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Interior point methods

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1 Introduction

The interest in interior point methods for linear programming emerged from Karmarkar's contribution in 1984. This field has soon become one of the most active in the area of mathematical programming. It introduced new ideas and techniques that now have received their own place among the basic tools in optimization. In this chapter we intend to provide an introduction to the theory of interior point methods.

To get a good understanding of interior point methods, one must keep in mind that at the heart of this theory is the analysis of complexity of algorithms. For a long time the simplex algorithm was the only practical algorithm for linear programming. Many people tried to justify the remarkable efficiency of the method by providing a theoretical bound on the number of simplex iterations. To date, no one has yet given a polynomial bound for general problems. The more basic question of the mere existence of an algorithm that would solve any instance of linear programming in polynomial time remained open until Khachiyan [27] settled the issue in 1979. Khachiyan's contribution was fundamental for the theory of linear programming and related topics. Unfortunately his ellipsoid method did not fulfil its promise as a practical solution method for linear programming. Shortly after Khachiyan's contribution, Karmarkar, in a seminal paper [26], patched the gap between theory and practice. He proposed a totally new method that enjoyed both polynomial complexity and practical efficiency. Since then, it has been the rule for each new algorithmic contribution that the author provides a complexity analysis and at least a bound on the total number of iterations. Consequently, a good deal of the developments in this chapter will be devoted to establishing bounds on the number of iterations.

In presenting interior point methods we are faced with a difficult choice. Due to the intense research activity in the field, literally hundreds of algorithms have been devised and analysed. The bibliography of Kranich [31, 32], in its most recent update of mid-1993, mentions approximately 1,300 papers, almost all of them dealing with some "new" interior point method. Although those methods obviously share similarities, they still look pretty much different. This very diversity discourages the non-specialist, and in the meantime challenges the specialist in search of a unifying framework. In this chapter we hope to give at least a partial answer to that query.

We organize this chapter around four main themes. After a short review of the definition relative to complexity we shall first revisit, in Section 3, the theory of linear programming itself, in the light of the ideas of interior point methods. We will put special emphasis on the concept of the central path, a privileged continuous curve that is interior to the feasible set and converges to an optimal point. Most interior point methods follow, or are related to, the central path. To describe those methods we shall use the notion of target sequence, an idea that was informally introduced by Mizuno [36, 37] and systematized in [25]. The concept captures the basic ideas that underlie some of the most prominent interior point methods that appeared in the literature. It provides a unified framework for a variety of methods, by-passing the need for separate analyses. This will be dealt with in Section 4. The last two sections will address two important issues and discuss algorithms that give answers to them. The first issue, dealt with in Section 5, is the asymptotic behaviour of interior point methods. It is often observed that shortly before convergence interior point methods drastically accelerate. This phenomenon can be explained relatively easily for the method named predictor–corrector. Section 6 will be devoted to the issue of infeasible starts. Indeed the vast majority of interior point algorithms suppose that an initial interior feasible point is at hand. Some special procedure must be set to produce this initial feasible point. However interior point methods have the nice property of enforcing feasibility quite naturally. This has recently been an area of very active research. See, e.g. [29, 38, 41, 44, 53, 57]. Interestingly enough a variant of Karmarkar's algorithm provides a simple solution to the problem. This gives us the opportunity to discuss Karmarkar's contribution and to conclude the chapter at the point where the field of interior point methods for linear programming started.

2 Complexity and convergence

Much of the theoretical analysis of algorithms has to do with the convergence properties. In the analysis of interior point methods, essentially two types of convergence are used. The first one is global convergence: does the algorithm converge to a solution and after how many iterations? The second one has to do with the asymptotic rate of convergence: is the progress towards a solution steady or can we expect an acceleration after some stage?

2.1 Complexity and global convergence

The main point of a complexity analysis is to estimate the total computational effort to solve any given linear programming problem. (The memory space used