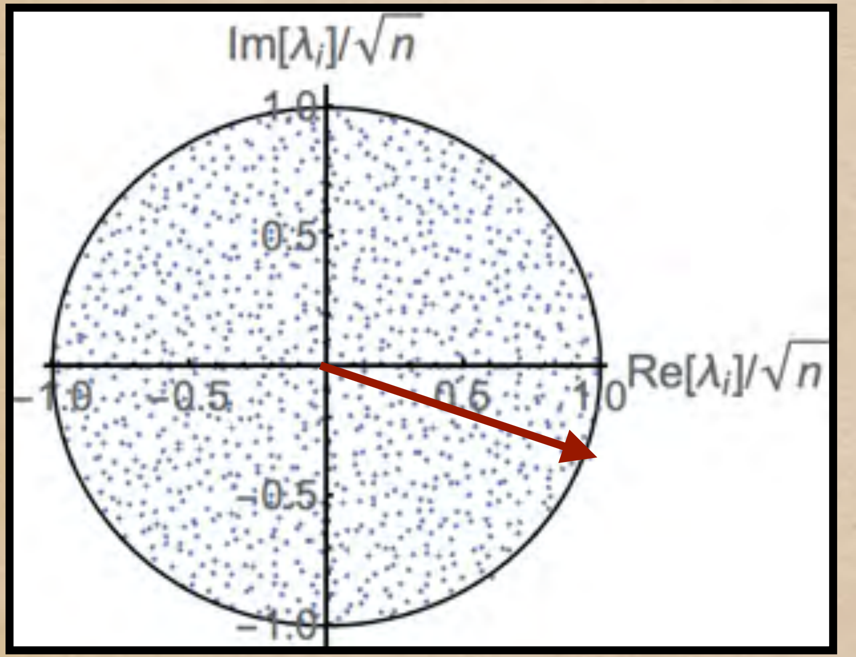
$$\frac{1}{u}\frac{\partial}{\partial R}V(Ru) = \frac{1}{R}\frac{\partial}{\partial u}V(Ru) \qquad \square$$

Interesting questions

1. Is Tracy-Widom necessary to induce a 3rd order PtP phase transition?

2. Is long-range nature of Coulomb interaction necessary to induce a 3rd order PtP phase transition?

1. Is <u>Tracy-Widom</u> necessary to induce a 3rd order PtP phase transition? No!



$$\lim_{N\to\infty} \Pr\left(\max|x_i| \leq R_\star + \frac{\gamma_N + t}{\sqrt{4N\gamma_N}}\right) = G(t)$$

[Rider 2003; Chafai and Peche' 2014]

$$F_{\mathrm{GinUE}}(R) = \begin{cases} \frac{1}{8}(4R^2 - R^4 - 4\log R - 3) & \text{if } R < R_\star \\ 0 & \text{if } R \geq R_\star \,. \end{cases}$$

[Cunden, Mezzadri & PV 2016; Allez, Touboul & Wainrib 2014]

Matching between Gumbel and large deviation function: novel intermediate regime

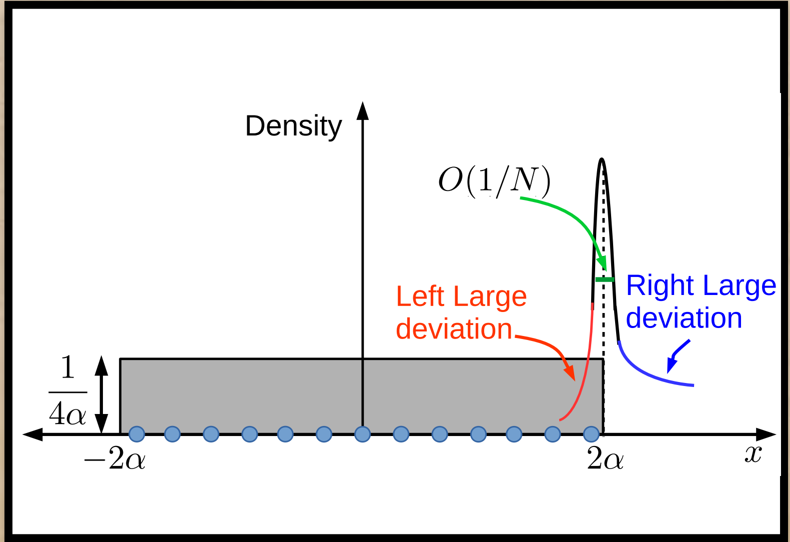
[Lacroix-A-Chez-Toine, Grabsch, Majumdar, Schehr 2017]

1. Is Tracy-Widom necessary to induce a 3rd order PtP phase transition? No!

PRL **119,** 060601 (2017) PHYSICAL REVIEW LETTERS week ending 11 AUGUST 2017

Exact Extremal Statistics in the Classical 1D Coulomb Gas

Abhishek Dhar,[1] Anupam Kundu,[1] Satya N. Majumdar,[2] Sanjib Sabhapandit,[3] and Grégory Schehr[2]
[1]*International Centre for Theoretical Sciences, TIFR, Bangalore 560089, India*
[2]*LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France*
[3]*Raman Research Institute, Bangalore 560080, India*
(Received 28 April 2017; published 8 August 2017)



2. Is long-range nature of Coulomb interaction necessary to induce a 3rd order PtP phase transition?

$$E_N(x_1,\ldots,x_N) = \frac{1}{2}\sum_{i\neq j}\Phi_d(x_i - x_j) + N\sum_k V(x_k), \quad x_i \in \mathbb{R}^d \ .$$

$$\mathrm{D}\Phi_d(x) = \Omega_d\,\delta(x), \quad x \in \mathbb{R}^d, \quad \text{where} \quad \mathrm{D} = -a^2\Delta + m^2 \ .$$

$$\Pr\left(x_i \in B_R,\, i = 1,\ldots,N\right) = \frac{\displaystyle\int_{x_i\in B_R} e^{-\beta E_N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N}{\displaystyle\int_{x_i\in\mathbb{R}^d} e^{-\beta E_N}\,\mathrm{d}x_1\cdots\mathrm{d}x_N} = \frac{Z_N(R)}{Z_N(\infty)} \ .$$

2. Is long-range nature of Coulomb interaction necessary to induce a 3rd order PtP phase transition?

$$\Pr(x_i \in B_R, i = 1, \ldots, N) = \frac{\int_{x_i \in B_R} e^{-\beta E_N} \mathrm{d}x_1 \cdots \mathrm{d}x_N}{\int_{x_i \in \mathbb{R}^d} e^{-\beta E_N} \mathrm{d}x_1 \cdots \mathrm{d}x_N} = \frac{Z_N(R)}{Z_N(\infty)} \approx e^{-\beta N^2 F_{d,a,m}(R)}$$

$$F_{d,a,m}(R) = \frac{1}{2} \int_{R \wedge R_\star}^{R_\star} \frac{c(r)^2}{a^2 r^{d-1}} \mathrm{d}r \, .$$

↓

- 3rd order phase transition
- Electrostatic interpretation

2. Is long-range nature of Coulomb interaction necessary to induce a 3rd order PtP phase transition?

$$\Pr\{x_i \in B_R, i=1,\ldots,N\} = \frac{\int_{x_i \in B_R} e^{-\beta E_N} \mathrm{d}x_1 \cdots \mathrm{d}x_N}{\int_{x_i \in \mathbb{R}^d} e^{-\beta E_N} \mathrm{d}x_1 \cdots \mathrm{d}x_N} = \frac{Z_N(R)}{Z_N(\infty)} \approx e^{-\beta N^2 F_{d,a,m}(R)} .$$

$$F_{d,a,m}(R) = \frac{1}{2} \int_{R \wedge R_\star}^{R_\star} \frac{c(r)^2}{a^2 r^{d-1}} \mathrm{d}r \ .$$

$$\mathrm{d}\rho_R(x) = \begin{cases} \frac{1}{\Omega_d}\left(\sigma_R(x) \mathbb{1}_{|x| \leq R} \mathrm{d}x + c(R) \mathbb{1}_{|x|=R}\right) & \textit{if } R \leq R_\star \textit{ (pushed phase)} \\ \frac{1}{\Omega_d} \sigma_{R_\star}(x) \mathbb{1}_{|x| \leq R_\star} \mathrm{d}x & \textit{if } R \geq R_\star \textit{ (pulled phase)}, \end{cases}$$

recovers GinUE for d=2, a=1, m=0

Summary and Outlook

- Exact formula for the excess free energy of a log-gas on a line constrained between two walls
- Electrostatic interpretation: order parameter is the electrostatic pressure of the constrained fluid
- Under rather general conditions, the excess free energy of the constrained gas has always a 3rd order discontinuity, in all dimensions. The spatial extent of the repulsion plays no role.

Third-order phase transition: random matrices and screened Coulomb gas with hard walls

Fabio Deelan Cunden · Paolo Facchi · Marilena Ligabò · Pierpaolo Vivo

Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

PAPER: Disordered systems, classical and quantum

Universality of the third-order phase transition in the constrained Coulomb gas

Fabio Deelan Cunden[1], Paolo Facchi[2,3], Marilena Ligabò[4] and Pierpaolo Vivo[5]

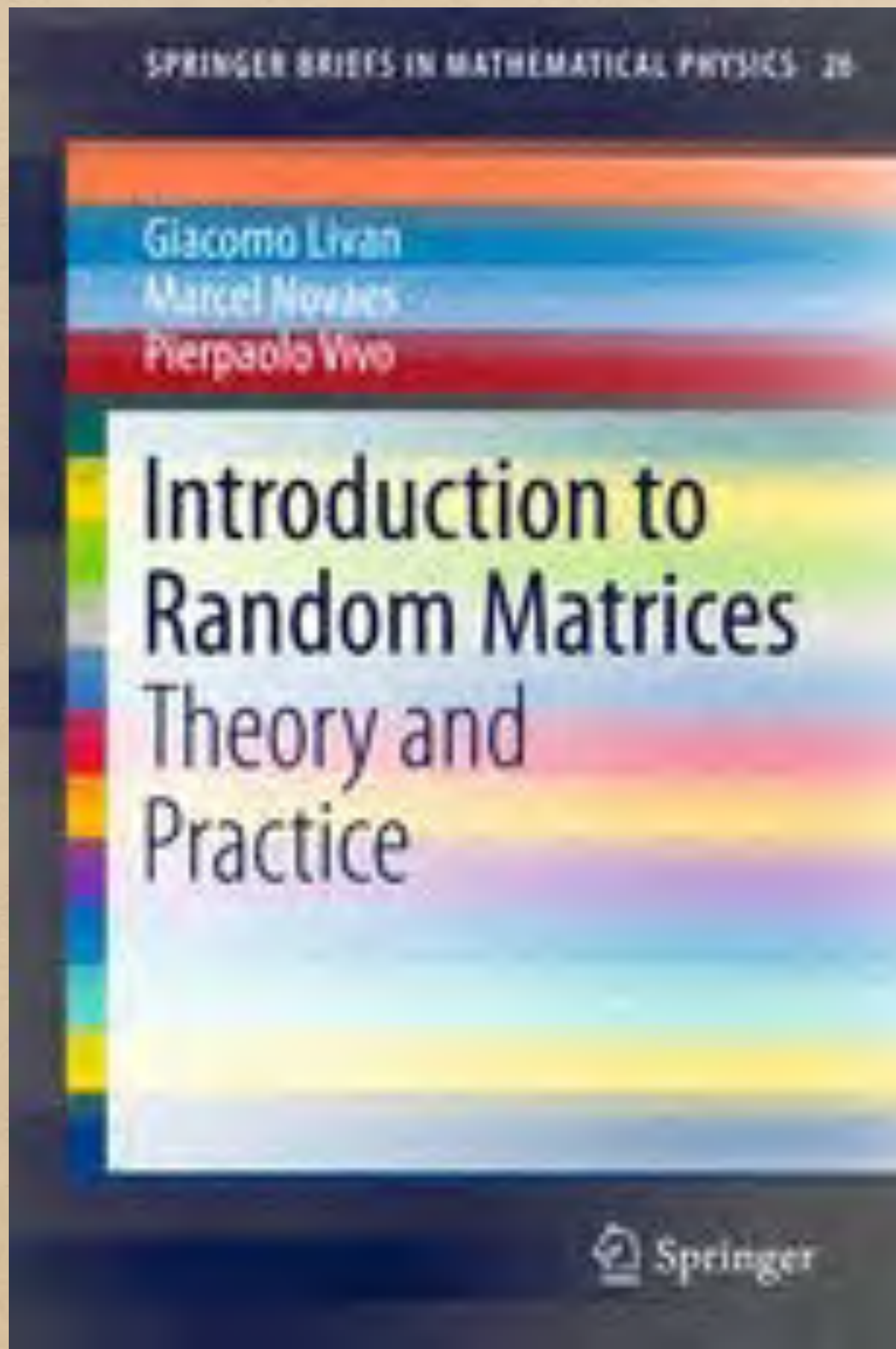
IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **51** (2018) 35LT01 (11pp) https://doi.org/10.1088/1751-8121/aad147

Letter

Universality of the weak pushed-to-pulled transition in systems with repulsive interactions

Fabio Deelan Cunden[1,6], Paolo Facchi[2,3], Marilena Ligabò[4] and Pierpaolo Vivo[5]



Freely available
on arXiv!

Journal of Physics A: Mathematical and Theoretical

Random Matrices: the first 90 years

Guest Editors

Giulio Biroli CEA - Paris, France
Zdzisław Burda AGH, Kraków, Poland
Pierpaolo Vivo King's College London, UK

Scope

2018 marks the 90th birthday of Random Matrix Theory (RMT). Originally introduced in multivariate statistics by Wishart, random matrices were first employed in physics at the end of the 50s to model Hamiltonians of heavy nuclei. In recent years, RMT techniques and insights have proven invaluable in deepening our understanding of:

- Fermions in harmonic traps
- Models of interface growth (e.g. KPZ equation and Tracy-Widom), also from the experimental point of view
- Models of delocalized, non-ergodic phases in quantum systems
- Financial applications, like cleaning of correlation matrices
- Condensed matter, like topological insulators
- Stability of ecosystems
- Wireless telecommunications

to mention just a few. This special issue is devoted to recent theoretical advancements, with a focus on universal and cross-disciplinary results. With this issue, we hope to extend the traditional boundaries of the discipline and to stimulate discussions about the most interesting directions for future cross-fertilized research.

Extra slides

[1]

PHYSICAL REVIEW D VOLUME 21, NUMBER 2 15 JANUARY 1980

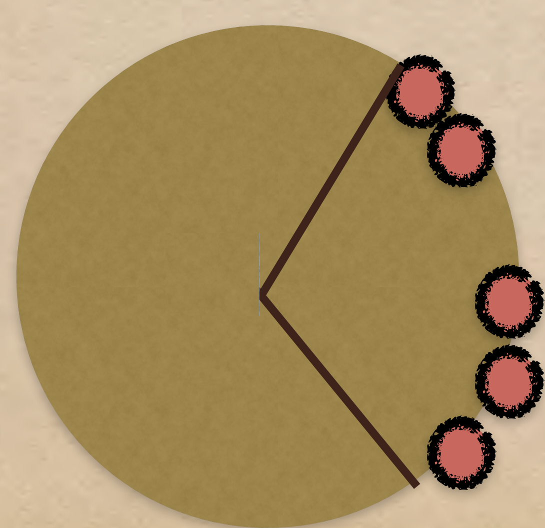
Possible third-order phase transition in the large-N lattice gauge theory

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

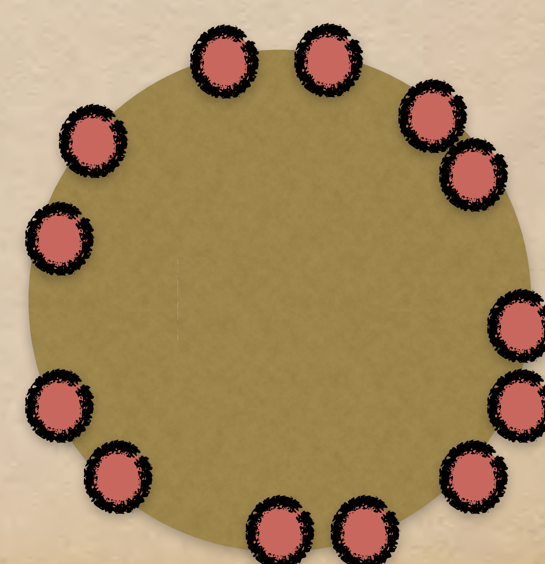
Edward Witten
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 10 July 1979)

2-d lattice QCD with Wilson action

$$\zeta = \int \mathcal{D}U \ \exp[(N/g)\,\mathrm{Tr}(U + U^\dagger)]$$



g<2



g>2

[2]

GENERAL

Will a Large Complex System be Stable?

Gardner and Ashby[1] have suggested that large complex systems which are assembled (connected) at random may be expected

[R. M. May, 1972]

A population with N non-interacting distinct species...

$$x_i(t) = \rho_i(t) - \rho_i^*$$

$$\frac{dx_i(t)}{dt} = -x_i(t)$$

$$x_i(t \to \infty) = 0$$

GENERAL

Will a Large Complex System be Stable?

Gardner and Ashby[1] have suggested that large complex systems which are assembled (connected) at random may be expected

[R. M. May, 1972]

A population with N distinct species, with pairwise interactions

$$x_i(t) = \rho_i(t) - \rho_i^*$$

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \sum_{j=1}^{N} J_{ij} x_j(t)$$

Coupling Strength

Random matrix

GENERAL

Will a Large Complex System be Stable?

Gardner and Ashby[1] have suggested that large complex systems

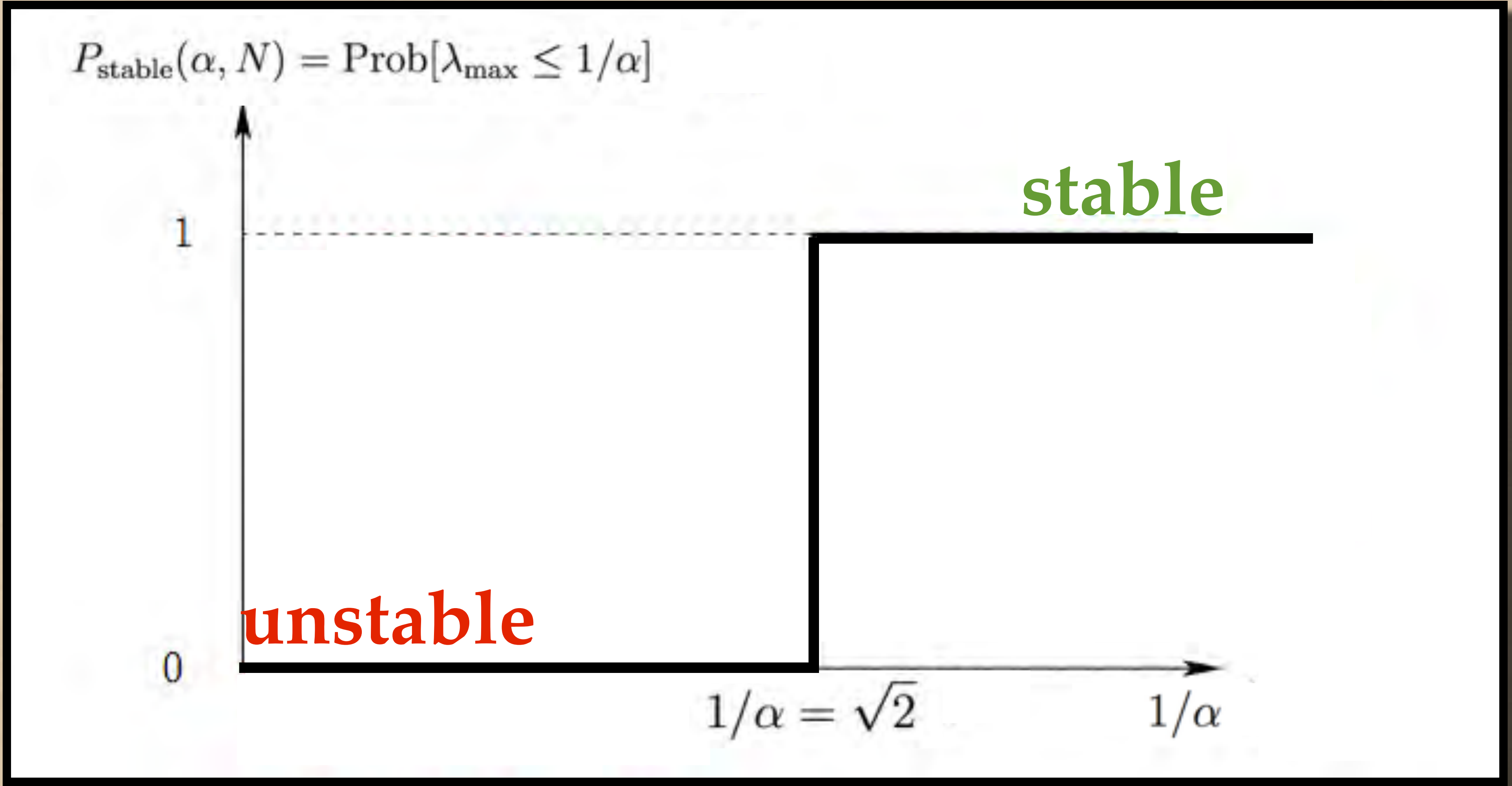
[R. M. May, 1972]

A population with N distinct species, with pairwise interactions

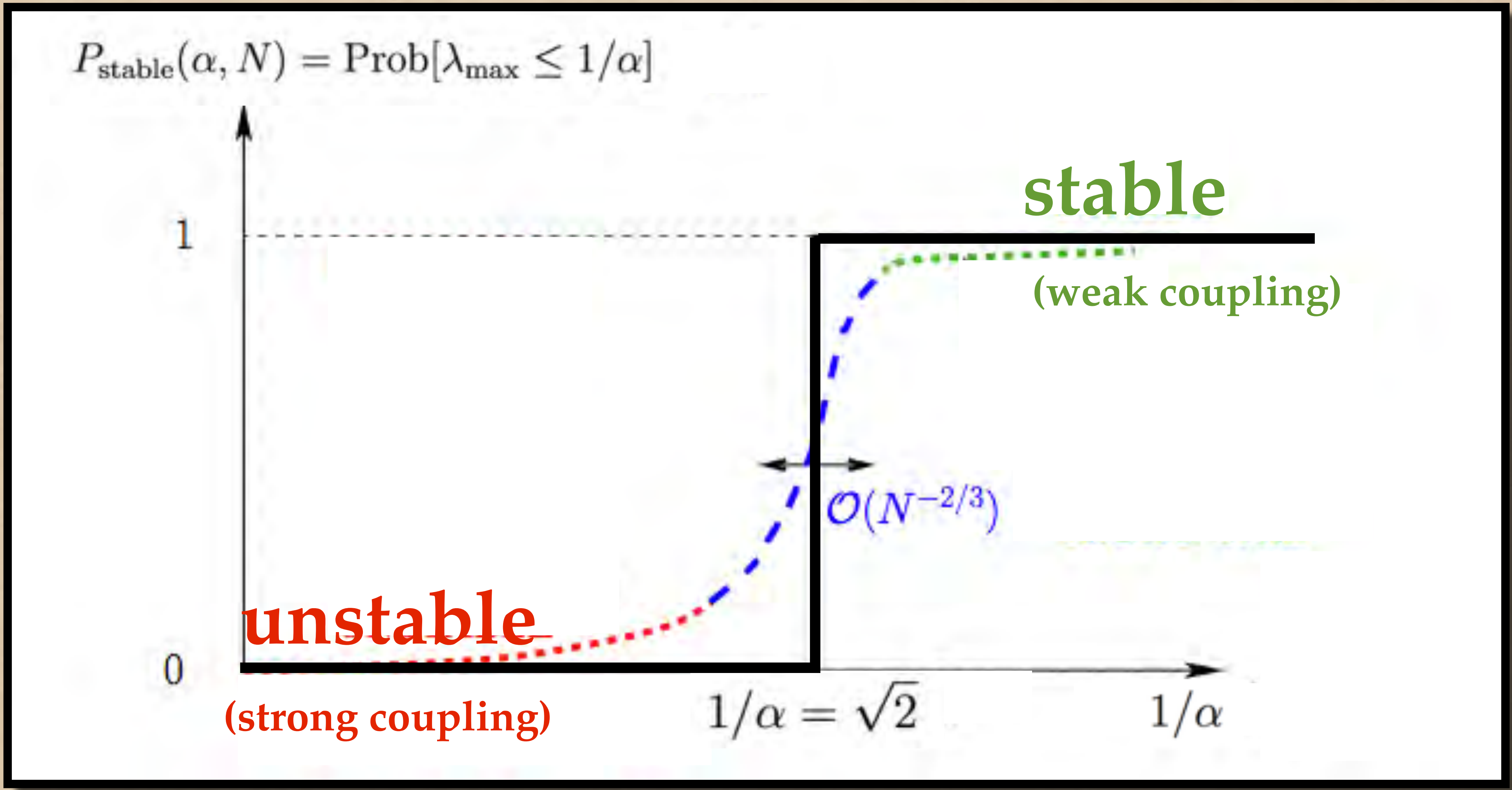
$$x_i(t) = \rho_i(t) - \rho_i^*$$

$$\frac{dx_i(t)}{dt} = -x_i(t) + \alpha \sum_{j=1}^{N} J_{ij} x_j(t)$$

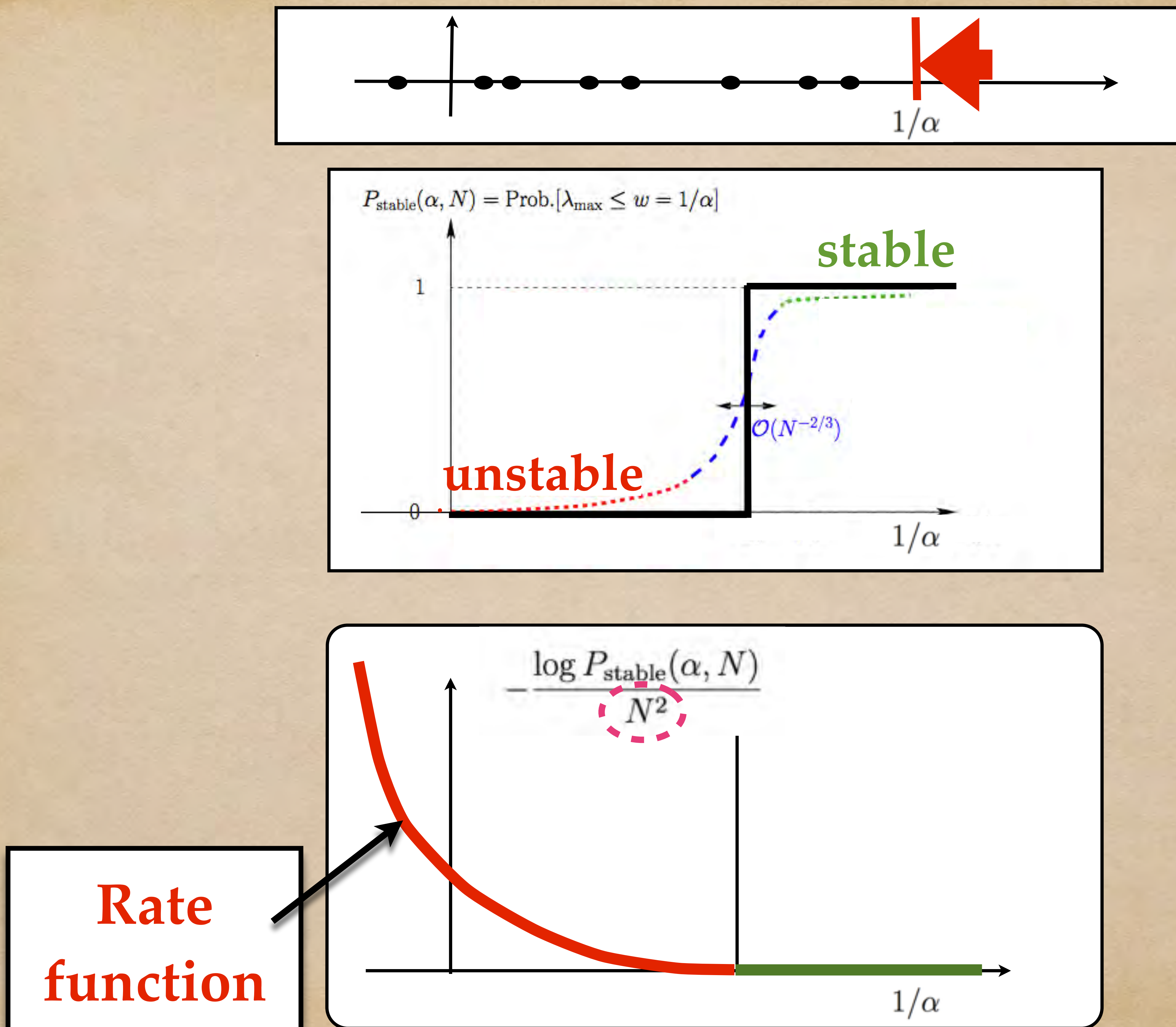
$$P_{\text{stable}}(\alpha, N)$$?



$$P_{stable}(\alpha, N) = \text{Prob}[-1 + \alpha\lambda_i \leq 0, \quad \forall i] = \text{Prob}[\lambda_{max} \leq 1/\alpha]$$



``...too large an average interaction strength...leads to instability. The larger the number of species, the more pronounced the effect. "

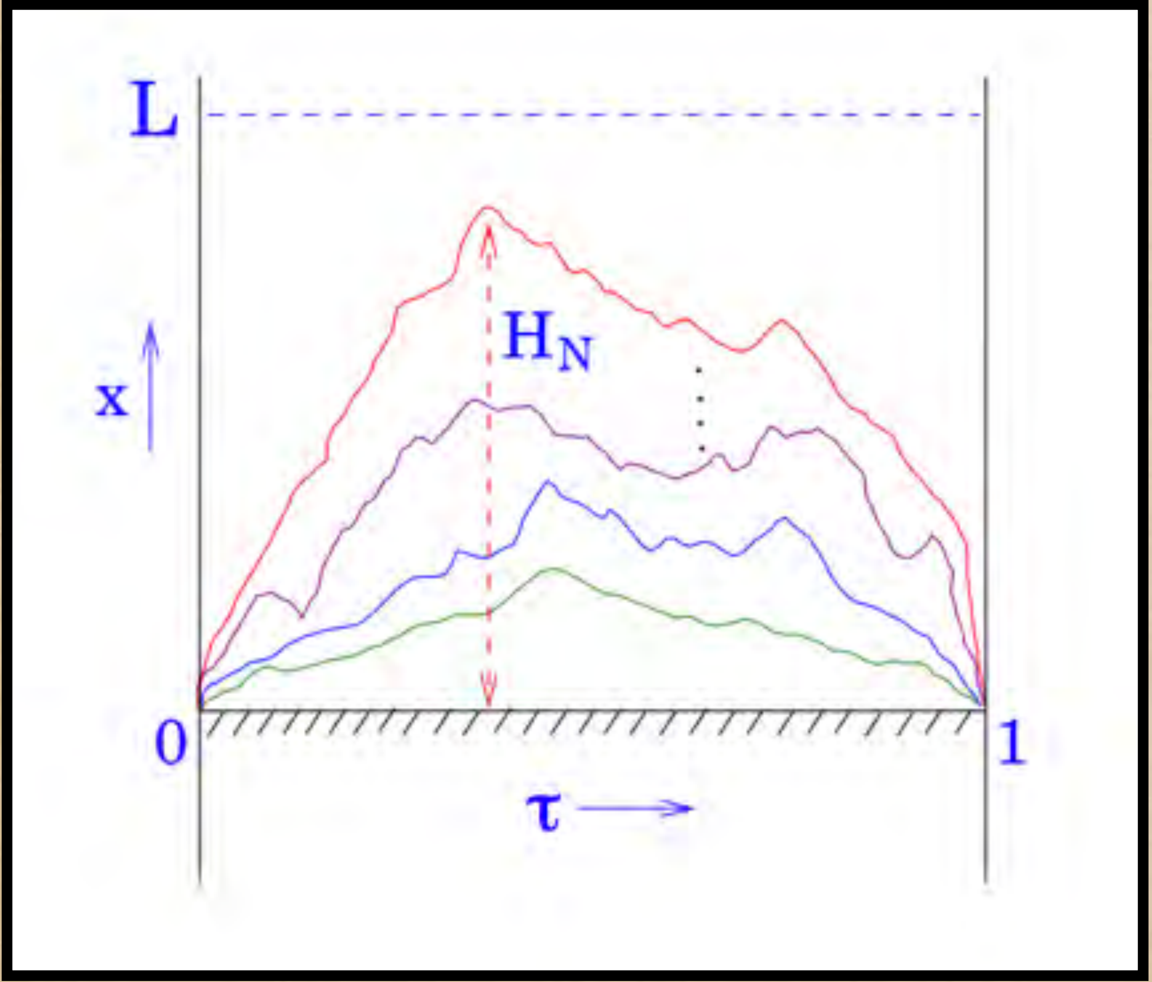


$1/\alpha$
$P_{\text{stable}}(\alpha, N) = \text{Prob.}[\lambda_{\max} \leq w = 1/\alpha]$
stable
1
$\mathcal{O}(N^{-2/3})$
unstable
0
$1/\alpha$
$-\frac{\log P_{\text{stable}}(\alpha, N)}{N^2}$
Rate function
$1/\alpha$

[3]

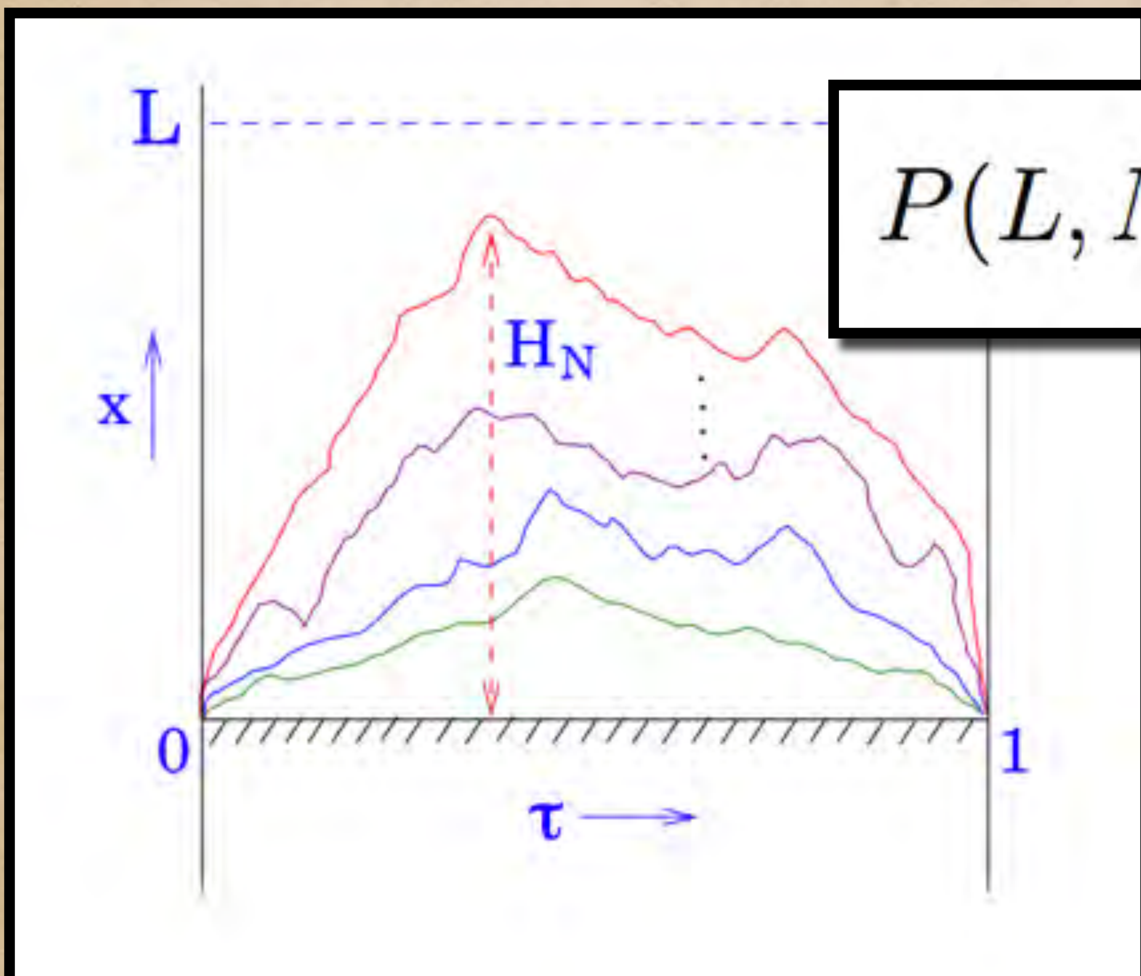
...gnment of post-collision weights by which we characterize vicious ...unken walkers.

Vicious drunks shoot on sight—we refrain from speculating as to their national origins—but they are short-sighted: thus on arriving at the same site they shoot each other dead: otherwise, they do not interact. In formal terms, the allowed steps carry the same weights, w_1, w_0, or w_{-1}, as before



assignment of post-collision weights by which we characterize vicious drunken walkers.

Vicious drunks shoot on sight—we refrain from speculating as to their national origins—but they are short-sighted: thus on arriving at the same site they shoot each other dead: otherwise, they do not interact. In formal terms, the allowed steps carry the same weights, w_1, w_0, or w_{-1}, as before

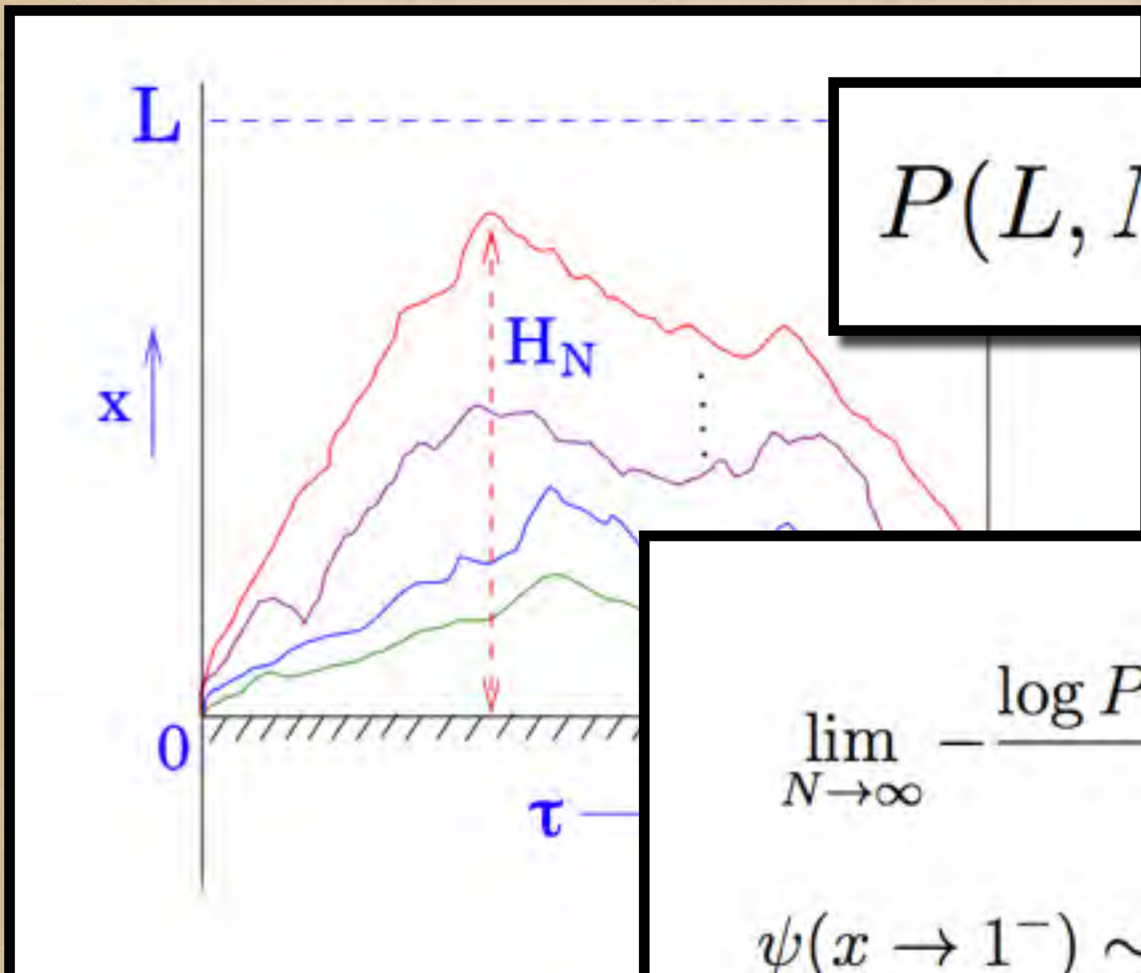


$$P(L, N) := \text{Prob}[H_N \leq L]$$

?

assignment of post-collision weights by which we characterize vicious drunken walkers.

Vicious drunks shoot on sight—we refrain from speculating as to their national origins—but they are short-sighted: thus on arriving at the same site they shoot each other dead: otherwise, they do not interact. In formal terms, the allowed steps carry the same weights, w_1, w_0, or w_{-1}, as before

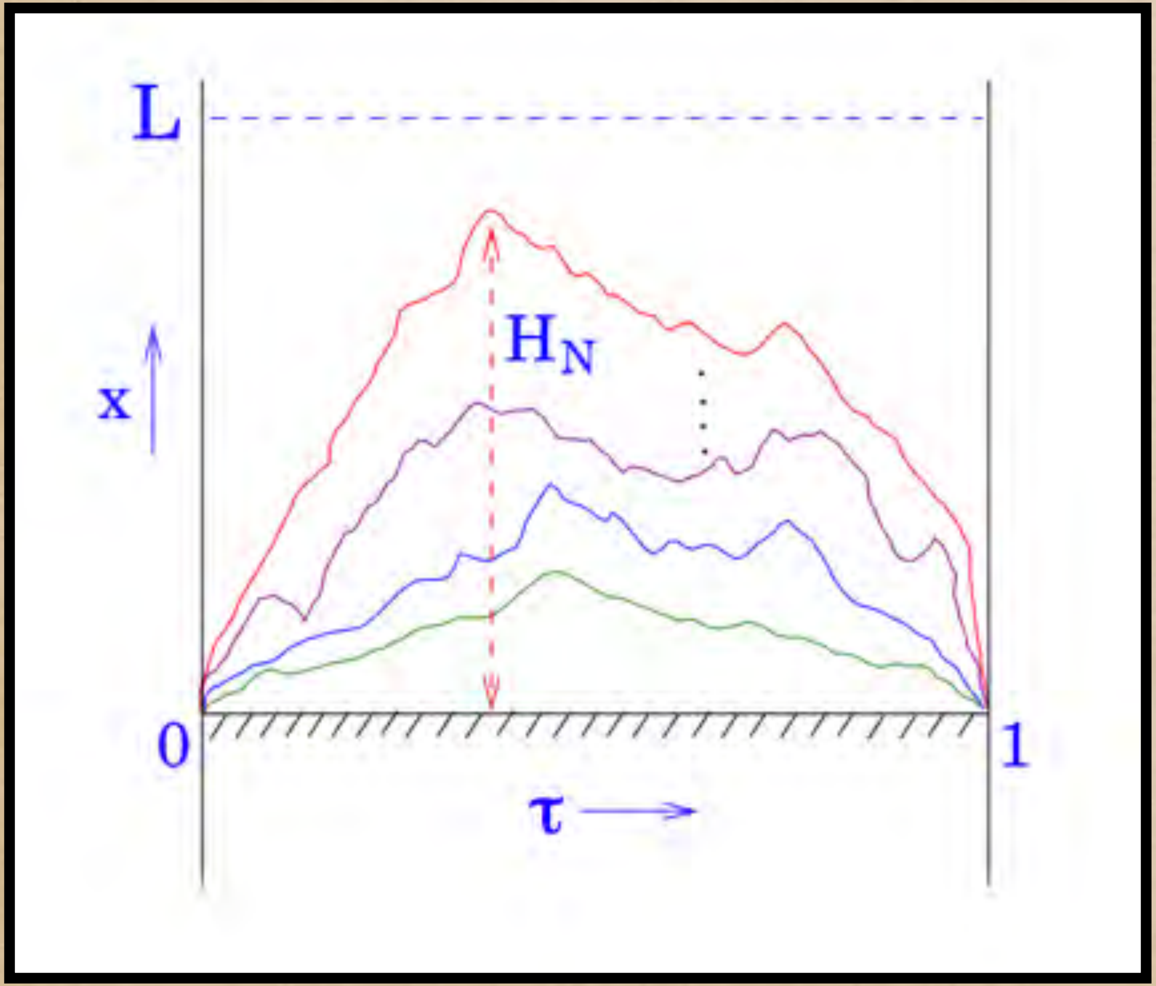


$$P(L, N) := \text{Prob}[H_N \leq L]$$

?

$$\lim_{N\to\infty} -\frac{\log P(x\sqrt{2N}, N)}{N^2} = \begin{cases} \psi(x), & x < 1 \\ 0, & x > 1 \end{cases}$$

$$\psi(x \to 1^-) \sim \frac{16}{3}(1-x)^3$$


L
x
H_N
0
1
τ

Largest eigenvalue of Gaussian ensembles

$$F_N(w) = \text{Prob}[x_{\max} \leq w]$$

$$F_N(w) = \frac{Z_N(w)}{Z_N(w \to \infty)}$$

$$Z_N(w) = \int_{-\infty}^{w} \mathrm{d}x_1 \cdots \int_{-\infty}^{w} \mathrm{d}x_N \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N} x_i^2 - \sum_{i \neq j} \ln|x_i - x_j|\right)\right]$$