Hierarchical Random Matrices and Operators

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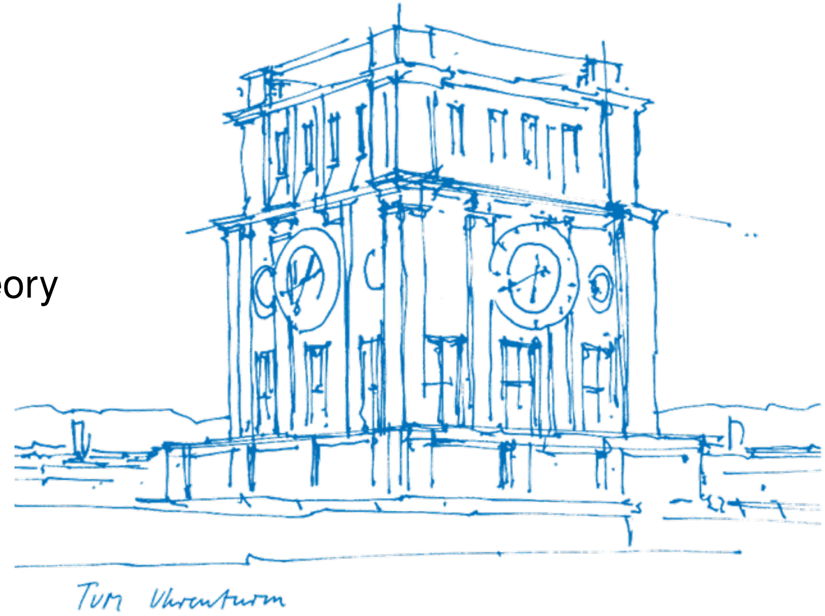
Joint works with Per von Soosten (TU München)

XIV Brunel-Bielefeld Workshop on Random Matrix Theory



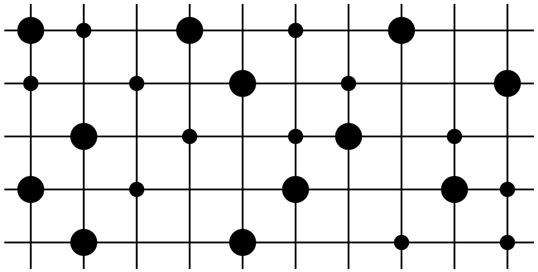
The localization-delocalization challenge

Understand the various facets of metal-insulator transitions in disordered systems, *e.g.*

- **Anderson Model** Anderson '58

$$H(\omega) = \Delta + V(\omega) \quad \text{on} \quad \ell^2(\Lambda)$$

with $\Lambda \subseteq \mathbb{Z}^d$



- **Power-Law Random Band Matrices (PRBM)** Mirlin/Fyodorov/Dittes/Quezada/Seligman '96

$$\mathbb{E}\left[H_{xy}\right] = 0$$

$$\mathbb{E}\left[H_{xy}^2\right] = \begin{cases} 1 & \text{if } d(x,y) \leq 1 \\ d(x,y)^{-2\alpha} & \text{else} \end{cases}$$

e.g. real, symmetric $N \times N$ matrix (H_{xy})

The localization-delocalization challenge

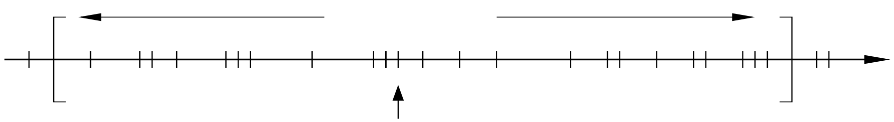
Predicted features – as a function of parameters in the model (α) or energy/disorder strength (E):

- **Eigenvectors** ψ undergo **localization-delocalization transition**

  Inverse participation ratios for ℓ^2-normalized function: $\|\psi\|_\infty = \begin{cases} \mathcal{O}(1) & \text{localization} \\ \mathcal{O}(N^{-1/2}) & \text{delocalization} \end{cases}$

- **Infinite-volume** self-adjoint operator has a.s. **pure point** or **continuous spectrum**.

- **Eigenvalue statistics** changes from Poisson to Random Matrix (GOE)



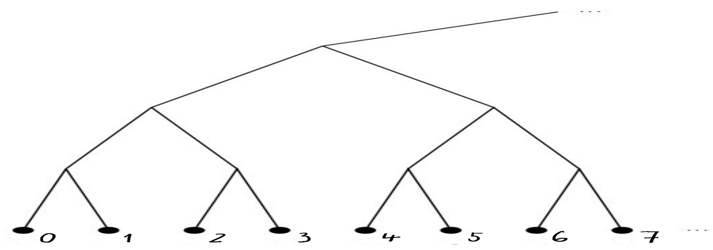
Rescaled random process of eigenvalues close to some energy E:

$$\sum_{\lambda \in \sigma(H)} \delta_{N(\lambda - E)} \xrightarrow[N \to \infty]{} \begin{cases} \text{Poisson process} & \text{localization} \\ \text{GOE process} & \text{delocalization} \end{cases}$$

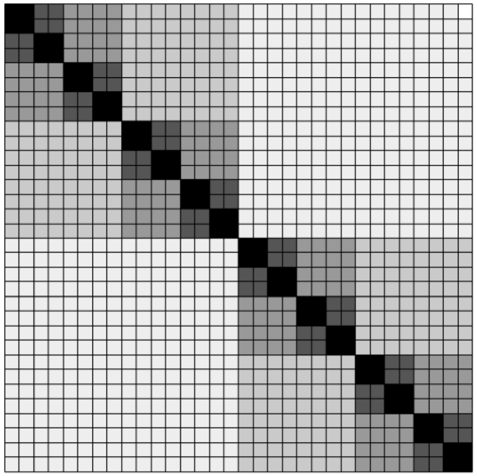
Hierarchical random matrices & operators

Hierarchy on set $X = \mathbb{N}$ or $X = \{1, 2, \ldots, 2^n\}$

Ultra-Metric: $d(x, y) = \min\{r \mid x, y \text{ lie in the same partition of size } 2^r\}$



Nested sequence of partitions $\mathcal{P}_r$, $r \geq 0$

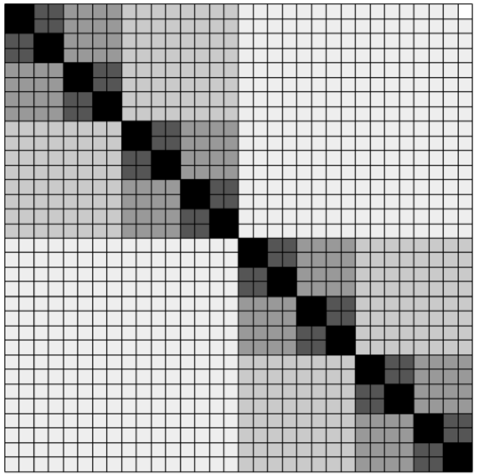


Hierarchical operator H on $\ell^2(X)$

$$H = \sum_{r \geq 0} \sum_{B \subset \mathcal{P}_r} H(B)$$

with self-adjoint $H(B)$ on $\ell^2(B)$.

Hierarchical or ultrametric ensemble (UE)



$N \times N$ random matrices $\qquad (N = 2^n)$

$$Z_{n,c}\, H_n = \sum_{r=0}^{n} 2^{-\frac{(1+c)}{2} r} \sum_{B \in \mathcal{P}_r} \Phi_B$$

with $c \in \mathbb{R}$ and (Φ_B) independent **GOE matrices.**

- Hierarchical analogue of PRBM:

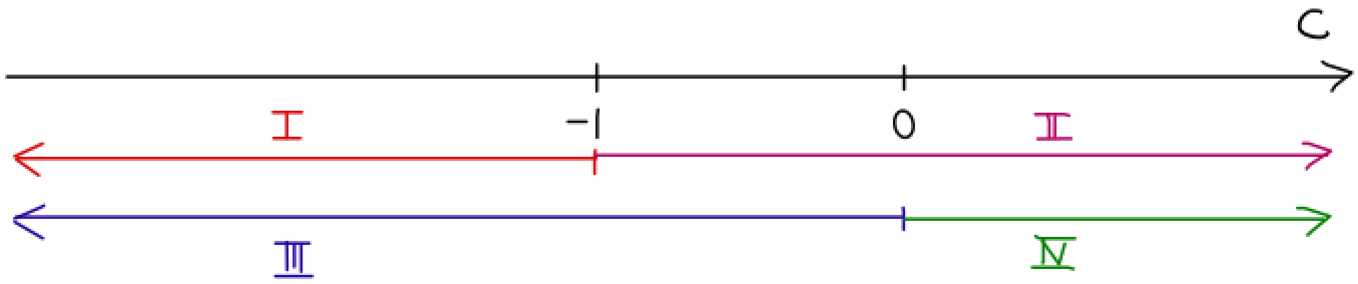
$$Z_{n,c}^2\, \mathbb{E}\left[(H_n)_{xy}^2\right] \approx 2^{-(2+c)d(x,y)}.$$

i.e. $\alpha > \mathbf{1}$ corresponds to $\mathbf{c} > \mathbf{0}$!

- Normalization $Z_{n,c}$ is chosen s.t. the variance matrix is doubly stochastic, *i.e.*

$$\sum_y \mathbb{E}\left[(H_n)_{xy}^2\right] = 1\,.$$

The parameter regimes of UE



I Mean Field Regime (GOE)

II Self-adjoint infinite-volume operator H_∞ on $\ell^2(\mathbb{N})$ exists

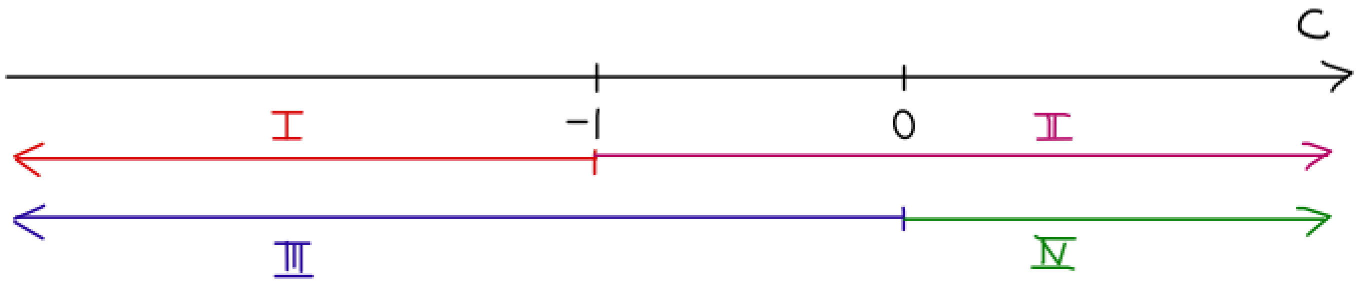
III Delocalisation Regime

IV Localisation Regime

Predictions: Fyodorov/Ossipov/Rodriguez '09, . . . , Bogomolny/Sieber '18

- All bulk eigenvectors undergo a localization-delocalization transition at $c = 0$
- Accompanying transition of eigenvalue statistics from Poisson to GOE

Summary of results



I Complete delocalization of eigenvectors & GOE eigenvalue statistics — Lee/Schnelli '13, Landon/Soesoe/Yau '16

II Spectrum of H_∞ on $\ell^2(\mathbb{N})$
- $\sigma_{ac}(H_\infty) = \emptyset$ a.s. if $c > 1$.
- **bulk spectral measure is a.s. θ-Hölder continuous with $\theta \in (0,1)$ if** $-1 < c < -1/2$

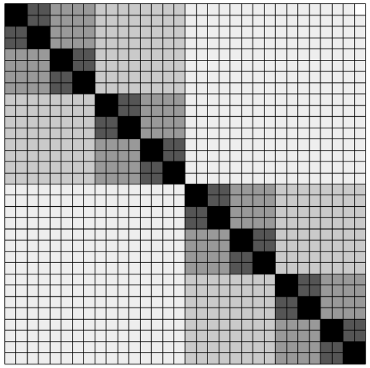
III If $c \in (-1, -1/2)$ **bulk eigenvectors are completely delocalized & GOE eigenvalue statistics** — Soosten/W. '18

IV Localization of eigenvectors & Poisson eigenvalue statistics — Soosten/W. '17

Renormalization approach

Iterative construction:

$$H_n = H_{n-1} \oplus H'_{n-1} + \sqrt{t_n}\,\Phi_n$$



- Independent copies H_{n-1}, H'_{n-1} plus GOE Φ_n
- Time scale $t_n = 2^{-n(1+c)}$ with $c \in (-1, 0)$ is
  - too short for macroscopic delocalization,
  - long enough for eigenfunction hybridization and local GOE equilibration.

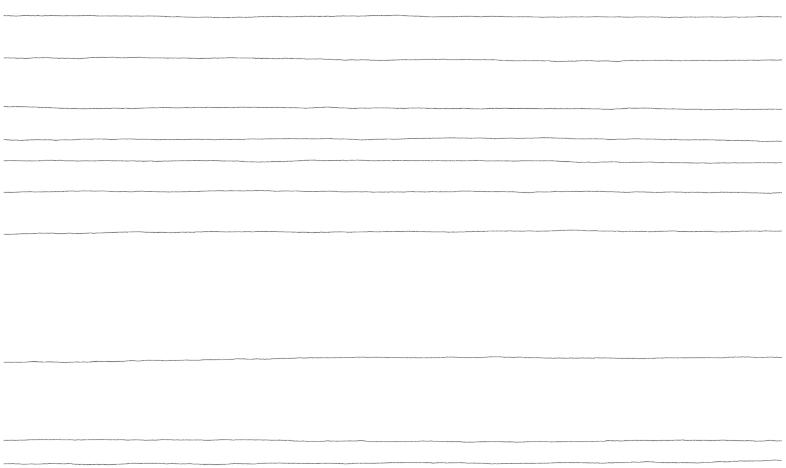
Renormalization step

Building blocks: **Rosenzweig-Porter Ensemble (RP)**

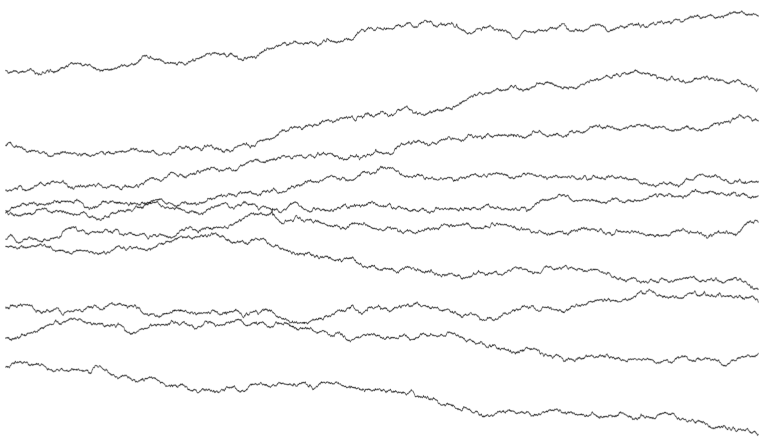
$$H(t) = V + \sqrt{t}\,\Phi \qquad t \geq 0\,,$$

with $N \times N$ initial matrix V and GOE matrix Φ.

- Eigenvalues undergo **Dyson Brownian Motion** $\quad \sqrt{t}\,\Phi_{x,y} \equiv \sqrt{\frac{1+\delta_{xy}}{N}}\,B_{xy}(t)$



10 Trajectories until $t = N^{-1}$ of 100-particle DBM with independent initial conditions



Trajectories until $t = 1$ of 10-particle DBM with independent initial conditions

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Time scales	**Results for RP**		
$t \ll N^{-1}$	pertubative regime	$c > 0$	Soosten/W. '17
$N^{-1} \ll t$	local equilibration regime	$c < 0$	(*)
$1 \ll t$	global equilibration regime uniform delocalisation	$c < -1$	(*)
$N^{-1} \ll t \ll 1$	**intermediate regime in RP** **non-uniform delocalization**	$c \in (-1, 0)$	Soosten/W. '17 Begnini '17

see also: Facoetti/Vivo/Biroli '16, . . .
Altshuler/Cuevas/Ioffe/Kravtsov '16

(*) *Dynamical approach to random matrix theory – local & global equilibration:* Erdős, Yau, . . .
. . . , Schlein, Bourgade, Yin, Lee, Schnelli, Huang, Landon, Sosoe, . . . '09-'17

Proof ideas for delocalization

Going beyond the mean-field regime to $c \in (-1, -1/2)$ requires:

- **Local law** for $S_n(z) = 2^{-n} \operatorname{Tr} (H_n - z)^{-1}$ down to scales $\operatorname{Im} z \gg 2^{-n}$ in case $c < -1/2$.
- Iterative control of **Dyson Brownian Motion (DBM)** flow of $G_n(x; z) = \langle \delta_x, (H_n - z)^{-1} \delta_x \rangle$

Key: One-step DBM flow of $H_t = H_0 + \sqrt{t}\,\Phi_N$:

$$dG_t(x, x, z) = \Big(S_t(z) \partial_z G_t(x, x, z) + \frac{1}{2N} \partial_z^2 G_t(x, x, z) \Big)\, dt + dM_t(x, z)$$

$$dS_t(z) = \Big[S_t(z) \partial_z S_t(z) + \frac{1}{2N} \partial_z^2 S_t(z) \Big]\, dt + dN_t(z)$$

with explicit martingals $dM_t(x, z)$, $dN_t(z)$,

combined with the **method of characteristics**.

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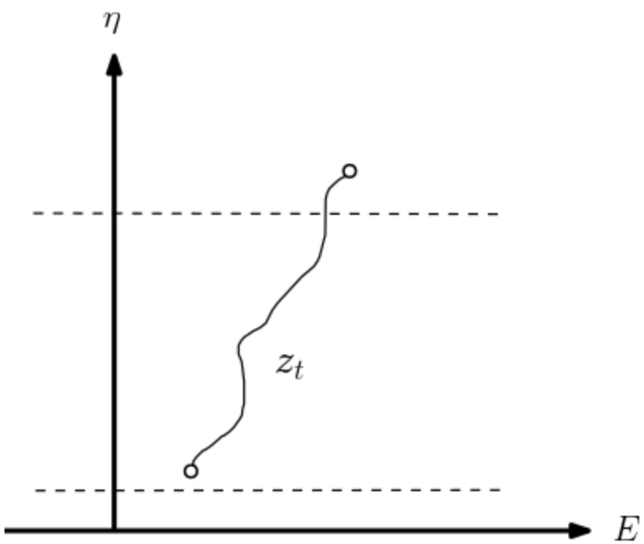


$$\dot z_t = -S_t(z_t)\,, \quad z_0 = z \in \mathbb{C}^+$$

In sense made more precise in Soosten/W. '17

$$S_t(z_t(z)) \approx S_0(z)$$
$$G_t(x,x;z_t(z)) \approx G_0(x,x;z)$$

provided $\operatorname{Im} z \gg N^{-1}$.

Remaining task: invert characteristics

Local laws for Schrödinger matrices

Schrödinger $N \times N$ matrices with any $A = A^*$ non-random and iid (ω_x) with density $\varrho \in L^\infty$:

$$V = A + \mathrm{diag}(\omega_1, \ldots, \omega_N)$$

Lemma (Soosten/W. '18)

Let $z = E + i\eta$. For any $\mu > 0$

$$\mathbb{P}\left(|S_0(z) - \mathbb{E}[S_0(z)]| > \mu\right) \leq C \exp\left(-c\mu^2 N\eta^2\right)$$

Proof idea: McDiarmid concentration inequality + rank-one perturbation theory.

Remark: Optimal upper bound also holds:

$$\mathbb{P}\left(\mathrm{Im}\, S_0(z) > e\pi\|\rho\|_\infty + \mu\right) \leq \exp\left(-\mu N\eta\right).$$

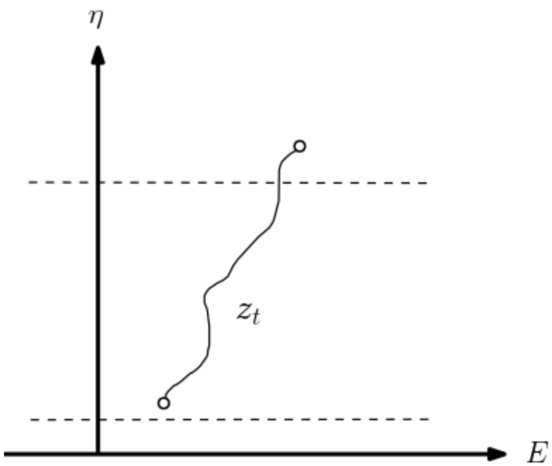
Open problem: Local law down to scale $\mathrm{Im}\, z \gg N^{-1}$.

Delocalization thru stochastic energy renormalization

Iterative construction: $H_n = H_{n-1} \oplus H'_{n-1} + \sqrt{t_n}\,\Phi_n$

For eigenvectors corresponding to eigenvalues $E \in W$ of H_n at any $\eta \gg 2^{-n}$

$$\|\psi\|_\infty^2 \leq \sup_{E\in W} \max_x \eta \operatorname{Im} G_n(x,x;E+i\eta) \approx \sup_{E\in W} \max_x \eta \; \operatorname{Im} G_{n-1}\left(x,x; z_{t_n}^{-1}(E+i\eta)\right).$$



Given the local law

$$\inf_{E\in W} \operatorname{Im} S_n(E+i\eta) > 0$$

the characteristics travels $\mathcal{O}(t_n)$ in the upper half plane. Infinite iteration and the apriori bound $|G(x;x;z)| \leq (\operatorname{Im} z)^{-1}$ yield uniform delocalization of eigenvectors.

Thank You!

P. von Soosten, S.W.:

- ***Delocalization and continuous spectrum for ultrametric random operators***
  arXive:1811.10517.
- *Non-Ergodic Delocalization in the Rosenzweig-Porter Model*
  arXiv:1709.10313. To appear in: LMP.
- *The phase transition in the ultrametric ensemble and local stability of Dyson Brownian motion*
  Electron. J. Probab., 23:1–24 (2018).
- *Singular spectrum and recent results on hierarchical operators*
  pp. 215–225 in: Contemp. Math. 717, AMS (2018)